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SOME SEARCH DESIGNS FOR SYMMETRIC AND ASYMMETRIC FACTORIALS

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Abstract: This paper describes the construction of search designs involving a larger numer of factors from those involving smaller numers of factors. Some search designs for symmetric factorials have been proposed. These results may be utilized for the construction of search designs for the series $s^r \times w^{m-r}$ where s and w are any positive integers.

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1. Introduction

The pioneering work in the area of search designs is due to Srivastava (1975) who introduced the notion of search designs and gave the basic mathematical formulation of the problem. Comprehensive construction procedures for the 2^n series were suggested by Srivastava (1975, 1976a), Ghosh (1975), Srivastava and Ghosh (1977) and Srivastava and Gupta (1979). Srivastava (1976b) considered optimality criteria, bias and some interesting applications of search models. Anderson and Thomas (1980) suggested construction procedures for search designs for the p^n series, where p is a prime or a prime power.

The present paper deals with the construction of search designs for general symmetric and asymmetric factorials that allow estimation of the main effects and detect at most one two-factor interaction. The results do not require that the number of levels of a factor should be a prime or a prime power. The suggested designs keep the number of runs considerably low.

The linear model considered here closely resembles that of Srivastava (1975). Let $Y^{(N\times 1)}$ be the observational vector with $E(Y)=X\tau$, $\mathrm{Disp}(Y)=\sigma^2 I$, where

$$X=[X_1^{(N\times n_1)},X_2^{(N\times n_2)}], \tag{1.1}$$

is the known design matrix and $\tau=(\tau_1',\tau_2')'$, τ_1 being an $n_1\times 1$ vector of unknown fixed parameters and τ_2 being an $n_2\times 1$ vector of fixed parameters which is partially known in the following sense. It is known that τ_2 may be partitioned as

$$\tau_2=(\beta_1',\beta_2',\dots,\beta_a')', \tag{1.2}$$

and among $\beta_1,\beta_2,\dots,\beta_a$ at most k are non-null, k being small relative to a. The problem is to search the non-null vectors among $\beta_1,\beta_2,\dots,\beta_a$ and to draw inference regarding them and τ_1. It is at this point where we deviate slightly from Srivastava (1975) who considers a set-up where each of $\beta_1,\beta_2,\dots,\beta_a$ involves a single component. This change, however, is imperative since this paper deals with general factorial designs and in such settings a factorial effect may be represented by more than one independent parameters.

Let the partitioning of X_2 corresponding to (1.2) be $X_2=[X_{21},X_{22},\dots,X_{2a}]$ and assume that the error variance is negligible (i.e. $\sigma^2=0$). Then one obtains the following theorem which is virtually another version of Theorem 1 of Srivastava (1975) and may be proved along the same line.

Theorem 1.1. *A necessary and sufficient condition that the above search design problem will be completely solved is that for arbitrary* $g_1,g_2,\dots,g_{2k}$ $(1\le g_1<g_2<\dots<g_{2k}\le a)$, *the matrix* $[X_1,X_{2g_1},X_{2g_2},\dots,X_{2g_{2k}}]$ *has full column rank.*

2. Notations and preliminaries

Consider a factorial set-up involving m factors $F_1,F_2,\dots,F_m$ at $s_1,s_2,\dots,s_m$ levels respectively $(s_j\ge 2;\ 1\le j\le m)$. Throughout this paper the $v=\prod s_j$ level combinations will be assumed to be lexicographically ordered. The main effect of factor F_j carrying s_j-1 degrees of freedom will also be denoted by F_j, while F_jF_l will represent a typical two-factor interaction with $(s_j-1)(s_l-1)$ degrees of freedom $(1\le j,\ l\le m,\ j<l)$.

Suppose prior information is available regarding the absence of all interactions involving three or more factors (these interactions will be assumed to be absent throughout). Also, as in many practical situations, there may be a natural partitioning of the set of factors into two groups, namely $F_1,\dots,F_r$ and $F_{r+1},\dots,F_m$ $(1\le r<m)$. A two-factor interaction F_jF_l will be of type (11) if $1\le j<l\le r$, of type (22) if $r+1\le j<l\le m$ and of type (12) if $1\le j\le r$, $r+1\le l\le m$. This paper develops some results on the estimation of the general effect and main effects and the detection of the possibly present two-factor interaction parameters when among the two-factor interactions at most one is present and that one is of (i) type (12), (ii) type (11), (iii) type (22), (iv) type (11) or type (22), (v) type (11) or type (12), (vi) type (22) or type (12), (vii) any one of the three types.

