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Some Inequalities for Norm Ideals

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Abstract. Several inequalities for norms of operators are extended to more operators and/or to more norms. These include results of Halmos and Bouldin on approximating a normal operator by another with restricted spectrum, the Powers–Stømer and the van Hemmen–Ando inequalities for the distance between the square roots of two positive operators and also some recent generalisations of these latter results by Kittaneh.

1. Introduction

In this note we obtain extensions of some inequalities for Hilbert space operators which are of importance in some problems of quantum physics and quantum chemistry.

The first problem we consider is called the *spectral approximation problem* for normal operators. Let K be a given closed subset of the complex plane $\mathbb{C}$, and let $\mathcal{N}(K)$ denote the set of all normal operators with spectrum contained in K. Given any normal operator A what element of $\mathcal{N}(K)$ is nearest to A? Halmos [8] showed that if F is any Borel measurable distance minimising retract onto K (i.e. a Borel measurable map from $\mathbb{C}$ onto K which satisfies the inequality $|z-F(z)| \leqq |z-w|$ for all z in $\mathbb{C}$ and w in K) then $\|A-F(A)\| \leqq \|A-N\|$ for every $N \in \mathcal{N}(K)$. This result was extended by Bouldin [3] who showed that for all Schatten p-norms, $p \geqq 2$, $\|A - F(A)\|_p \leqq \|A-N\|_p$. (This statement is to be interpreted to mean that if there exists an N in $\mathcal{N}(K)$ such that $A-N$ is in the Schatten class C_p for some $p \geqq 2$, then $A-F(A)$ also belongs to this class and the above inequality holds.) We refer the reader to the bibliography in [3] for the connection between this problem and some problems arising in molecular orbital calculations in quantum chemistry.

The question for the cases $1 \leqq p < 2$ has been left unanswered by Bouldin. We give below an example to show that the Halmos–Bouldin inequality does not extend to these cases. We then show that there is an interesting class of norms, which includes the p-norms for $p \geqq 2$, to which this inequality can be extended. Further, we show that if the set K is convex then this inequality holds for all unitarily invariant norms.

After this, we consider the inequality of Powers and Stømer [14], derived in the course of their work on free states of the canonical anti-commutation relations. They proved that if A, B are positive operators then $\|A^{1/2}-B^{1/2}\|_2^2 \leqq \|A-B\|_1$. Recently, Kittaneh [12] has generalised this to show that $\|A^{1/2}-B^{1/2}\|_{2p}^2 \leqq \|A-B\|_p$ for $1 \leqq p \leqq \infty$.

Now note that for any operator T, we have $\|T\|_{2p}^2 = \|T^*T\|_p$, $1 \leqq p \leqq \infty$. So, this result of Kittaneh can be restated as $\|(A-B)^2\|_p \leqq \|A^2-B^2\|_p$ for all positive operators A, B and for all $1 \leqq p \leqq \infty$. We first show that this inequality is valid for the larger class of all unitarily invariant norms. Next we show that the same inequality holds when the exponent 2 is replaced by any power of 2. When specialised to the p-norms, this gives the following interesting generalisation of the Powers–Størmer inequality. We have,

$$\|A^{1/m}-B^{1/m}\|_{mp}^m \leqq \|A-B\|_p, \quad \text{for all integers } m \text{ of the form } 2^k,\ k=1,2,\ldots.$$

Another inequality concerning the distance between square roots of positive operators was proved by van Hemmen and Ando [18]. They showed that if A, B are positive and if $A^{1/2}+B^{1/2} \geqq aI \geqq 0$ for some a, then for every unitarily invariant norm $|||\cdot|||$, we have $a\,|||A^{1/2}-B^{1/2}||| \leqq |||A-B|||$. Kittaneh [11] proved some related results, one of which says that if A is any bounded operator with its real part Re $A \geqq aI \geqq 0$, then for all X and for all $1 \leqq p \leqq \infty$, we have $2a\|X\|_p \leqq \|AX+XA^*\|_p$. We show that this inequality is valid, more generally, for all unitarily invariant norms. We point out how this and the crucial estimate in van Hemmen and Ando [18, Lemma 3.1] are related to the classical work of Heinz [9]. Further, these results are shown to be valid not only for positive (bounded) operators but also for maximal accretive (unbounded) operators.

2. Norms and Norm Ideals

We quickly sum up some facts which could be found in any of the references [6, 15, 16, 17].

