Application of Least Square Estimation Technique for Image Restoration Using Signal-Noise Correlation Constraint

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Abstract—A linear degradation model for digital image restoration has been considered. It is assumed that the noise is additive, uncorrelated and independent of the degradation process and also of original image. A new objective criterion has been used for the least square restoration method proposed here. The criterion is to minimize the correlation between image and noise gray levels. The method can be implemented in the Fourier domain and is computationally attractive. The test results on a chromosome picture have been demonstrated for different signal-to-noise power ratio.

I. Introduction

The least square estimation or Wiener filtering has been found useful in image restoration problems. However, there is experimental evidence that the least square technique does not always leads to a visually satisfactory restoration [1]. Of the modifications proposed to get around this difficulty, the constrained least square estimation (CLSE) has a demonstrated capability of improving the result [2]. The constraint used in [2] is the second difference of the restored image. A constraint that matches the psychophysics of the human visual system has also been considered [3].

In the present work, a different objective function has been used to restore the image blurred by mild defocusing and corrupted by random noise. It has been assumed that the noise is independent of the blurring process. A linear model has been chosen to represent the defocusing. The criterion for optimality used here is the minimization of the correlation between the image and noise gray levels.

The CLSE [2], [3] are reasonably economical in computer complexity. To improve the computer speed Dines and Kak [4] treated the problem in Fourier domain. Their idea is applicable in the present case also. An added advantage of the present technique is that no singularity in the transfer function can occur for nonzero noise power in the degraded picture. Furthermore, the block circulant matrix description [5] of the problem has been generalized here to include the case when the pels outside the image field have nonzero but uniform gray levels. The usefulness of this generalization is discussed in Section II-C.

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The method and related algorithm have been described in Section II and Section III, respectively. The assumptions made here are realistic for a variety of pictures and the method is quite useful. It has been tested on a chromosome picture, and the test results for different signal-to-noise power ratio have been given in Section IV.

II. Problem Formulation

A. The Restoration Problem

Let the degradation process be given by a product operator H, which together with an additive noise $n(x,y)$ operates on an original input image $f(x,y)$ to produce a degraded image $g(x,y)$. In discrete matrix-vector notations

$$g = Hf + n \tag{1}$$

where g, f, and n are all MN-dimensional column vectors formed by lexicographic ordering of the gray levels in an $M \times N$ grid structure and H is the operator matrix of order $MN \times MN$.

The problem is to obtain an approximation to f, say $\hat{f}$, given g and a knowledge about the nature of H and n, subject to a constraint and optimized through a predefined criterion. One popular approach is to use CLSE [2] where the criterion for optimality is the second derivative of $\hat{f}$. We define a different criterion based on a measure of independence between the estimated original image and noise process. Let the correlation coefficient $R(k,m)$ between the image and noise be given by

$$R(k,m) = \frac{1}{MN}\sum_{j=0}^{N-1}\sum_{i=0}^{M-1}\hat{f}(i,j)n(i+k,j+m) \tag{2}$$

and $k = 0,1,2,\cdots,M-1$ and $m = 0,1,2,\cdots,N-1$. By lexicographic ordering, (2) can be represented in matrix-vector notation as

$$R = A\hat{f} \tag{3}$$

where A is a block circulant matrix of order $MN \times MN$ and is composed of noise gray levels. Since the correlation between two independent processes should be zero, here the criterion is to minimize $\|A\hat{f}\|^2$ subject to the constraint $\|g - H\hat{f}\|^2 = \|n\|^2$. The objective function $J(\hat{f})$ is given by

$$J(\hat{f}) = \|A\hat{f}\|^2 + \alpha\left(\|g - H\hat{f}\|^2 - \|n\|^2\right) \tag{4}$$

where α is the Lagrange's multiplier. Setting the differentiation of (4) with respect to $\hat{f}$ equal to zero we get the solution

$$\hat{f} = (H'H + \gamma A'A)^{-1}H'g \tag{5}$$

where $\gamma = 1/\alpha$. The assumption of independence between the picture and noise processes is practical in many situations. Also, as shown later, the estimation using (5) has other computational advantages.

