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On resolvable and affine resolvable variance-balanced designs

BY RAHUL MUKERJEE
Statistics and Mathematics Division, Indian Statistical Institute, Calcutta 700 035, India

AND SANPEI KAGEYAMA
Department of Mathematics, Hiroshima University, Hiroshima 734, Japan

SUMMARY

This paper introduces the notion of affine $(\mu_1, \ldots, \mu_t)$-resolvability and explores the interrelations between: (a) affine $(\mu_1, \ldots, \mu_t)$-resolvability, (b) variance-balance, and (c) the relation $b = v+t-1$, where b is the number of blocks. It is seen that, while (a) and (b) imply (c), and (b) and (c) imply (a), the relation (a) and (c) imply (b) is not in general true. A necessary and sufficient condition under which (a) and (c) imply (b) has been derived and certain nonexistence results follow. The last section states an open problem in this connexion and indicates the link with a problem in factorial designs.

Some key words: Affine; Orthogonal main effect design; Proportional array; Resolvability; Variance-balanced block design.

1. INTRODUCTION AND PRELIMINARIES

A binary variance-balanced block design with parameters v, b, r_i $(i = 1, \ldots, v)$ and k_j $(j = 1, \ldots, b)$ is given by a $v \times b$ incidence matrix N satisfying

$$C = R - NK^{-1}N' = \rho(I_v - v^{-1}J_v),$$

where $R = \mathrm{diag}(r_1, \ldots, r_v)$, $K = \mathrm{diag}(k_1, \ldots, k_b)$, $\rho = (n-b)/(v-1)$, $n = \Sigma r_i$, I_v is the identity matrix of order v, J_v is a $v \times v$ matrix with all elements unity, and N' is the transpose of N. A block design is said to be $(\mu_1, \ldots, \mu_t)$-resolvable if the blocks can be separated into $t \geqslant 2$ sets of $m_1, \ldots, m_t \geqslant 1$ blocks such that the set consisting of m_l blocks contains every treatment $\mu_l \geqslant 1$ times $(l = 1, \ldots, t)$. Clearly a $(\mu_1, \ldots, \mu_t)$-resolvable design is equireplicated, that is $r_1 = \ldots = r_v$.

The importance of variance-balance and resolvability in the context of experimental planning is well known; the former yields optimal designs apart from ensuring simplicity in the analysis and the latter is helpful, among other respects, in the recovery of interblock information. Also practical situations sometimes demand designs with varying block sizes (Pearce, 1964) or resolvable designs with unequal replication numbers between sets of blocks; for a practical example, see Kageyama (1976). These considerations indicate the importance of $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced block designs with possibly varying block sizes and having $\mu_1, \ldots, \mu_t$ possibly not all equal.

Generalizing the results of Raghavarao (1962; 1971, p. 61), Kageyama (1973) and Hughes & Piper (1976), Kageyama (1984) established that for a $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced block design with $\rho < r = \mu_1 + \ldots + \mu_t$, the inequality $b \geqslant v+t-1$ holds. Kageyama (1984) also obtained the following.

THEOREM 1·1. *In a $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced block design with $b = v+t-1$, except when $\mu_1 = \ldots = \mu_t = 1$ block sizes of blocks belonging to the same set are always equal.*

Whether the above holds for the case $\mu_1 = \ldots = \mu_t = 1$ as well, is an open problem and has been considered in the last section of the present paper.

2. CONNEXION BETWEEN AFFINE RESOLVABILITY, VARIANCE-BALANCE AND THE RELATION $b = v+t-1$

The following result is due to Shrikhande & Raghavarao (1963).

THEOREM 2·1. *For a μ-resolvable incomplete block design involving b blocks in t sets and v treatments with constant block size, any two of the following imply the third:* (a) *affine μ-resolvability,* (b) *variance-balance,* (c) $b = v+t-1$.

It is interesting to examine how far this result can be extended to $(\mu_1, \ldots, \mu_t)$-resolvable block designs. Throughout this section, attention will be restricted to only those $(\mu_1, \ldots, \mu_t)$-resolvable block designs which have a constant block size within each set. In view of Theorem 1·1, this is justified at least when $(\mu_1, \ldots, \mu_t) \neq (1, \ldots, 1)$. The constant block size within the lth set may be denoted by k_l^* for $l = 1, \ldots, t$.