Denote by $\mathscr{B}(\mathscr{H})$ the space of all bounded operators on the Hilbert space $\mathscr{H}$. For any $A \in \mathscr{B}(\mathscr{H})$ the symbol $\|A\|$ denotes its usual operator bound norm. In addition there are other interesting norms defined on ideals contained in $\mathscr{B}(\mathscr{H})$: Each proper ideal of $\mathscr{B}(\mathscr{H})$ is contained in the ideal of compact operators. For any compact operator A, denote by $s_1(A) \geqq s_2(A) \geqq \ldots$ the singular values of A, i.e. the eigenvalues of $(A^*A)^{1/2}$. Each "*symmetric gauge function*" Φ on sequences gives rise to a *symmetric norm* or a *unitarily invariant norm* on operators defined by $\|A\|_\Phi = \Phi(\{s_j(A)\})$. We will denote by the symbol $|||\cdot|||$ any such norm. Each such norm satisfies the invariance property $|||UAV||| = |||A|||$ for all A and unitary U, V. With each such norm is associated a "norm ideal" of $\mathscr{B}(\mathscr{H})$ on which it is bounded, and this ideal is closed in the topology generated by this norm.

Two special families of unitarily invariant norms are the *Schatten p-norms* defined as $\|A\|_p = \left(\sum_{j=1}^{\infty} s_j(A)^p\right)^{1/p}$, $1 \leqq p \leqq \infty$, where by convention $\|A\|_\infty = \max s_j(A) = s_1(A) = \|A\|$ and the *Ky Fan norms* defined as $\|A\|_k = \sum_{j=1}^{k} s_j(A)$, $k=1,2,\ldots$.

This latter family is important because of the following theorem of Fan [5], (called the Dominance Property in [6, p. 82]). If B belongs to the norm ideal associated with a unitarily invariant norm $|||\cdot|||$ and if $\|A\|_k \leqq \|B\|_k$ for $k = 1, 2, \ldots$, then A also belongs to this ideal and $|||A||| \leqq |||B|||$.

We will consider another subclass of unitarily invariant norms which we call the *Q-norms*. A norm $|||\cdot|||$ is a Q-norm if there exists some unitarily invariant norm $|||\cdot|||'$ such that $|||A||| = (|||A^*A|||')^{1/2}$. Since $\|A\| = \|A^*A\|^{1/2}$ and $\|A\|_p = (\|A^*A\|_{p/2})^{1/2}$ for $p \geqq 2$, the Schatten p-norms are all Q-norms for $2 \leqq p \leqq \infty$. However, for $1 \leqq p < 2$ they are not Q-norms. The family of Q-norms contains several other norms as well. For example, for each $k = 1, 2, \ldots$ define the "root mean square" of the top k singular values as $\|A\|_{k,sq} = \left(\sum_{j=1}^{k} s_j^2(A)\right)^{1/2}$. It can be seen that this is a symmetric gauge function of the $s_j(A)$, and hence defines a unitarily invariant norm. Also, it is a Q-norm because it is equal to $\|A^*A\|_k^{1/2}$, where $\|A\|_k$ is the Ky Fan norm. We will use the symbol $\|\cdot\|_Q$ to denote any of these Q-norms.

3. On the Halmos–Bouldin Inequality

The example below shows that the Halmos–Bouldin inequality can not be extended to Schatten p-norms for $1 \leqq p < 2$.

Let $\mathscr{H}$ be the two-dimensional Hilbert space $\mathbb{C}^2$ and let K be the closed set $\{i, -i\}$ in the plane, Let A and N be the operators having the matrix representations

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $N \in \mathscr{N}(K)$ and $\|A - N\|_p = 2$ for any $1 \leqq p \leqq \infty$. But if F is *any* function of the plane onto K, then $\|A - F(A)\|_p = 2^{1/p + 1/2}$, for $1 \leqq p \leqq \infty$. (To see this note that since A and $F(A)$ commute they can be simultaneously diagonalised.) So for $1 \leqq p < 2$, $\|A - F(A)\|_p > \|A - N\|_p$. Further no function of A can be a best approximant to A from $\mathscr{N}(K)$ for these norms. (A "worse" example may be based on [2, Example 4.2].)

The following theorem says that the Halmos–Bouldin inequality can be extended to all Q-norms for arbitrary closed sets K and to all unitarily invariant norms for closed convex sets K.

Theorem 1. *Let $A \in \mathscr{B}(\mathscr{H})$ be a normal operator. Let F be a Borel measurable distance minimising retract onto a given closed subset K of the complex plane. Suppose there exists an N in the class $\mathscr{N}(K)$ of all normal operators whose spectrum is contained in K, such that $A - N$ lies in the norm ideal associated with any Q-norm $\|\cdot\|_Q$. Then $A - F(A)$ also belongs to this ideal and*

$$\|A - F(A)\|_Q \leqq \|A - N\|_Q. \tag{1}$$

Further, if the set K is convex, then under the above conditions we have

$$|||A - F(A)||| \leqq |||A - N||| \tag{2}$$

for every unitarily invariant norm.