It may be remarked that among the seven cases mentioned above, the last one is the most general and is meaningful even when there is no grouping of factors. Still then a study of the first six cases remains important since they may be relevant in particular practical situations. For example, the m factors may represent m fertilizers of which the first r are traditional and the last $m-r$ are newly introduced ones. Now if prior knowledge is available that at most one two-factor interaction is present and further that the traditionally used fertilizers do not interact among themselves then an investigation of case (vi) becomes appropriate.

In the following, designs suitable for cases (i)–(vii) will be called search designs of types A_1–A_7 respectively. The next section describes the construction of search designs involving all the m factors by suitably combining designs involving only $F_1, \ldots, F_r$ with those involving only $F_{r+1}, \ldots, F_m$. In this connection, a subset of the level combinations of $F_1, \ldots, F_r$ alone (ignoring $F_{r+1}, \ldots, F_m$) will be called

(a) a type B_{11} design if it represents a saturated main effect plan in terms of $F_1, \ldots, F_r$,

(b) a type B_{12} (search) design if given that among the main effects $F_1, \ldots, F_r$ at most one is present it allows estimation of the general effect and detection of the possibly present main effect,

(c) a type B_{13} (search) design if given that among the two-factor interactions F_jF_l $(1 \leq j < l \leq r)$ at most one is present it allows estimation of the general effect and the main effects and detection of the possibly present two-factor interaction.

Similarly considering subsets of level combinations of $F_{r+1}, \ldots, F_m$ alone (ignoring $F_1, \ldots, F_r$) one may define a type B_{21} (saturated) design and (search) designs of types B_{22} and B_{23}. The (symbolic) direct product, $S_1 * S_2$, of a subset S_1 of the level combinations of $F_1, \ldots, F_r$ (ignoring $F_{r+1}, \ldots, F_m$) and a subset S_2 of the level combinations of $F_{r+1}, \ldots, F_m$ (ignoring $F_1, \ldots, F_r$) will be a subset of the level combinations of $F_1, \ldots, F_m$ obtained by combining each member of S_1 with each member of S_2. The next section considers the construction of search designs of types A_1–A_7 employing direct products involving designs of types B_{j1}, B_{j2}, B_{j3} $(j=1,2)$.

3. Some general results

Let S_{11}, S_{12}, S_{13} (S_{21}, S_{22}, S_{23}) be not necessarily disjoint subsets of distinct level combinations of $F_1, \ldots, F_r$ $(F_{r+1}, \ldots, F_m)$ forming designs of types B_{11}, B_{12}, B_{13} (B_{21}, B_{22}, B_{23}) respectively. For each $u, u', h_{uu'}$ is the cardinality of $S_{uu'}$, $S'_{uu'}$ is any singleton subset of $S_{uu'}$, $\bar{S}_{uu'} = S_{uu'} - S'_{uu'}$ and $\mathbf{1}$ is a vector of appropriate order with all elements unity.

Theorem 3.1. *Each of the unions* (a) $(S_{11} * S'_{21}) \cup (S_{12} * \bar{S}_{21})$, (b) $(S'_{11} * S_{21}) \cup (\bar{S}_{11} * S_{22})$ *gives a type* A_1 *search design in* $h_{11} + h_{12}(h_{21} - 1)$ *and* $h_{21} + h_{22}(h_{11} - 1)$ *runs respectively.*

Proof. Consider first (a). Let for $u=1,2$, the design matrix for S_{1u} be $[1, L_{1u}, \ldots, L_{ru}]$ where 1 corresponds to the general effect and L_{ju} involves (s_j-1) columns corresponding to parameters representing main effect F_j $(1\le j\le r)$. Similarly, let the design matrix for S_{21} be $[1, D_{r+1}, \ldots, D_m]$ with a partitioning

$$\begin{pmatrix} 1 & d'_{r+1} & \cdots & d'_m \\ 1 & D^*_{r+1} & \cdots & D^*_m \end{pmatrix}, \tag{3.1}$$

as dictated by S'_{21} and $\bar{S}_{21}$. Then under the absence of two-factor interactions of types (11) and (22), following (1.1) the design matrix of the plan (a) may be expressed as $X=[X_1, X_2]$, where

$$X_1=\begin{pmatrix} X_{11} & 1\otimes d'_{r+1} & \cdots & 1\otimes d'_m \\ X_{12} & 1\otimes D^*_{r+1} & \cdots & 1\otimes D^*_m \end{pmatrix},$$

$$X_2=[G_{1,r+1}, G_{1,r+2}, \ldots, G_{rm}], \qquad X_{11}=[1, L_{11}, \ldots, L_{r1}],$$

$$X_{12}=[1\otimes 1, L_{12}\otimes 1, \ldots, L_{r2}\otimes 1],$$

$$G_{jt}=[L'_{j1}\otimes d_t, L'_{j2}\otimes D^{*\prime}_t]' \quad (1\le j\le r,\ r+1\le t\le m),$$

and $\otimes$ denotes Kronecker product.