B. The Degradation Model

A linear model has been used for degradation. Let a point object at (α,β) with gray level $f(\alpha,\beta)$ in the object plane be spread over a circular region D of radius $r = \frac{1}{2}|x_1 - x_2|$ in the blurred plane as shown in Fig. 1. Let $h_{0,\alpha,\beta}$ be the peak intensity after spreading and $h_{\alpha,\beta}(x,y)$ be the intensity at (x,y) due to point object at (α,β). If $f(\alpha,\beta)$ is unity then $h_{\alpha,\beta}(x,y)$ is known as point spread function (PSF) due to blurring. It is assumed that the blurred plane has a one-to-one relative correspondence with positions at the object plane, and no energy is lost due to blurring, i.e.,

$$\iint_D f(\alpha,\beta)h_{\alpha,\beta}(x,y)\,dx\,dy = f(\alpha,\beta).$$

A picture contains infinitely many point objects, and linear blurring distributes the intensity of each point over the circular region of radius r. The observed intensity at any point in the blurred plane is the aggregate of components of blurring distribution of intensities of point lying in the circular area around the corresponding point.

For discrete formulation, it is convenient to consider a square of side $(2r+1)$ centered at the (i,j)th pel. Then the observed intensity $g(i,j)$ in the blurred plane gets contribution from the pels lying within this area and is given by

$$g(i,j) = \sum_{k=i-r}^{i+r} \sum_{m=j-r}^{j+r} f(k,m) h_{k,m}(i,j) \qquad (6)$$

where

$$h_{k,m}(i,j) = \begin{cases} h_{0,k,m}\left[1 - \frac{1}{r}\left(|k-i|^2 + |m-j|^2\right)^{1/2}\right], \\ \quad \text{for } \left\{|k-i|^2 + |m-j|^2\right\}^{1/2} < r \\ 0, \\ \quad \text{otherwise.} \end{cases} \qquad (7)$$

Hence $h_{k,m}(i,j)$ is the PSF in discrete domain and, in a more general way, can be expressed as $h(|k-i|,|m-j|)$ and is space invariant. Considering noise term, $g(i,j)$ is finally represented as

$$g(i,j) = \sum_{k=0}^{M-1} \sum_{m=0}^{N-1} f(k,m) h(|k-i|,|m-j|) + n(i,j) \qquad (8)$$

i.e., in matrix-vector notation

$$g = H_{bt} f + n$$

where H_{bt} is a symmetric banded block Toeplitz matrix. Each block of H_{bt} is a symmetric banded toeplitz matrix with $2p$ (where $p = r-1$) nonzero subdiagonal elements, i.e., the elements $h_{i,j}$ of each block are

$$h_{i,j} = \begin{cases} h_{|i-j|}, & \text{for } |i-j| \leq p \\ 0, & \text{otherwise.} \end{cases}$$

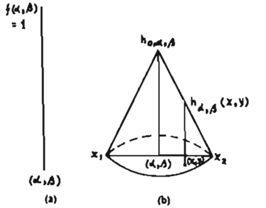

Fig. 1. (a) Point object of intensity $f(\alpha,\beta)$ in object plane (b) Intensity spread of (a) in blurred plane.

$$g_e(i,j) = \begin{cases} g(i,j), & \text{if } 0 \leq i \leq N-1 \text{ and } 0 \leq j \leq N-1 \\ g(i,N-1), & \text{if } 0 \leq i \leq N-1 \text{ and } N \leq j \leq N+p-1 \\ g(N-1,j), & \text{if } N \leq i \leq N+p-1 \text{ and } 0 \leq j \leq N-1 \\ g(N-1,N-1), & \text{if } N \leq i \leq N+p-1 \text{ and } N \leq j \leq N+p-1. \end{cases} \qquad (10)$$