Proof. If our norm is the operator norm this is exactly the result of Halmos. In the case of any other norm ideal, $A - N$ is compact. So A and N have the same essential spectrum, (see [10]), i.e. the part of the spectrum of A which is outside K just consists of eigenvalues of finite multiplicity. So to prove the inequality we might assume that A is a compact operator. (See Bouldin [3] who uses the same argument.)

Now, by the definition of a Q-norm and by the Fan Dominance Property cited in Sect. 2, the inequality (1) will be proved if we show that for every $k = 1, 2, \ldots$ we have

$$\|(A - F(A))^*(A - F(A))\|_k \leqq \|(A - N)^*(A - N)\|_k. \tag{3}$$

To prove this, it suffices to prove that if $\alpha_1, \alpha_2, \ldots$ are the eigenvalues of A, each counted as many times as its multiplicity, then

$$\sum_{j=1}^{k} |\alpha_{i_j} - F(\alpha_{i_j})|^2 \leqq \sum_{j=1}^{k} s_j^2(A - N), \tag{4}$$

for any choice of k indices $i_1, \ldots, i_k$. To prove (4) we appeal to a minimax principle of Fan [4], which says

$$\sum_{j=1}^{k} s_j^2(A - N) = \max \sum_{j=1}^{k} \|(A - N)v_j\|^2, \tag{5}$$

where the maximum is taken over all possible choices of k orthonormal vectors $v_1, \ldots, v_k$. In particular if e_j are the eigenvectors of A such that $Ae_j = \alpha_j e_j$, then (5) gives

$$\sum_{j=1}^{k} s_j^2(A - N) \geqq \sum_{j=1}^{k} \|(\alpha_{i_j} - N)e_{i_j}\|^2. \tag{6}$$

But, if α is any complex number and e any unit vector, then $\|(\alpha - N)e\|$ is greater than the distance of α from the spectrum of N. (See [3] or [8].) Using this fact and the definition of F, one obtains the inequality (4) from (6).

Now assume that K is a convex set. Once again to prove the inequality (2) we need to prove it only for the special class of Ky Fan norms, i.e. we need to prove that for $k = 1, 2, \ldots$ we have

$$\sum_{j=1}^{k} |\alpha_{i_j} - F(\alpha_{i_j})| \leqq \sum_{j=1}^{k} s_j(A - N), \tag{7}$$

for any choice of k indices $\alpha_{i_1}, \ldots, \alpha_{i_k}$. By a theorem of Fan ([4], [6, p. 47]) we have

$$\sum_{j=1}^{k} s_j(A - N) \geqq \sum_{j=1}^{k} |\langle (A - N)v_j, v_j \rangle| \tag{8}$$

for all k-tuples of orthonormal vectors $v_1, \ldots, v_k$. Since N is normal and K is a convex set containing the spectrum of N, the points $\langle Nv_j, v_j \rangle$ all lie in K. (See, e.g. [7].) Now choose v_j to be the eigenvectors e_{i_j} as before. Then the inequality (7) follows from (8). □

4. On the Powers–Størmer–Kittaneh Inequality

By a modification of the arguments of Kittaneh [12] we can derive the following generalisation of his Theorems 1 and 2.

Theorem 2. *Let $A, B \in \mathscr{B}(\mathscr{H})$ and let X be a self-adjoint element of $\mathscr{B}(\mathscr{H})$. Let $A + B \geqq \pm X$. Suppose $AX + XB$ belongs to the norm ideal associated with any unitarily invariant norm $|||\cdot|||$. Then X^2 also belongs to this ideal and*

$$|||AX + XB||| \geqq |||X^2|||. \tag{9}$$

Proof. For the case of the operator norm this is Kittaneh's Theorem 1 [12]. For the case of any other norm, $AX + XB$ is compact. Then the assumption $A + B \geqq \pm X$, together with the operator norm case of this theorem implies that X is compact. (See [12], Proof of Theorem 2.) Arrange the eigenvalues of X in descending order of modulus as $|\lambda_1| \geqq |\lambda_2| \geqq \ldots$. Then $s_j(X) = |\lambda_j|$. Let e_j be the eigenvectors of X corresponding to λ_j. Once again we need to prove (9) only for the special class of Ky Fan norms. Using, again, the theorem of Fan ([4, 6, p. 47]) we have

$$\begin{aligned} &\|AX + XB\|_k \\ &= \sum_{j=1}^{k} s_j(AX + XB) \geqq \sum_{j=1}^{k} |\langle (AX + XB)e_j, e_j\rangle| \\ &= \sum_{j=1}^{k} |\langle AXe_j, e_j\rangle + \langle Be_j, Xe_j\rangle| = \sum_{j=1}^{k} |\lambda_j| |\langle (A+B)e_j, e_j\rangle| \\ &\geqq \sum_{j=1}^{k} s_j(X) |\langle Xe_j, e_j\rangle| = \sum_{j=1}^{k} s_j^2(X) = \|X^2\|_k. \quad \square \end{aligned}$$

