Fuzzy Sets and Systems 15 (1985) 249–254
North-Holland

SOME OBSERVATIONS ON FUZZY RELATIONS OVER FUZZY SUBSETS

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Received October 1983
Revised March 1984

This paper considers fuzzy relations defined over fuzzy subsets and settles some open problems regarding the distributivity and transitivity of such relations.

Keywords: Fuzzy relations, Distributivity, Transitivity.

1. Introduction

Since the introduction of the theory of fuzzy sets in 1965 [4], the study of fuzzy relations over two ordinary sets has attracted the attention of many researchers. A recent paper [1] (see also [5] in this context) considers a generalisation of fuzzy relations, namely fuzzy relations on fuzzy subsets, develops some properties of such generalised relations and leaves some open problems on some properties. The present paper attempts to settle all these open problems. The relevant definitions are presented below, following [1].

Let U be an initial set and $\mathscr{A}$, $\mathscr{B}$ fuzzy subsets of U defined by the membership functions $f_{\mathscr{A}}$ and $f_{\mathscr{B}}$ respectively.

Definition 1.1. Let $\mathscr{R} \subset \mathscr{A} \times \mathscr{B}$, i.e.

$$f_{\mathscr{R}}(x, y) \leq \min[f_{\mathscr{A}}(x), f_{\mathscr{B}}(y)].$$

Then $\mathscr{R}$ is a fuzzy relation from $\mathscr{A}$ to $\mathscr{B}$.

Definition 1.2. Let $\mathscr{R}_1$, $\mathscr{R}_2$ be fuzzy relations from $\mathscr{A}$ to $\mathscr{B}$. Then $\mathscr{R}_1 \hat{+} \mathscr{R}_2$, $\mathscr{R}_1 \cdot \mathscr{R}_2$, $\mathscr{R}_1 \Cup \mathscr{R}_2$ and $\mathscr{R}_1 \Cap \mathscr{R}_2$ are defined as follows:

$$f_{\mathscr{R}_1 \hat{+} \mathscr{R}_2}(x, y) = f_{\mathscr{R}_1}(x, y) + f_{\mathscr{R}_2}(x, y) - f_{\mathscr{R}_1}(x, y) \cdot f_{\mathscr{R}_2}(x, y),$$

$$f_{\mathscr{R}_1 \cdot \mathscr{R}_2}(x, y) = f_{\mathscr{R}_1}(x, y) \cdot f_{\mathscr{R}_2}(x, y),$$

$$f_{\mathscr{R}_1 \Cup \mathscr{R}_2}(x, y) = \min[1, f_{\mathscr{R}_1}(x, y) + f_{\mathscr{R}_2}(x, y)],$$

$$f_{\mathscr{R}_1 \Cap \mathscr{R}_2}(x, y) = \max[0, f_{\mathscr{R}_1}(x, y) + f_{\mathscr{R}_2}(x, y) - 1],$$

for all $x, y \in U$.

Definition 1.3. The max–min composition $\circ$ of two fuzzy relations $\mathcal{R}_1$ and $\mathcal{R}_2$ is defined by

$$f_{\mathcal{R}_1\circ\mathcal{R}_2}(x,y)=\max_{z\in U}[\min(f_{\mathcal{R}_1}(x,z),f_{\mathcal{R}_2}(z,y))],$$

where $\mathcal{R}_1$ is a relation from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{R}_2$ from $\mathcal{B}$ to $\mathcal{C}$.

Definition 1.4. Let $\mathcal{R}$ be a fuzzy relation on $\mathcal{A}\subseteq U$. Then $\mathcal{R}$ is transitive if $\mathcal{R}\circ\mathcal{R}\subseteq\mathcal{R}$.

Specifically, the authors of [1, Section 4] left open the question of distributivity of $\mathcal{R}_1\circ(\mathcal{R}_2\cdot\mathcal{R}_3)$, $\mathcal{R}_1\circ(\mathcal{R}_2\hat{+}\mathcal{R}_3)$, $\mathcal{R}_1\circ(\mathcal{R}_2\cup\mathcal{R}_3)$, $\mathcal{R}_1\circ(\mathcal{R}_2\cap\mathcal{R}_3)$, where $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ are fuzzy relations over fuzzy subsets. Also nothing was stated about the transitivity of $\mathcal{R}_1\cdot\mathcal{R}_2$, $\mathcal{R}_1\cup\mathcal{R}_2$, $\mathcal{R}_1\cap\mathcal{R}_2$, when $\mathcal{R}_1$ $\mathcal{R}_2$ are transitive. This paper considers the open problems regarding distributivity and transitivity in Sections 2 and 3 respectively.