C. Circulant Matrix Formulation

Consider, for simplicity, $M = N$. We define elements $g_e(i,j)$ of extended image matrix (generated due to defocusing) from $g(i,j)$ as follows:

$$g_e(i,j) = \begin{cases} \left[s_{iN+j} \times g(i,j) + \left(S - s_{iN+j}\right) \times g(N-1,N-1)\right]/S, \\ \quad \text{if } 0 \leq i \text{ or } j \leq p-1 \text{ and } 0 \leq i,j \leq N-1 \\ g(i,j), \\ \quad \text{if } p \leq i,j \leq N-1 \\ \left[s_{(N+p-1-i)N+j} \times g(N-1,N-1) + \left(S - s_{(N+p-1-i)N+j}\right) \times g(0,0)\right]/S, \\ \quad \text{if } N \leq i \leq N+p-1 \text{ and } 0 \leq j \leq N-1 \\ \left[s_{(i+1)N+p-1-j} \times g(N-1,N-1) + \left(S - s_{(i+1)N+p-1-j}\right) \times g(0,0)\right]/S, \\ \quad \text{if } 0 \leq i \leq N-1 \text{ and } N \leq j \leq N+p-1 \\ \left[s_{(N+p-1-i)N+N+p-1-j} \times g(N-1,N-1) + \left(S - s_{(N+p-1-i)N+N+p-1-j}\right) \times g(0,0)\right]/S, \\ \quad \text{if } N \leq i,j \leq N+p-1 \end{cases} \qquad (9)$$

where

$$s_k = \sum_{m=0}^{N^2-1} h(k,m).$$

$h(k,m)$ is the (k,m)th element of H_{bt} and should not be confused with $h_{i,j}$. Also,

$$S = \max_k \{s_k\}.$$

Under the assumption that the gray levels of the pels outside the image region of interest are uniform and are equal to those of the background of the image (which is more general than the assumption made in [5], where these gray levels are assumed to be zero), elements $g_e(i,j)$ can be defined in a more simplified form. Since in circulant matrix formulation the pels outside the image field are to be considered and since the image restoration problem is ill-conditioned, it is better to assume the gray-levels of pels outside the image field equal to that of background rather than zero.

If at least r rows and r columns of pels of equal gray levels lie between the object and the ends of the original image, we can simplify (9) as (10).

The extended observed image is related to extended original image in the following manner:

$$g_e = H_{bc} f_e + n_e. \qquad (11)$$

g_e, f_e, and n_e are L^2-dimensional column vectors formed by lexicographic ordering of $g_e(i,j)$, $f_e(i,j)$, and $n_e(i,j)$, respectively, and $L = N + p$. $f_e(i,j)$ is defined by

$$f_e(i,j) = \begin{cases} f(i,j), & \text{if } 0 \leq i \leq N-1 \text{ and } 0 \leq j \leq N-1 \\ f(N-1,N-1), & \text{if } 0 \leq i \leq N+p-1 \text{ and } N \leq j \leq N+P-1 \\ f(N-1,N-1), & \text{if } N \leq i \leq N+p-1 \text{ and } 0 \leq j \leq N+p-1. \end{cases} \quad (12)$$

H_{bc} is a symmetric block circulant matrix of dimension $L^2 \times L^2$.

D. Diagonalization of Operator

The technique, which has already been used to obtain (5) from (1), yields $\hat{f}_e$ from (11) as

$$\hat{f}_e = \left(H_{bc}'H_{bc} + \gamma A_e'A_e\right)^{-1} H_{bc}'g_e \quad (13)$$

where A_e is constructed from n_e. Now H_{bc} can be diagonalized in the following manner [2]:

$$D_H = W^{-1}H_{bc}W$$

where $D_H = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{L^2-1})$, with λ_i, the ith eigenvalue of H_{bc} being given by

$$\lambda_i = \mathscr{H}\left(\left[\frac{i}{L}\right], i \bmod L\right)$$

$\mathscr{H}(\cdot,\cdot)$s are two-dimensional Fourier transform coefficients of PSF, $h_e(\cdot,\cdot)$ and are real as H_{bc} is symmetric. When W^{-1} is multiplied with a lexicographically ordered image vector, we obtain the lexicographically ordered Fourier transform coefficients of that image.