By Theorem 1.1, it is enough to show that for every j_1, t_1, j_2, t_2, where $1\le j_1, j_2\le r$, $r+1\le t_1, t_2\le m$ and $(j_1, t_1)\ne(j_2, t_2)$, the matrix $[X_1, G_{j_1t_1}, G_{j_2t_2}]$ has full column rank. Observe that elementary column transformations reduce this matrix to the form $[V'_1, V'_2]'$, where $V_1=[X_{11}, 0]$, $V_2=[X_{12}, H]$,

$$H=[1\otimes C_{r+1}, \ldots, 1\otimes C_m, L_{j_12}\otimes C_{t_1}, L_{j_22}\otimes C_{t_2}]$$

and $C_t=D^*_t-1d'_t$ $(r+1\le t\le m)$. Since S_{11} is a type B_{11} (saturated) design, X_{11} has full column rank and hence it remains to show that the matrix H also has full column rank. First suppose $j_1\ne j_2$. Since S_{12} is a type B_{12} design, by Theorem 1.1, the matrix $[1, L_{j_12}, L_{j_22}]$ is of full column rank. Further, as S_{21} is a type B_{21} design, the matrix (3.1) and hence the matrix

$$\begin{pmatrix} 1 & 0' & \ldots & 0' \\ 1 & C_{r+1} & \ldots & C_m \end{pmatrix},$$

which may be obtained thereof by elementary column transformations, is of full column rank. Therefore, $[C_{r+1}, \ldots, C_m]$ and consequently,

$$[1, L_{j_12}, L_{j_22}]\otimes[C_{r+1}, \ldots, C_m] \tag{3.2}$$

has full column rank. Since H may be obtained from (3.2) by deleting some of the columns of the latter, linear independence of the columns of H is immediate. The case $j_1=j_2$ $(t_1\ne t_2)$ may be treated similarly noting that $[1, L_{j_12}]$ has full column rank. This proves the result for the design (a). The proof for the design (b) is similar. □

The following results hold for search designs of types A_2–A_7. The proofs that follow along the line of proof of Theorem 3.1 are somewhat lengthy and will not be presented here.

Theorem 3.2. (a) *The union* $(S'_{13} * S_{21}) \cup (\bar{S}_{13} * S'_{21})$ *gives a type* A_2 *search design in* $h_{21} + h_{13} - 1$ *runs.*

(b) *The union* $(S_{11} * S'_{23}) \cup (S'_{11} * \bar{S}_{23})$ *gives a type* A_3 *search design in* $h_{11} + h_{23} - 1$ *runs.*

Theorem 3.3. *Each of the unions* (a) $(S_{13} * S'_{23}) \cup (S'_{13} * \bar{S}_{23})$, (b) $(S'_{13} * S_{23}) \cup (\bar{S}_{13} * S'_{23})$ *gives a type* A_4 *search design in* $h_{13} + h_{23} - 1$ *runs.*

Theorem 3.4. (a) *The union* $(S'_{13} * S_{21}) \cup (\bar{S}_{13} * S_{22})$ *gives a type* A_5 *search design in* $h_{21} + h_{22}(h_{13} - 1)$ *runs.*

(b) *The union* $(S_{11} * S'_{23}) \cup (S_{12} * \bar{S}_{23})$ *gives a type* A_6 *search design in* $h_{11} + h_{12}(h_{23} - 1)$ *runs.*

Theorem 3.5. *Each of the unions* (a) $(S_{13} * S'_{23}) \cup (S_{12} * \bar{S}_{23})$, (b) $(S'_{13} * S_{23}) \cup (\bar{S}_{13} * S_{22})$ *gives a type* A_7 *search design in* $h_{13} + h_{12}(h_{23} - 1)$ *and* $h_{23} + h_{22}(h_{13} - 1)$ *runs respectively.*

In any particular situation, among the two designs suggested in Theorems 3.1, 3.3, 3.5, the one with a smaller number of runs should be used.

