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ON CERTAIN UNBIASED PRODUCT ESTIMATORS

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ABSTRACT

When data on an auxiliary variate is available on all the units of the population, negatively correlated with the study variate, Robson (1957) and Murthy (1964) proposed 'product method' of estimation for the estimation of the population total (mean) of the study variate. In this paper, we discuss a method given in Rao(1983) and obtain a simpler derivation of the class of unbiased product estimators for the case of Simple Random Sampling WithOut Replacement design as well as for the case of interpenetrating subsamples design which follows as a limiting case. Finally, we shall illustrate the results by means of two simple numerical examples from live data.

1. INTRODUCTION

Consider a finite population of size N consisting of units $(U_1, U_2, \ldots, U_N)$. Let y be the study variate taking values Y_i on units U_i, $i = 1,2,\ldots,N$. When auxiliary information on a variate x taking values X_i on U_i is available which is negatively correlated with y, it was shown that this information could be profitably used to construct a 'product estimator' for the population total $Y = \sum_{i=1}^{N} Y_i$ (or equivalently, the mean $\bar{Y} = Y/N$) which is in certain cases far superior to the conventional estimator which does not use the information on x (Robson (1957), Murthy (1964)). During the years that followed, several studies were made on the extent of bias and mean square error of these estimators and attempts were made to construct 'unbiased product estimators'. Srivenkataramana and Tracy (1979) reviewed some of these methods while at the same time offering some more estimates. Vos (1980) generalized these methods to obtain 'mixing estimators' and considered the efficiency of these estimators subsequently. Following the recent approach given in Rao (1981 b), an analogous method of constructing a new class of unbiased product estimators under a general setup which can also identify the class of unbiased ratio estimators considered earlier was discussed by Rao (1983). Singh, Iachan and Upadhyaya (SIU(1985)) explored this approach further and obtained certain generalizations. In this paper, we shall give a simpler construction of the general class of unbiased product estimators for the case of Simple Random Sampling WithOut Replacement (SRSWOR) design. We shall also make some corrections in the estimators given by SIU (1985). The corresponding results for the interpenetrating subsamples design then follow as a limiting case.

2. UNBIASED PRODUCT ESTIMATORS FOR SRSWOR DESIGN

For the case of SRSWOR, let y_i and x_i denote respectively the y and x values of the i th sampled unit, $i=1,2,\ldots,n$. Let

$$\hat{\bar{Y}}_1 = \bar{y}\,\bar{x}/\bar{X} \tag{2.1}$$

and

$$\hat{\bar{Y}}_n = \sum_{i=1}^{n} y_i\, x_i/n\,\bar{X} \tag{2.2}$$

where $\bar{y} = \sum_{1}^{n} y_i/n$, $\bar{x} = \sum_{1}^{n} x_i/n$ and $\bar{X} = \sum_{1}^{N} X_i/N$.

We then have

$$B(\hat{\hat{Y}}_n) = (1-c')^{-1} B(\hat{\hat{Y}}_1) \tag{2.3}$$

where

$$c' = N(n-1)/n(N-1).$$

Motivated by Rao (1981 b, 1983) we now rewrite (2.3) as

$$E(k \hat{\hat{Y}}_1) = c'k \bar{Y} + (1-c')k E(\hat{\hat{Y}}_n)$$

where k is any arbitrary constant

$$\text{i.e.} \quad E(k \hat{\hat{Y}}_1 - (1-c')k \hat{\hat{Y}}_n + (1-c'k)\bar{y}) = \bar{Y}. \tag{2.4}$$

Thus we have

Theorem 2.1: A general class of unbiased product estimators for the population mean $\bar{Y}$ is given by

$$\hat{\hat{Y}}_p = k \hat{\hat{Y}}_1 - (1-c')k \hat{\hat{Y}}_n + (1-c'k)\bar{y}$$

where k is any arbitrary constant and $c' = N(n-1)/n(N-1)$.

Remark. The above estimator $\hat{\hat{Y}}_p$ was derived by Rao (1983) in a different way and Singh, Ischan and Upadhyaya (SIU(1985)) rephrased the same estimator as

$$\hat{\hat{Y}}_{PC} = \phi_1 \bar{y} + \phi_2 \hat{\hat{Y}}_n + \phi_3 \hat{\hat{Y}}_1 \tag{2.5}$$

in (5.1) of their paper. For different choices of k we get the different estimators considered by SIU. Some of the coefficients ϕ_i given in SIU are incorrect and wrongly referenced. We shall now give here a table of these estimators correcting the coefficients and references given in SIU.

Estimator I was given by Robson (1957) and Murthy (1964), estimators II and III were given by Gupta and Adhvaryu (1982), estimators IV and V were suggested by Ischan, Singh and Upadhyaya (1983) while estimators VI and VII were given by Rao (1983) and Sastry (1974) respectively. Notice that in the above table all the estimators are non-convex combinations (cf. Rao (1981 a)).