Before presenting the main results, the following notation is introduced which will be helpful in presenting the numerical examples in this paper. If the initial set $U=\{a_1,\ldots,a_n\}$ be finite and if $\mathcal{F}$ be a fuzzy subset of $U\times U$ with membership function $f_{\mathcal{F}}(x,y)$, then $\mathcal{F}$ will be described in matrix notation simply as

$$\mathcal{F}:\begin{bmatrix} f_{\mathcal{F}}(a_1,a_1) & f_{\mathcal{F}}(a_2,a_1) & \cdots & f_{\mathcal{F}}(a_n,a_1)\\ f_{\mathcal{F}}(a_1,a_2) & f_{\mathcal{F}}(a_2,a_2) & & f_{\mathcal{F}}(a_n,a_2)\\ \vdots & & & \vdots\\ f_{\mathcal{F}}(a_1,a_n) & f_{\mathcal{F}}(a_2,a_n) & \cdots & f_{\mathcal{F}}(a_n,a_n)\end{bmatrix}. \tag{1.1}$$

2. The results on distributivity

Let $\mathcal{R}_1$ be a fuzzy relation from $\mathcal{A}$ to $\mathcal{B}$ and $\mathcal{R}_2$, $\mathcal{R}_3$ be fuzzy relations from $\mathcal{B}$ to $\mathcal{C}$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are fuzzy subsets of the initial set. Then the following results hold.

Theorem 2.1. *None of the following is necessarily true:*

$$\begin{aligned}&\mathcal{R}_1\circ(\mathcal{R}_2\cdot\mathcal{R}_3)\subseteq(\mathcal{R}_1\circ\mathcal{R}_2)\cdot(\mathcal{R}_1\circ\mathcal{R}_3),\\&\mathcal{R}_1\circ(\mathcal{R}_2\cdot\mathcal{R}_3)\supseteq(\mathcal{R}_1\circ\mathcal{R}_2)\cdot(\mathcal{R}_1\circ\mathcal{R}_3),\\&\mathcal{R}_1\circ(\mathcal{R}_2\cap\mathcal{R}_3)\subseteq(\mathcal{R}_1\circ\mathcal{R}_2)\cap(\mathcal{R}_1\circ\mathcal{R}_3),\\&\mathcal{R}_1\circ(\mathcal{R}_2\cap\mathcal{R}_3)\supseteq(\mathcal{R}_1\circ\mathcal{R}_2)\cap(\mathcal{R}_1\circ\mathcal{R}_3),\end{aligned} \tag{2.1}$$

and consequently $\mathcal{R}_1\circ(\mathcal{R}_2\cdot\mathcal{R}_3)$ *and* $\mathcal{R}_1\circ(\mathcal{R}_2\cap\mathcal{R}r_3)$ *are not distributive.*

Proof. This will be proved by an example. Let $U=\{a_1,a_2,a_3\}$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be given by

$$f_{\mathscr{A}}(a_1)=f_{\mathscr{A}}(a_2)=0.97,\quad f_{\mathscr{A}}(a_3)=0,$$
$$f_{\mathscr{B}}(a_1)=f_{\mathscr{B}}(a_2)=0.95,\quad f_{\mathscr{B}}(a_3)=0,$$
$$f_{\mathscr{C}}(a_1)=f_{\mathscr{C}}(a_2)=0.93,\quad f_{\mathscr{C}}(a_3)=0.$$

Then following (1.1) and [1],

$$\mathscr{A}\times\mathscr{B}:\begin{bmatrix}0.95 & 0.95 & 0\\ 0.95 & 0.95 & 0\\ 0 & 0 & 0\end{bmatrix},\qquad \mathscr{B}\times\mathscr{C}:\begin{bmatrix}0.93 & 0.93 & 0\\ 0.93 & 0.93 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Let $\mathscr{R}_1$ be a relation from $\mathscr{A}$ to $\mathscr{B}$ and $\mathscr{R}_2$, $\mathscr{R}_3$ be relations from $\mathscr{B}$ to $\mathscr{C}$ defined by

$$\mathscr{R}_1:\begin{bmatrix}0.6 & 0.4 & 0\\ 0.9 & 0.4 & 0\\ 0 & 0 & 0\end{bmatrix},\qquad \mathscr{R}_2:\begin{bmatrix}0.7 & 0.8 & 0\\ 0.3 & 0.5 & 0\\ 0 & 0 & 0\end{bmatrix},\qquad \mathscr{R}_3:\begin{bmatrix}0.5 & 0.1 & 0\\ 0.3 & 0.5 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

With $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ as above, one can follow the definitions above to obtain the different expressions in (2.1) and check that none of the relations in (2.1) holds. □

Theorem 2.2. *$\mathscr{R}_1\circ(\mathscr{R}_2\hat{+}\mathscr{R}_3)$ is not distributive but only 'subdistributive' in the sense that although*

$$\mathscr{R}_1\circ(\mathscr{R}_2\hat{+}\mathscr{R}_3)\subseteq(\mathscr{R}_1\circ\mathscr{R}_2)\hat{+}(\mathscr{R}_1\circ\mathscr{R}_3) \tag{2.2}$$

is true,

$$\mathscr{R}_1\circ(\mathscr{R}_2\hat{+}\mathscr{R}_3)=(\mathscr{R}_1\circ\mathscr{R}_2)\hat{+}(\mathscr{R}_1\circ\mathscr{R}_3) \tag{2.3}$$

is not necessarily true.