Corollary 3. *Let $A, B \in \mathscr{B}(\mathscr{H})$ be positive. Suppose $A^2 - B^2$ belongs to the norm ideal associated with any unitarily invariant norm $|||\cdot|||$. Then $(A - B)^2$ also belongs to this ideal and*

$$|||(A - B)^2||| \leqq |||A^2 - B^2|||. \tag{10}$$

Proof. Choose $X = A - B$ in Theorem 2. $\square$

As explained in the Introduction the Powers–Størmer inequality and its generalisation by Kittaneh are special cases of (10) for the p-norms.

Inequality (10) can be further generalised as follows. Let $x_1 \geqq x_2 \geqq \cdots \geqq x_n \geqq 0$ and $y_1 \geqq y_2 \geqq \cdots \geqq y_n \geqq 0$ be real numbers such that $\sum_{j=1}^{k} x_j \leqq \sum_{j=1}^{k} y_j$ for $k = 1, 2, \ldots, n$. Then by standard results in the theory of majorisation we have $\sum_{j=1}^{k} x_j^2 \leqq \sum_{j=1}^{k} y_j^2$. (See [1 or 13].) In particular, this implies that if A and B are self-adjoint operators with $\|A\|_k \leqq \|B\|_k$ for $k = 1, 2, \ldots, n$, then $\|A^2\|_k \leqq \|B^2\|_k$ for $k = 1, 2, \ldots, n$. Using this fact together with Fan's Theorem we get from (10) by iteration:

Corollary 4. *Let m be any integer of the form $m = 2^k$, $k = 1, 2, \ldots$. Then with the same notation as in the statement of Corollary 3 we have*

$$|||(A - B)^m||| \leqq |||A^m - B^m|||. \tag{11}$$

Next note that if T is self-adjoint and m is any integer, then $\|T^m\|_p = \|T\|_{mp}^m$ for $1 \leqq p \leqq \infty$. So specialising (11) to the case of p-norms we obtain

$$\|A^{1/m} - B^{1/m}\|_{mp}^m \leqq \|A - B\|_p \tag{12}$$

for all $m = 2^k$, $k = 1, 2, \ldots$.

A special case of the above inequality (when $A \geqq B \geqq 0$) has been proved by J. Philips (unpublished) using a different argument.

5. On the van Hemmen–Ando–Kittaneh Inequalities

Recall [10] that a closed operator A defined on a dense domain $\mathscr{D}(A)$ is called *maximal accretive* if $\operatorname{Re}\langle Au, u\rangle \geqq 0$ for all $u \in \mathscr{D}(A)$ and if A has no proper extension to an operator which also satisfies the above condition. Every maximal accretive operator has a unique maximal accretive square root (and also other roots). For bounded operators maximal accretiveness just means that $\operatorname{Re} A = (A + A^*)/2$ is a positive operator.

The following theorem, essentially due to Heinz [9], can be found in [2].

Theorem 5. *Let $A - aI$ and $-B - aI$ be maximal accretive operators for some real $a \geqq 0$. Then for every S belonging to the norm ideal corresponding to any unitarily invariant norm $|||\cdot|||$, the equation $AX - XB = S$ has a unique solution X in the same norm ideal, and*

$$2a|||X||| \leqq |||S|||.$$

(*Note.* The statement about the operator equation above is to be interpreted to mean that for all $u \in \mathscr{D}(A^*)$ and $v \in \mathscr{D}(B)$ we have

$$\langle v, X^* A^* u\rangle - \langle XBv, u\rangle = \langle Sv, u\rangle.)$$

Corollary 6. *Let $A - aI$ be maximal accretive for some $a \geqq 0$. Then for all $X \in \mathscr{B}(\mathscr{H})$ and for every unitarily invariant norm we have*

$$2a|||X||| \leqq |||AX + XA^*|||.$$

One of the main results in Kittaneh [11] is a special case of Corollary 6 when A is bounded and the norm is a Schatten p-norm for $1 \leqq p \leqq \infty$. Kittaneh's result, in turn, is an extension of an inequality of van Hemmen and Ando [18, Lemma 3.1] which is the special case when $A = A^*$.

Using these inequalities and the arguments of van Hemmen–Ando [18] one can easily extend their results to maximal accretive operators. For example, their Proposition 3.2 can be generalised as:

Proposition 7. *If A, B are maximal accretive operators and if for some $a \geqq 0$, the operator $A^{1/2} + B^{1/2} - aI$ is also maximal accretive, then for every unitarily invariant norm*

$$a|||A^{1/2} - B^{1/2}||| \leqq |||A - B|||.$$

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