Table I

Unbiased Product Estimators in the class (2.5) under SRSWOR

Estimator	k	ϕ_1	ϕ_2	ϕ_3
I	$\frac{1}{c'}$	0	$-\frac{(N-n)}{N(n-1)}$	$\frac{n(N-1)}{N(n-1)}$
II	$\frac{1}{c'-1}$	$\frac{n(N-1)}{N-n}$	1	$-\frac{n(N-1)}{N-n}$
III	1	$\frac{N-n}{n(N-1)}$	$-\frac{N-n}{n(N-1)}$	1
IV	$\frac{1}{1+c'}$	$-\frac{n(N-1)}{N(1-2n)+n}$	$\frac{N-n}{N(1-2n)+n}$	$\frac{-n(N-1)}{N(1-2n)+n}$
V	$\frac{1}{2c'-1}$	$-\frac{N-n}{N(n-2)+n}$	$-\frac{N-n}{N(n-2)+n}$	$\frac{n(N-1)}{N(n-2)+n}$
VI	k	$1-c'k$	$-(1-c')k$	k
VII	$-\frac{q}{c'}$	$q+1$	$\frac{(1-c')q}{c'}$	$-\frac{q}{c'}$

3. OPTIMUM ESTIMATOR IN THE CLASS

The estimator $\hat{\bar{Y}}_p$ obtained in Theorem 2.1 can be written as

$$\hat{\bar{Y}}_p = \bar{y} - k\,\hat{\bar{Y}}' \tag{3.1}$$

where $\hat{\bar{Y}}' = c'\bar{y} + (1-c')\hat{\bar{Y}}_n - \hat{\bar{Y}}_1$. (3.2)

From (3.1) it follows that $\text{Var}(\hat{\bar{Y}}_p)$ is minimized by

$$k_{opt.} = \text{Cov}(\bar{y}, \hat{\bar{Y}}')/V(\hat{\bar{Y}}') \tag{3.3}$$

which gives

$$V(\hat{\bar{Y}}_{P,opt.}) = V(\bar{y})\ (1-\rho_1^2) \tag{3.4}$$

where $\rho_1 = \text{Corr.}(\bar{y}, \hat{\bar{Y}}')$.

From (3.4) it is clear that

$$V(\hat{\bar{Y}}_{P,opt.}) < V(\bar{y}). \tag{3.5}$$

Furthermore, it may be recalled that

$$M(\hat{\bar{Y}}_1) = V(\bar{y}) + 2R\ \mathrm{Cov}\ (\bar{y},\bar{x}) + R^2V(\bar{x}) \qquad (3.6)$$

where $R = \bar{Y}/\bar{X}$. Thus $\hat{\hat{\bar{Y}}}_{P,opt.}$ will be better than the usual product estimator (biased) $\hat{\bar{Y}}_1 = \bar{y}\ \bar{x}/\ \bar{X}$ if

$$V(\hat{\hat{\bar{Y}}}_{P,opt.}) < M(\hat{\bar{Y}}_1) \qquad (3.7)$$

which can be rewritten as

$$V(\bar{y})(1-\rho_1^2) < V(\bar{y}) + 2R\ \rho\sigma(\bar{y})\sigma(\bar{x}) + R^2V(\bar{x}) \qquad (3.8)$$

where $\rho = \mathrm{Corr.}(\bar{y},\bar{x})$ and $\sigma(\bar{y}) = (V(\bar{y}))^{1/2}$ and $\sigma(\bar{x})=(V(\bar{x}))^{1/2}$

The inequality (3.8) can be simplified as

$$(\rho\sigma(\bar{y}) + R\sigma(\bar{x}))^2 + V(\bar{y})\ (\rho_1^2 - \rho^2) > 0. \qquad (3.9)$$

Thus it follows from (3.9) that a sufficient condition for the estimator $\hat{\hat{\bar{Y}}}_{P,opt.}$ to be better than $\hat{\bar{Y}}_1$ is that $\rho_1^2 > \rho^2$. It may be mentioned here that the sufficient condition can be verified in practice based on their estimated values or from a pilot study. The two illustrative examples show the relative performance of different estimators. The mse of $\ddot{\bar{Y}}_{P,opt.}$ defined as $V(\hat{\hat{\bar{Y}}}_{P,opt.})/\ \bar{Y}^2$ is the smallest compared to others. However the optimum value of k is not known and has to be estimated from a pilot study.