A is also a block circulant matrix. So it also can be diagonalized in the same way as H_{bc}. Let the noise terms be zero mean uncorrelated, i.e.,

$$\sum_i \sum_j n_e(i,j) n_e(i+k, j+l) = 0 \text{ for all } k, l$$

but not both equal to zero. Then a straightforward analysis leads to

$$A_e'A_e = \frac{1}{L^2} S_n I \quad (14)$$

where I is the identity matrix and

$$S_n = \frac{1}{L^2} \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} n_e^2(i,j) = \sigma_n^2 \quad (15)$$

where σ_n^2 is the variance of noise gray levels.

Substituting (14) in (13) and taking Fourier transform on both sides, we obtain

$$\hat{F}_e(u,v) = \frac{\mathscr{H}(u,v)}{\mathscr{H}^2(u,v) + \gamma S_n} G_e(u,v) \quad (16)$$

for $u = 0, 1, 2, \cdots, L-1$ and $v = 0, 1, 2, \cdots, L-1$. $F_e(u,v)$ and $G_e(u,v)$ are the (u,v)th coefficient of two-dimensional Fourier transform of $\hat{f}_e(i,j)$ and $g_e(i,j)$, respectively.

Solving (16) under the constraint $\|g_e - H_{bc}\hat{f}_e\|^2 = \|n_e\|^2$, we obtain the value of γ and $\hat{F}_e(u,v)$. To avoid computations required to find Fourier inverse transform of $\hat{F}_e(u,v)$ and to find $H_{bc}\hat{f}_e$ during the estimation of γ, we apply Perseval's theorem to constraint identity to obtain

$$\sum_{u=0}^{L-1} \sum_{v=0}^{L-1} \left|G_e(u,v) - \mathscr{H}(u,v)\hat{F}_e(u,v)\right|^2 = \sum_{u=0}^{L-1} \sum_{v=0}^{L-1} \left|N_e(u,v)\right|^2 = L^4 S_n. \quad (17)$$

$N_e(u,v)$ is the (u,v)th Fourier transform coefficient of noise.

Restoration through (16) has several advantages. First, for finite noise energy, singularity in the right-hand side of (16) does not occur, and Fourier inverse transform can be computed uniquely. Secondly, the noise energy can be calculated only through its variance. Thus a complete knowledge of first-order statistics and subsequent Fourier transform is not necessary. This fact, along with (17), improves the speed of restoration.

It is useful to compare the present criterion and those used previously. The unconstrained least square criterion does not utilize the properties of noise, and hence its performance is poor under certain noise condition. The Wiener filter does not guarantee $\|g - H\hat{f}\|^2 = \|n\|^2$ and the noise power spectrum as well as noise-free image power spectrum needed to realize the filter may not be available a priori. Also, the restored image may not be visually acceptable, although Wiener filter minimizes the mean square restoration error in a statistical sense. Neither CLSE nor the present criterion are optimal like Wiener filter. In CLSE smoothness of the restored image is emphasized and hence the sharpness of the picture is affected. In the present method, the independence of noise and image is emphasized. The effect of the process is to decorrelate the noise from the picture. CLSE and the present method are comparable in computer complexity, and the algorithms of the two methods are similar. A hybrid of the two methods may lead to interesting results.

III. Concise Algorithm

The method of image restoration by constrained least square estimation given in Section II can be summarized by the following steps.

1) Construct and store the first row of block Toeplitz matrix H_{bc} from point spread function defined by (7).

2) Modify the first row of H_b, to the first row of a block circulant matrix H_{bc}.

3) From the elements of the first row of H_{bc} construct $h_e(i,j)$ of size $L \times L$ by the process just reverse to lexicographic ordering.

4) Find $\mathscr{H}(u,v)$ by two-dimensional Fourier transform of $h_e(i,j)$.

5) Construct extended image matrix $g_e(i,j)$ of size $L \times L$ using (10) (or in some cases, (9)) from observable image $g(i,j)$ of size $(L-p) \times (L-p)$.

6) Find $G_e(u,v)$ by two-dimensional Fourier transform of $g_e(i,j)$.

7) If average energy S_n (or power) of noise function is not known, calculate it from the variance using (15).