Lemma. *If ξ, μ and ϕ are mappings from U to $[0, 1]$, then*

$$\begin{aligned}&\max_{z\in U}[\min(\xi(z), \mu(z)+\phi(z)-\mu(z)\phi(z))]\\ &\quad\leq \max_{z\in U}[\min(\xi(z), \mu(z))]+\max_{z\in U}[\min(\xi(z), \phi(z))]\\ &\quad -\max_{z\in U}[\min(\xi(z), \mu(z))]\cdot\max_{z\in U}[\min(\xi(z), \phi(z))].\end{aligned} \tag{2.4}$$

Proof of the lemma. For any particular $z\in U$, a complete enumeration of all possible situations reveals that

$$\begin{aligned}&1-\min(\xi(z), \mu(z)+\phi(z)-\mu(z)\phi(z))\\ &\quad\geq[1-\min(\xi(z), \mu(z))]\cdot[1-\min(\xi(z), \phi(z))].\end{aligned}$$

This gives

$$\left[1-\max_{z\in U}\{\min(\xi(z),\mu(z))\}\right]\cdot\left[1-\max_{z\in U}\{\min(\xi(z),\phi(z))\}\right]$$
$$=\min_{z\in U}[1-\min(\xi(z),\mu(z))]\cdot\min_{z\in U}[1-\min(\xi(z),\phi(z))]$$
$$\leq[1-\min(\xi(z),\mu(z))]\cdot[1-\min(\xi(z),\phi(z))]$$
$$\leq 1-\min(\xi(z),\mu(z)+\phi(z)-\mu(z)\phi(z)),\quad \forall z\in U.$$

Since the left-hand side is a fixed quantity,

$$\left[1-\max_{z\in U}\{\min(\xi(z),\mu(z))\}\right]\cdot\left[1-\max_{z\in U}\{\min(\xi(z),\phi(z))\}\right]$$
$$\leq\min_{z\in U}[1-\min(\xi(z),\mu(z)+\phi(z)-\mu(z)\phi(z))]$$
$$=1-\max_{z\in U}[\min(\xi(z),\mu(z)+\phi(z)-\mu(z)\phi(z))],$$

whence, on simplification, the lemma follows.

Proof of the theorem. For any fixed $x, y\in U$, taking $\xi(z)=f_{\mathcal{R}_1}(x,z)$, $\mu(z)=f_{\mathcal{R}_2}(z,y)$, $\phi(z)=f_{\mathcal{R}_3}(z,y)$, it is easy to see that the left- and right-hand sides of (2.4) give the membership functions of $\mathcal{R}_1\circ(\mathcal{R}_2\hat{+}\mathcal{R}_3)$ and $(\mathcal{R}_1\circ\mathcal{R}_2)\hat{+}(\mathcal{R}_1\circ\mathcal{R}_3)$ respectively. Hence (2.2) follows.

That (2.3) is not necessarily true follows if in particular one considers $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ as in the example given in the context of Theorem 2.1. □

Theorem 2.3. *$\mathcal{R}_1\circ(\mathcal{R}_2\uplus\mathcal{R}_3)$ is not distributive but only 'subdistributive' in the sense that although*

$$\mathcal{R}_1\circ(\mathcal{R}_2\uplus\mathcal{R}_3)\subseteq(\mathcal{R}_1\circ\mathcal{R}_2)\uplus(\mathcal{R}_1\circ\mathcal{R}_3) \tag{2.5}$$

is true,

$$\mathcal{R}_1\circ(\mathcal{R}_2\uplus\mathcal{R}_3)=(\mathcal{R}_1\circ\mathcal{R}_2)\uplus(\mathcal{R}_1\circ\mathcal{R}_3) \tag{2.6}$$

is not necessarily true.