Definition 2·1. A $(\mu_1, \ldots, \mu_t)$-resolvable block design with a constant block size in each set will be said to be affine $(\mu_1, \ldots, \mu_t)$-resolvable if:

(i) for $l = 1, \ldots, t$, every two distinct blocks from the lth set intersect in the same number, say q_{ll}, of treatments;

(ii) for $l \neq l' = 1, \ldots, t$, every block from the lth set intersects every block of the l'th set in the same number, say $q_{ll'}$, of treatments.

With m_l, k_l^* $(l = 1, \ldots, t)$ defined as above, it is evident from elementary considerations that for affine $(\mu_1, \ldots, \mu_t)$-resolvable block designs

$$q_{ll}(m_l - 1) = k_l^*(\mu_l - 1), \quad q_{ll'} m_{l'} = k_l^* \mu_{l'} \quad (l \neq l' = 1, \ldots, t). \tag{2·1}$$

The following two theorems present generalizations of some of the ideas of Theorem 2·1 in the context of $(\mu_1, \ldots, \mu_t)$-resolvable designs.

THEOREM 2·2. *A $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced block design with parameters v, $b = v+t-1 = \Sigma m_l$, $r = \Sigma \mu_l$, k_l^* $(l = 1, \ldots, t)$ must be affine $(\mu_1, \ldots, \mu_t)$-resolvable with*

$$q_{ll} = (k_l^{*2}/v)[1 - (b-r)/\{\mu_l(v-1)\}]$$

provided $m_l \geqslant 2$,

$$q_{ll'} = k_l^* k_{l'}^*/v \quad (l \neq l' = 1, \ldots, t).$$

Proof. Denote by $s_{jj'}$ the intersection number of jth and j'th blocks $(j \neq j' = 1, \ldots, b)$. When $b = v+t-1$, as in the proof of Theorem 3 of Kageyama (1984), one can obtain, after some calculation,

$$N'N = v^{-1} kk' + (r-\rho) K - (r-\rho) v^{-1} \operatorname{diag}(\mu_1^{-1} k_1^{*2} J_{m_1}, \ldots, \mu_t^{-1} k_t^{*2} J_{m_t}),$$

where

$$k = (k_1, \ldots, k_b)' = (k_1^* 1'_{m_1}, \ldots, k_t^* 1'_{m_t})',$$

$1_{m_l} = (1, \ldots, 1)'$ being of size $m_l \times 1$, $J_{m_l} = 1_{m_l} 1'_{m_l}$ $(l = 1, \ldots, t)$. Comparing the off-diagonal elements of the above, one obtains that:

(i) if the jth and j'th blocks belong to the same lth set ($l = 1, \ldots, t$), then

$$s_{jj'} = k_l^{*2}/v - (r-\rho)\, k_l^{*2}/(v\mu_l) = (k_l^{*2}/v)\,[1-(b-r)/\{\mu_l(v-1)\}] \tag{2·2}$$

provided $m_l \geqslant 2$;

(ii) if the jth and j'th blocks belong to different sets, say to the lth and l'th sets, then

$$s_{jj'} = k_l^* k_{l'}^*/v. \tag{2·3}$$

From (2·2) and (2·3), the required result follows. □

Note in particular that, if $\mu_1 = \ldots = \mu_t = 1$, then $r = t$, and the right-hand side of (2·2) vanishes, i.e. one gets $q_{ll} = 0$ for $l = 1, \ldots, t$.

THEOREM 2·3. *An incomplete block affine* $(\mu_1, \ldots, \mu_t)$*-resolvable variance-balanced design must have* $b = v+t-1$.