8) Choose and fix an initial value of γ. (Dines and Kak [4] have shown that if the noise power is less than signal power then the value of γ must be greater than zero. It is true for the present case also.) Estimate the value of $\hat{F}_e(u,v)$ using (16), and find the total estimation error given by

$$\text{error} = \sum_{u=0}^{L-1} \sum_{v=0}^{L-1} \left|G_e(u,v) - \mathscr{H}(u,v)F_e(u,v)\right|^2$$

a) increase the value of γ, if error $< L^4S_n - \epsilon$

b) decrease the value of γ, if error $> L^4S_n + \epsilon$

where ϵ is permissible error in estimation.

9) For acceptable value of γ, determine $\hat{f}_e(i,j)$ by applying two-dimensional Fourier inverse transform on $\hat{F}_e(u,v)$, and then find $f(i,j)$ (see (12)).

IV. Results and Discussion

To test the validity of the present approach to a practical problem, the algorithm has been implemented on a microphoto-

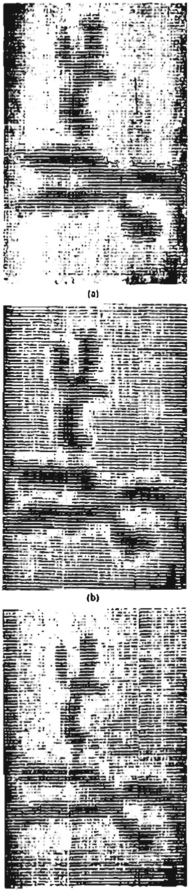

Fig. 2. (a) Original defocussed picture of chromosome containing small noise (SNR = 24.8 dB). (b) Restored picture of (a) using $r = 2$ and $\gamma = 0.025$. (c) Restored picture of (a) using $r = 3$ and $\gamma = 0.022$.

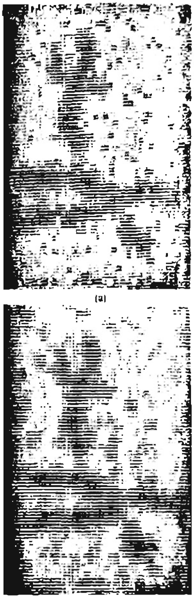

Fig. 3. (a) Degraded picture of chromosome after adding large noise to Fig. 2(a) (SNR = 17.1 dB). (b) Restored picture of (a) using $r = 2$ and $\gamma = 0.048$.

graph of human chromosome. A small rectangular portion of the entire picture is taken by windowing, which satisfies the condition that the background of the extracted portion has uniform gray level. The size of the extracted portion is 48 × 48. The portion has been extended on both dimension by adding pels with gray level equal to that of the background so that the present size is $N \times N$ and $N + p = 2^l$, where $l = 6$. The number of gray levels chosen for the image is 32.

The program for the algorithm of Section III has been written in Fortran language and executed in EC 1033 computer system. The hard copies of input and output pictures have been made through a line printer by overprinting of common characters because of the unavailability of adequate instruments.

The original defocused picture contains small noise (SNR = 24.8 dB) due to granularity in the developing paper and quantisation effect. Fig. 2(b) and (c) are the restored images of the original defocused image of Fig. 2(a) obtained for $r = 2$, $\gamma = 0.025$, and $r = 3$, $\gamma = 0.022$, respectively. To see the efficiency of the present algorithm on large noise, uniformly distributed random noise has been added to Fig. 2(a) to get Fig. 3(a) where SNR is decreased to 17.1 dB (approximately). Fig. 3(b) represents the restored image with $r = 2$ and $\gamma = 0.048$. It can be seen that the noise has been removed fairly well, but the edges of the objects, i.e., chromosomes, are smoother than in Fig. 2(b). This is because the noise power term in (16) reduces heavily the value of high-frequency component of the filter function, and hence the high-frequency component of the image is greatly reduced. This effect smooths the edges.

The time required to process the image is less than that for the standard constrained least square estimation process [2]. Furthermore, ill-conditioning of the problem due to singularity in the filter function of (16) is absent in the present case. If the noise is uncorrelated, the present method may be applied to other nonlinear space-invariant degradation functions.

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