4. UNBIASED PRODUCT ESTIMATORS FOR INTERPENETRATING SUBSAMPLES DESIGN

Consider the case of interpenetrating subsamples discussed by Murthy (1964). Let y_i and x_i be unbiased estimates of the population totals Y and X respectively based on the i th independent interpenetrating subsample, $i = 1,2,\ldots,n$. As before, consider now the two product estimators

$$\hat{\bar{Y}}_1 = \bar{y}\ \bar{x}/\ \bar{X} \qquad (4.1)$$

Table II

Unbiased Product Estimators in the class (4.3) under IPS design

Estimator	k	ϕ_1	ϕ_2	ϕ_3
(i)	$\frac{1}{c}$	0	$-\frac{1}{n-1}$	$\frac{n}{n-1}$
(ii)	-n	n	1	-n
(iii)	1	$\frac{1}{n}$	$-\frac{1}{n}$	1
(iv)	$\frac{n}{2n-1}$	$\frac{n}{2n-1}$	$-\frac{1}{2n-1}$	$\frac{n}{2n-1}$
(v)	$\frac{n}{n-2}$	$-\frac{1}{n-2}$	$-\frac{1}{n-2}$	$\frac{n}{n-2}$
(vi)	k	1-ck	-(1-c)k	k
(vii)	n	2-n	-1	n
(viii)	$-\frac{q}{c}$	q+1	$\frac{q}{n-1}$	$-\frac{qn}{n-1}$

and

$$\hat{\bar{Y}}_n = \sum_{i=1}^{n} y_i \, x_i / n \bar{X} \tag{4.2}$$

where $\bar{y} = \sum_1^n y_i/n$, $\bar{x} = \sum_1^n x_i/n$ and $\bar{X} = \sum_1^N X_i/N$. It now easily follows from Theorem 2.1 as a limiting case(as $N \to \infty$) for the interpenetrating subsamples design that

<u>Theorem 4.1</u>: A general class of unbiased product estimators for the population mean $\bar{Y}$ is given by

$$\hat{\bar{Y}}_p = k\,\hat{\bar{Y}}_1 - (1-c)k\,\hat{\bar{Y}}_n + (1-ck)\bar{y}$$

where k is any arbitrary constant and $c = (n-1)/n$.

SIU (1985) rephrased the above estimator as

$$\hat{\bar{Y}}_{PC} = \phi_1 \bar{y} + \phi_2 \hat{\bar{Y}}_n + \phi_3 \hat{\bar{Y}}_1. \tag{4.3}$$

For the sake of completeness we shall give below a table of estimators for the case of interpenetrating subsamples design similar to Table I for the SRSWOR case. Some of the coefficients given in SIU (1985) are wrong which are corrected in this table.

Estimator (i) was given by Murthy (1964). Estimators (ii) and (iii) are variations of Gupta and Adhvaryu's (1982) while (iv) and (v) were introduced by Iachan, Singh and Upadhyaya (1983). Estimators (vi) and (vii) were discussed in Rao (1983) where as (viii) is a variation of Sastry's (1974) estimator.

5. ILLUSTRATIVE EXAMPLES

In this section we shall illustrate the results by two simple empirical examples using live data of population of size 4 when samples of size 2 are drawn by SRSWOR from each. We have chosen 4 cities each with a population 100,000 and above from Census of India (1971) document. In these cities situated in India, high (low) female literacy rates correspond to low (high) female work participation rates. We treat female literacy rate as the auxiliary variate x and the female work participation rate as the study variate y. The data corresponding to these two variates and the relevant results for the two chosen populations are given below :

Population I

Name of city	y	x
1. Bhavnagar	5.51	45.54
2. Berhampur	7.48	37.94
3. Jullundhar	2.78	50.98
4. Kanpur	3.71	41.21
	$\bar{Y} = 4.87$	$\bar{X} = 43.9175$

Table III

Relative mean square error (rmse) of the estimator $\hat{\bar{Y}}_p$

k	estimator	$V(\hat{\bar{Y}}_p)$	rmse
-3	II	2.42942	0.10243
0	$\bar{y}$	1.07795	0.04545
0.6	IV	0.91144	0.03843
1	III	0.81965	0.03456
1.5	I	0.72654	0.03063
2	-	0.65745	0.02772
3	V	0.59134	0.02493
3.18789	Optimum	0.58964	0.02486
4	-	0.62133	0.02620
	$\hat{\bar{Y}}_1$		0.02934
	$\hat{\bar{Y}}_n$		0.02829

Here rmse is defined as $V(\hat{\bar{Y}}_p)/\bar{Y}^2$.

Population II

Name of city	y	x
1. Asansol	4.38	48.95
2. Ranchi	4.83	49.39
3. Gauhati	8.26	41.87
4. Malegaon	9.83	31.73
	$\bar{Y}$ = 6.825	$\bar{X}$ = 42.985

Table IV

Relative mean square error of the estimator $\hat{\bar{Y}}_p$

k	estimator	$V(\hat{\bar{Y}}_p)$	rmse
-3	II	6.56692	0.14098
0	$\bar{y}$	1.75394	0.03765
0.6	IV	1.18879	0.02552
1	III	0.88561	0.01901
1.5	I	0.58944	0.01265
2	-	0.38527	0.00827
2.85961	Optimum	0.24932	0.00535
3	V	0.25293	0.00543
	$\hat{\bar{Y}}_1$		0.01195
	$\hat{\bar{Y}}_n$		0.01385

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