Proof. For $l = 1, \ldots, t$, let N_l denote the portion of the incidence matrix, N, arising from the lth set of blocks, that is $N = (N_1 : \ldots : N_t)$. Then with $1_v = (1, \ldots, 1)'$ of size $v \times 1$, defining

$$N^* = \begin{bmatrix} N_1 & N_2 & \ldots & N_t & 1_v \\ k_1^* 1'_{m_1} & 0' & \ldots & 0' & 0 \\ 0' & k_2^* 1'_{m_2} & \ldots & 0' & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0' & 0' & \ldots & k_t^* 1'_{m_t} & 0 \end{bmatrix}$$

of size $(v+t) \times (b+1)$, we obtain from the proof of Theorem 1 of Kageyama (1984) that

$$v+t = \operatorname{rank}(N^*) \leqslant b+1, \tag{2·4}$$

using the fact that the design under consideration is a $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced block design.

Next, the additional information regarding affine resolvability will be used to establish the linear independence of the columns of N^*. Under affine $(\mu_1, \ldots, \mu_t)$-resolvability, clearly

$$N_l' N_l = (k_l^* - q_{ll})\, I_{m_l} + q_{ll} J_{m_l}, \quad N_l' N_{l'} = q_{ll'}\, 1_{m_l} 1'_{m_{l'}} \quad (l \neq l' = 1, \ldots, t). \tag{2·5}$$

Now for any vector $\xi = (\xi_1', \ldots, \xi_t', \theta)'$ of size $(b+1) \times 1$, where ξ_l is of size $m_l \times 1$ ($l = 1, \ldots, t$),

$$N^* \xi = 0 \tag{2·6}$$

implies

$$\sum_{u=1}^{t} N_u \xi_u + \theta 1_v = 0, \tag{2·7}$$

$$1'_{m_l} \xi_l = 0 \quad (l = 1, \ldots, t). \tag{2·8}$$

Premultiplying (2·7) by N_l' and applying (2·5) and (2·6), one obtains on simplification

$$(k_l^* - q_{ll})\, \xi_l + \theta k_l^*\, 1_{m_l} = 0 \quad (l = 1, \ldots, t), \tag{2·9}$$

whence, premultiplying both sides by $1'_{m_l}$ and applying (2·8) again, one gets $\theta = 0$. Hence if we note that, for an incomplete block design $k_l^* > q_{ll}$, equation (2·9) yields

$\xi_l = 0$ $(l = 1, \ldots, t)$. Thus (2·6) implies $\xi = 0$, so that the columns of N^* are linearly independent. Therefore, rank $(N^*) = b+1$ and the required result follows from (2·4). This completes the proof. □

Note that an example of an affine $(\mu, \ldots, \mu, 1)$-resolvable variance-balanced block design can be constructed by a juxtaposition of a complete block and some affine μ-resolvable balanced incomplete block design.

Theorems 2·2 and 2·3 extend respectively the two implications '(b), (c) imply (a)' and '(a), (b) imply (c)' contained in Theorem 2·1. Thus, Theorem 2·1 can be partially extended to $(\mu_1, \ldots, \mu_t)$-resolvable block designs. The result '(a), (c) imply (b)' of Theorem 2·1 cannot, however, be extended in general, i.e. an incomplete block affine $(\mu_1, \ldots, \mu_t)$-resolvable design with $b = v+t-1$ is not necessarily variance-balanced. This point is illustrated by the following example.

Example 2·1. Consider an affine $(2, 2, 1, 1)$-resolvable incomplete block design with $v = 9$, $b = 12$, $r = 6$, $k_1^* = k_2^* = 6$, $k_3^* = k_4^* = 3$, $t = 4$, given by the following incidence matrix. Clearly here $b = v+t-1$, but it can be checked that the design is not variance-balanced; see also Corollary 2·2 below. In fact

$$N = \begin{bmatrix} 0\,1\,1 & 0\,1\,1 & 1\,0\,0 & 1\,0\,0 \\ 0\,1\,1 & 1\,0\,1 & 0\,1\,0 & 0\,1\,0 \\ 0\,1\,1 & 1\,1\,0 & 0\,0\,1 & 0\,0\,1 \\ 1\,0\,1 & 0\,1\,1 & 0\,1\,0 & 0\,0\,1 \\ 1\,0\,1 & 1\,0\,1 & 0\,0\,1 & 1\,0\,0 \\ 1\,0\,1 & 1\,1\,0 & 1\,0\,0 & 0\,1\,0 \\ 1\,1\,0 & 0\,1\,1 & 0\,0\,1 & 0\,1\,0 \\ 1\,1\,0 & 1\,0\,1 & 1\,0\,0 & 0\,0\,1 \\ 1\,1\,0 & 1\,1\,0 & 0\,1\,0 & 1\,0\,0 \end{bmatrix}$$