4. Examples

This section explains, through a series of examples, the construction of designs of types $B_{uu'}$ ($u = 1, 2$; $u' = 1, 2, 3$). An application in the construction of a type A_7 search design has also been illustrated. The mathematical details, which are essentially based on Theorem 1.1, are not difficult to work out and hence omitted.

Example 4.1. The level combinations

$$\{(0,0,\ldots,0), (i_1,0,\ldots,0)(1 \le i_1 \le s_1 - 1), (0,i_2,\ldots,0)(1 \le i_2 \le s_2 - 1), \ldots, (0,0,\ldots,i_r)(1 \le i_r \le s_r - 1)\}$$

of $F_1, \ldots, F_r$ constitute a type B_{11} design in $1 + \sum_{j=1}^{r} (s_j - 1)$ runs. A dual example of a type B_{21} design for $F_{r+1}, \ldots, F_m$ is easy to construct.

Turning to the construction of search designs of types B_{12}, B_{13}, consider the special case when $s_1 = s_2 = \cdots = s_r = s$ (say). For $r \ge 4$ and $1 \le b, g \le s - 1$, define the circulant

$$E_{bg}^{(r\times r)}=\begin{pmatrix} 0 & b & g & \cdots & g \\ g & 0 & b & \cdots & g \\ \vdots & & & & \vdots \\ b & g & g & \cdots & 0 \end{pmatrix}.$$

Example 4.2. If s $(=2\mu+1)$ be odd and $r\geq 4$, the columns of the maxtrix $[0, E_{21}, E_{43}, \dots, E_{2\mu,2\mu-1}]$, interpreted as level combinations of $F_1, \dots, F_r$, give a type B_{12} search design in $r\mu+1$ runs.

Example 4.3. If $s(=2\mu)$ be even and $r\geq 4$, the columns of the matrix

$$[0, E_{21}, E_{43}, \dots, E_{2\mu-2,2\mu-3}, E_{2\mu-1,2\mu-2}],$$

interpreted as level combinations of $F_1, \dots, F_r$, give a type B_{12} search design in $r\mu+1$ runs. To illustrate, if in particular $s=6$, say, then this matrix becomes $[0, E_{21}, E_{43}, E_{54}]$.

Example 4.4. If $r=3$, the level combinations $\{(0,0,0),(i,0,i),(i,i,0)\ 1\leq i\leq s-1\}$ give a type B_{12} search design in $2s-1$ runs.

Example 4.5. If $r\geq 4$, the level combinations

$$\{(0,0,\dots,0),(b,g,\dots,g),(g,b,\dots,g),\dots,(g,g,\dots,b);\ 0\leq g,b\leq s-1,\ g\neq b\}$$

of $F_1, \dots, F_r$ give a type B_{13} search design in $rs(s-1)+1$ runs.

Example 4.6. If $r=3$, the level combinations

$$\{(b,g,g),(g,b,g),(g,g,b);\ 0\leq g,b\leq s-1,\ g\neq b\}$$

give a type B_{13} search design in $3s(s-1)$ runs.

The search designs presented in Examples 4.2-4.5 involve considerably small numbers of runs. The corresponding regular resolution V plans require $r(s-1)+1$, $r(s-1)+1$, $3s-2$ and $\binom{r}{2}(s-1)^2+r(s-1)+1$ runs respectively. These regular plans are, indeed, capable of estimating many more parameters than those considered in this paper. Anyway, if prior knowledge is available that at most one two-factor interaction is present then such elaborate estimation is not required and, in such situations, the designs presented in this paper appear to be more economic. In a set-up similar to that in Example 4.5, Anderson and Thomas (1980) describe search designs in at least $rs(s-1)+s$ runs. Their designs are capable of detecting even more than one two-factor interactions but require that s be a prime or a prime power. If $s_{r+1}=s_{r+2}=\cdots=s_m=w$, say, it is easy to develop duals of Examples 4.2-4.6 yielding search designs of types B_{22} and B_{23}.

The examples in this section are relevant if one deals with symmetric factorials. Further, when combined with Theorems 3.1-3.5, they yield search designs for asymmetric factorials, particularly for factorials of the type $s^r \times w^{m-r}$ where $s, w\ (\geq 2)$ are any positive integers. These search designs involve fairly small numbers of runs especially when the number of factors is large.

Example 4.7. With a $2^5 \times 3^7$ factorial, Examples 4.2, 4.3, 4.5, when combined with Theorem 3.5(a) or (b), give type A_7 search designs in 263 or 123 runs respectively. The smaller design, obtained through Theorem 3.5(b), involves 123 runs. It may be checked that the corresponding regular resolution V fraction requires as many as 184 runs.

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