Proof. Following the definitions in [1], the membership function of $\mathcal{R}_1\circ(\mathcal{R}_2\uplus\mathcal{R}_3)$ is

$$\begin{aligned}f_{\mathcal{R}_1\circ(\mathcal{R}_2\uplus\mathcal{R}_3)}(x,y)&=\max_{z\in U}[\min\{f_{\mathcal{R}_1}(x,z),\min(1,f_{\mathcal{R}_2}(z,y)+f_{\mathcal{R}_3}(z,y))\}]\\&=\max_{z\in U}[\min\{1,f_{\mathcal{R}_1}(x,z),f_{\mathcal{R}_2}(z,y)+f_{\mathcal{R}_3}(z,y)\}]\\&=\max_{z\in U}[\min\{f_{\mathcal{R}_1}(x,z),f_{\mathcal{R}_2}(z,y)+f_{\mathcal{R}_3}(z,y)\}],\end{aligned} \tag{2.7}$$

since $f_{\mathcal{R}_1}(x,z)\leq 1$. Now, by a complete enumeration of all possible situations, for any particular x, y, z,

$$\begin{aligned}
&\min[f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_2}(z, y)+f_{\mathcal{R}_3}(z, y)] \\
&\quad\leqslant \min[f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_2}(z, y)]+\min[f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_3}(z, y)] \\
&\quad\leqslant \max_{z\in U}[\min\{f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_2}(z, y)\}]+\max_{z\in U}[\min\{f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_3}(z, y)\}].
\end{aligned}$$

Since, trivially, $\min[f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_2}(z, y)+f_{\mathcal{R}_3}(z, y)]\leqslant 1$, this gives

$$\begin{aligned}
&\min[f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_2}(z, y)+f_{\mathcal{R}_3}(z, y)] \\
&\quad\leqslant \min\Big[1, \max_{z\in U}\{\min(f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_2}(z, y))\} \\
&\qquad\qquad +\max_{z\in U}\{\min(f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_3}(z, y))\}\Big] \\
&\quad= f_{(\mathcal{R}_1\circ\mathcal{R}_2)\cup(\mathcal{R}_1\circ\mathcal{R}_3)}(x, y) \qquad (2.8)
\end{aligned}$$

where the right-hand side of (2.8) gives the membership function of $(\mathcal{R}_1\circ\mathcal{R}_2)\cup(\mathcal{R}_1\circ\mathcal{R}_3)$. Since this does not depend on z, it follows by (2.8) that

$$\max_{z\in U}[\min\{f_{\mathcal{R}_1}(x, z), f_{\mathcal{R}_2}(z, y)+f_{\mathcal{R}_3}(z, y)\}]\leqslant f_{(\mathcal{R}_1\circ\mathcal{R}_2)\cup(\mathcal{R}_1\circ\mathcal{R}_3)}(x, y)$$

whence, applying (2.7), it is clear that (2.5) is true.

That (2.6) is not necessarily true follows considering $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ as in the example given in the context of Theorem 2.1. □

3. The results on transitivity

Theorem 3.1. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are transitive relations on a fuzzy subset $\mathcal{A}$, then* (a) $\mathcal{R}_1\cdot\mathcal{R}_2$, (b) $\mathcal{R}_1\cup\mathcal{R}_2$, (c) $\mathcal{R}_1\cap\mathcal{R}_2$ *are not necessarily transitive.*

Proof. This will be proved by an example. Let $U=\{a_1, a_2, a_3\}$ and $\mathcal{A}$ be given by $f_{\mathcal{A}}(a_1)=f_{\mathcal{A}}(a_2)=0.9$, $f_{\mathcal{A}}(a_3)=0$. Then

$$\mathcal{A}\times\mathcal{A}: \begin{bmatrix}0.9 & 0.9 & 0\\ 0.9 & 0.9 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Let $\mathcal{R}_1$, $\mathcal{R}_2$ be relations on $\mathcal{A}$ defined by

$$\mathcal{R}_1: \begin{bmatrix}0.4 & 0.3 & 0\\ 0.8 & 0.4 & 0\\ 0 & 0 & 0\end{bmatrix}, \qquad \mathcal{R}_2: \begin{bmatrix}0.4 & 0.8 & 0\\ 0.3 & 0.4 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

It is a routine matter to check that $\mathcal{R}_1$, $\mathcal{R}_2$ are transitive fuzzy relations while $\mathcal{R}_1\cdot\mathcal{R}_2$, $\mathcal{R}_1\cup\mathcal{R}_2$, $\mathcal{R}_1\cap\mathcal{R}_2$ are not transitive. □

4. Concluding remarks

The authors of [1], in their concluding remarks, stress on the need of evolving a general methodology for handling expressions involving symbols like max, min, +, ×, 1 and 0. It may be noted that the lemma in the context of Theorem 2.7 of [1] is a version of the Minimax Theorem well known in statistical decision theory and game theory. Also the techniques employed in proving the first parts of Theorems 2.2, 2.3 of this paper are similar to those used in the proof of the Minimax Theorem. Hence it appears that such techniques may constitute, at least partly, the required general methodology.

Acknowledgement

The author is thankful to the referees for their constructive suggestions.

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