In view of the above example, it is interesting to examine the possible existence of necessary and sufficient conditions under which an incomplete block $(\mu_1, \ldots, \mu_t)$-resolvable design with $b = v+t-1$ becomes variance-balanced. This is given by the following with the notation defined above.

THEOREM 2·4. *An incomplete block affine $(\mu_1, \ldots, \mu_t)$-resolvable design satisfying $b = v+t-1$ is variance-balanced if and only if*

$$(\mu_l - 1)/(m_l - 1) = (r-t)/(v-1) \quad (l = 1, \ldots, t). \tag{2·10}$$

Proof. First consider necessity. Suppose the design is variance-balanced. Then by (2·1) and Theorem 2·2,

$$(k_l^{*2}/v)\,[1-(b-r)/\{\mu_l(v-1)\}] = k_l^*(\mu_l-1)/(m_l-1) = q_{ll} \quad (l = 1, \ldots, t),$$

from which, noting that $b = v+t-1$ and applying the obvious relations

$$\mu_l v = k_l^* m_l \quad (l = 1, \ldots, t), \tag{2·11}$$

one gets (2·10).

For sufficiency, note that, under (2·10), by (2·1) and (2·11),

$$q_{ll} = k_l^*(r-t)/(v-1), \quad q_{ll'} = k_l^* k_{l'}^*/v \quad (l \neq l' = 1, \ldots, t). \tag{2·12}$$

With 1_v as in the proof of Theorem 2·3, define P as a $(v-1)\times v$ matrix such that $(v^{-\frac{1}{2}}1_v : P')$ is orthogonal, so that $PP' = I_{v-1}$ and $P'P = I_v - v^{-1}J_v$. If one evaluates the determinant

$$\begin{vmatrix} K & N'P' \\ PN & rI_{v-1} \end{vmatrix}$$

in two ways and equates the corresponding expressions, then one gets

$$|K|\,|PCP'| = r^{v-1}\,|K - r^{-1}N'(I_v - v^{-1}J_v)N|, \qquad (2\cdot13)$$

the matrix C being as in (1·1). Clearly

$$N'J_vN = N'1_v1_v'N = kk',$$

where $k = (k_1^* 1_{m_1}', \ldots, k_t^* 1_{m_t}')'$. Hence, because the design is affine $(\mu_1, \ldots, \mu_t)$-resolvable and by (2·5) and (2·12), it follows, after some simplification, that

$$K - r^{-1}N'(I_v - v^{-1}J_v)N = \operatorname{diag}(W_1, \ldots, W_t),$$

where, for $l = 1, \ldots, t$,

$$W_l = \{k_l^* - r^{-1}(k_l^* - q_{ll})\}\, I_{m_l} + r^{-1}(v^{-1}k_l^{*2} - q_{ll})\, J_{m_l}. \qquad (2\cdot14)$$

One obtains after some calculation with (2·13) and (2·14) that

$$|PCP'| = \{(vr-b)/(v-1)\}^{v-1} \qquad (2\cdot15)$$

Since $P'P = I_v - v^{-1}J_v$, $CJ_v = 0$ and the design is binary,

$$\operatorname{tr}(PCP') = \operatorname{tr}(CP'P) = \operatorname{tr}(C) = vr - b.$$

Hence by (2·15), $|PCP'| = \{\operatorname{tr}(PCP')/(v-1)\}^{v-1}$ which implies that the eigenvalues of PCP' are all equal, in fact each being equal to $(vr-b)/(v-1)$. Now, from the definition of P, it is immediate that $C = [(vr-b)/(v-1)](I_v - v^{-1}J_v)$, so that the design is variance-balanced. Thus, the proof is completed. □

From Theorem 2·2 and the necessity of Theorem 2·4, the following corollary is immediate.

COROLLARY 2·1. *A necessary condition for the existence of an incomplete block $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced design with parameters v, $b = v+t-1 = \Sigma m_l$, $r = \Sigma \mu_l$, k_l^* $(l = 1, \ldots, t)$ is that (2·10) holds for each l, in which case $(r-t)(m_l-1)/(v-1) = \mu_l - 1$ must be integral for each l.*

The above corollary may be used to prove nonexistence results. In particular, under (2·10), $(\mu_l - 1)/(m_l - 1)$ is constant over l and hence $\mu_l = 1$ for some l implies $\mu_1 = \ldots = \mu_t = 1$. Thus one has established the following.

COROLLARY 2·2. *An incomplete block $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced design with parameters v, $b = v+t-1$, $r = \Sigma \mu_l$, k_l^* $(l = 1, \ldots, t)$ and having $\mu_l = 1$, $\mu_{l'} > 1$ for some $l \neq l' = 1, \ldots, t$ is nonexistent.*

The conclusion of Example 2·1 follows also from Corollary 2·2.

In the setting of Corollary 2·1, if $\mu_1 = \ldots = \mu_t \geqslant 2$, then, under (2·10), $m_1 = \ldots = m_t$. Consequently, by (2·11), $k_1^* = \ldots = k_t^*$, i.e. the design must then be a balanced incomplete block design. Thus, Theorem 4 of Kageyama (1984) follows as a corollary. The following example of an incomplete block affine 1-resolvable variance-balanced design with unequal block sizes and $b = v+t-1$, however shows that the last observation cannot be extended to the situation $\mu_1 = \ldots = \mu_t = 1$.

$$N = \begin{bmatrix} 1\,0\,0\,0 & 1\,0 & 1\,0 & 1\,0 & 1\,0 \\ 0\,1\,0\,0 & 1\,0 & 0\,1 & 1\,0 & 0\,1 \\ 0\,0\,1\,0 & 1\,0 & 1\,0 & 0\,1 & 0\,1 \\ 0\,0\,0\,1 & 1\,0 & 0\,1 & 0\,1 & 1\,0 \\ 1\,0\,0\,0 & 0\,1 & 0\,1 & 0\,1 & 0\,1 \\ 0\,1\,0\,0 & 0\,1 & 1\,0 & 0\,1 & 1\,0 \\ 0\,0\,1\,0 & 0\,1 & 0\,1 & 1\,0 & 1\,0 \\ 0\,0\,0\,1 & 0\,1 & 1\,0 & 1\,0 & 0\,1 \end{bmatrix} \qquad C = 4(I_8 - \tfrac{1}{8}J_8).$$

Some further nonexistence results follow from Theorem 2·2 as stated below.

COROLLARY 2·3. *There does not exist a $(\mu_1, \ldots, \mu_t)$-resolvable variance-balanced block design with parameters v, $b = v+t-1$, $r = \Sigma\mu_l$, k_l^* $(l = 1, \ldots, t)$ provided one of the following holds:*

(i) *there is a block size, k_l^*, say, of blocks belonging to the same lth set such that*

$$(k_l^{*2}/v)[1-(b-r)/\{\mu_l(v-1)\}]$$

is not integral;

(ii) *there are two block sizes k_l^*, $k_{l'}^*$, say, of blocks belonging to different sets such that $k_l^* k_{l'}^*$ is not divisible by v.*

3. AN OPEN PROBLEM

This section considers the problem of examining whether Theorem 1·1 is valid when $\mu_1 = \ldots = \mu_t = 1$, that is for 1-resolvable block designs. Note that for a 1-resolvable design $t = r$. Furthermore, it is clear that a 1-resolvable variance-balanced block design with $b = v+r-1$ is binary and even if there are some complete blocks, the design obtained by deleting these complete blocks will again be a 1-resolvable variance-balanced block design with the same property. Therefore, without loss of generality, attention will be restricted to incomplete block designs, so that each set involves at least two blocks, and the following problem will be considered.

Problem 3·1. Does there exist an incomplete block 1-resolvable variance-balanced design with $b = v+t-1$, having unequal block sizes within a set?

By establishing a correspondence between incomplete block 1-resolvable variance-balanced designs with $b = v+r-1$ and saturated proportional frequency plans for main effects, the above problem can be expressed in an equivalent form in the context of fractional factorial plans.

Definition 3·1. A proportional array A, with v assemblies, r constraints, $m_1, \ldots, m_r$ symbols and strength 2, is an $r \times v$ matrix with entries in the lth row coming from the set $\{1, \ldots, m_l\}$ such that

$$v_{j_l, j_{l^*}}^{(l, l^*)} = v_{j_l}^{(l)} v_{j_{l^*}}^{(l^*)}/v \quad (j_l = 1, \ldots, m_l;\ j_{l^*} = 1, \ldots, m_{l^*};\ 1 \leqslant l < l^* \leqslant r),$$

where $v_{j_l, j_{l^*}}^{(l, l^*)}$ is the number of times the ordered pair $(j_l, j_{l^*})'$ occurs as a column vector in the two-rowed submatrix of A given by its lth and l^*th rows, and $v_{j_l}^{(l)}$ is the number of times the symbol j_l occurs in the lth row of A.

Hereafter, a proportional array of strength 2, as defined above, will be denoted by PA $[v, r;\ m_1, \ldots, m_r]$. It is well known (Addelman, 1963; Raghavarao, 1971, Ch. 15) that interpreting the columns as level combinations, a PA $[v, r;\ m_1, \ldots, m_r]$ yields an orthogonal main effect fraction of an $m_1 \times \ldots \times m_r$ factorial in v runs. Such a fraction will be said to be saturated if it admits no error degrees of freedom, i.e. $v-1 = \Sigma(m_l - 1)$.

THEOREM 3·1. *Let* $r, m_1, \ldots, m_r, v$ *be given positive integers such that*

$$m_1 + \ldots + m_r = b = v + r - 1.$$

Then a 1-resolvable variance-balanced block design in v *treatments and* r *sets of blocks with* $m_1, \ldots, m_r$ *blocks in the* r *sets exists if and only if a* PA $[v, r;\ m_1, \ldots, m_r]$ *exists.*

Proof. For the necessity, let the stated variance-balanced block design exist. For $j = 1, \ldots, m_l;\ l = 1, \ldots, r$, denote by k_{jl} the size of the jth block in the lth set and write $k_l = (k_{1l}, \ldots, k_{m_l l})'$, $k = (k_1', \ldots, k_r')'$. Then, as in the proof of Theorem 2·2,

$$N'N = v^{-1} kk' + (r-\rho) K - (r-\rho) v^{-1} \operatorname{diag}(k_1 k_1', \ldots, k_r k_r').$$

Hence, if we define $\phi(jl, j'l')$ as the intersection number between the jth block of the lth set and the j'th block of the l'th set ($j = 1, \ldots, m_l;\ j' = 1, \ldots, m_{l'};\ l \neq l' = 1, \ldots, r$),

$$\phi(jl, j'l') = v^{-1} k_{jl} k_{j'l'}. \tag{3·1}$$

Form now an $r \times v$ array placing in its (l, i)th cell the symbol j if the treatment i occurs in the jth block of the lth set ($l = 1, \ldots, r;\ i = 1, \ldots, v$). By (3·1), the array so formed will be a PA $[v, r;\ m_1, \ldots, m_r]$.

Conversely, given a PA $[v, r;\ m_1, \ldots, m_r]$, form a 1-resolvable block design in r sets of blocks, there being m_l blocks in the lth set, putting the treatment i in the jth block of the lth set if the symbol j occurs in the (l, i)th cell of the proportional array ($l = 1, \ldots, r;\ i = 1, \ldots, v$). This 1-resolvable design clearly has $b = \Sigma m_l = v + r - 1$. It remains to show that the design is variance-balanced. This will be proved following the line of the sufficiency part of Theorem 2·4.

For the design constructed as above define k_l, k as in the proof of the necessity part, write $K_l = \operatorname{diag}(k_{1l}, \ldots, k_{m_l l})$, denote, as usual, the portion of the incidence matrix, N, arising from the lth set by N_l ($l = 1, \ldots, t$) and observe that (2·13) holds. Since $N'J_v N = kk'$, and, by construction,

$$N_l' N_l = K_l, \quad N_l' N_{l'} = v^{-1} k_l k_{l'}' \quad (l \neq l' = 1, \ldots, t),$$

it follows that

$$N'(I_v - v^{-1} J_v) N = \operatorname{diag}(K_1 - v^{-1} k_1 k_1', \ldots, K_r - v^{-1} k_r k_r'),$$

and hence, after some simplification, the determinant in the right-hand side of (2·13) reduces to $\{(r-1)/r\}^{v-1} |K|$, on making use of the fact that $\Sigma m_l = v + r - 1$. Thus (2·13) yields

$$|PCP'| = (r-1)^{v-1} = \{(vr-b)/(v-1)\}^{v-1},$$

since $b = v + r - 1$, and the rest follows as in Theorem 2·4. Thus, the proof is completed. □

In view of this theorem, the open problem posed in the beginning of this section may be stated equivalently as follows.

Problem 3·2. Does there exist a saturated proportional frequency plan for main effects with unequal replication numbers for the levels of at least one factor?

Since proportional frequency plans for main effects are in fact orthogonal main effect plans, there is yet another formulation of the problem as follows.

Problem 3·3. Does there exist a saturated orthogonal main effect plan with unequal replication numbers for the levels of at least one factor?

It should be clarified that 'orthogonality' in the last problem is in the sense of Addelman (1963); note that there is another definition of orthogonality (Yamamoto, Shirakura & Kuwada, 1975), which is not being followed here. Trivially, if v is a prime, then $k_{jl} k_{j'l'}/v$ cannot be an integer, since incomplete block designs are being considered, and by (3·1) nonexistence follows. Also, the existing methods of construction of proportional frequency plans involve the technique of collapsing of levels (Addelman, 1963) and cannot lead to a plan as stated in Problem 3·2. Therefore, in order to find out an example, if it exists, satisfying the conditions of Problem 3·2, or equivalently the other problems, one should look for a method for the construction of proportional frequency plans without applying the collapsing technique of Addelman. Our conjecture is, however, that there does not exist a variance-balanced block design as envisaged in Problem 3·1, or, equivalently, a fractional factorial plan as in Problems 3·2 and 3·3.

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REFERENCES

ADDELMAN, S. (1963). Techniques for constructing fractional replicate plans. *J. Am. Statist. Assoc.* **58**, 45–71.

HUGHES, D. R. & PIPER, F. C. (1976). On resolutions and Bose's theorem. *Geom. Dedicata* **5**, 129–33.

KAGEYAMA, S. (1973). On μ-resolvable and affine μ-resolvable balanced incomplete block designs. *Ann. Statist.* **1**, 195–203.

KAGEYAMA, S. (1976). Resolvability of block designs. *Ann. Statist.* **4**, 655–61.

KAGEYAMA, S. (1984). Some properties on resolvability of variance-balanced designs. *Geom. Dedicata* **15**, 289–92.

PEARCE, S. C. (1964). Experimenting with blocks of natural sizes. *Biometrics* **20**, 699–706.

RAGHAVARAO, D. (1962). On balanced unequal block designs. *Biometrika* **49**, 561–2.

RAGHAVARAO, D. (1971). *Constructions and Combinatorial Problems in Design of Experiments*. New York: Wiley.

SHRIKHANDE, S. S. & RAGHAVARAO, D. (1963). Affine α-resolvable incomplete block designs. In *Contributions to Statistics*, Ed. C. R. Rao, pp. 471–80. Calcutta: Pergamon.

YAMAMOTO, S., SHIRAKURA, T. & KUWADA, M. (1975). Balanced arrays of strength $2l$ and balanced fractional 2^m factorial designs. *Ann. Inst. Statist. Math.* **27**, 143–57.

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