PSYCHOMETRIKA—VOL. 29, NO. 3
SEPTEMBER, 1964

ANALYSIS OF PREFERENTIAL EXPERIMENTS

C. RAMANUJACHARYULU

INDIAN STATISTICAL INSTITUTE, CALCUTTA, INDIA

The problems of selecting the winner in a tournament, a leader in a society, or the most dominating or influential person in a group of individuals are not infrequent. Graph theory is successfully used in such situations in locating, by the use of associated matrices of graphs representing the individuals under the studied relation, the person with the greatest power to influence. In this paper one more important point is brought into consideration before selecting the leader or the most influencing personality—that is the consideration of weakness to be influenced by. The one with a nice blending of these two characters—possessing the highest power to influence a person and simultaneously having the least weakness to be influenced by—is to be selected. But, in practice, to locate such a man in a group is delicate. A solution is presented here by appealing to graph theoretic notions and using them.

1. *Introduction*

Suppose we have a set X of people and also pairwise comparisons of these people. We might be considering relations like dominates, influences, likes, favors, etc. In such situations one naturally is interested in knowing the most dominating or influential or liked person. It is easy to see that the problems involved in selecting such a person from the results of an experiment are delicate; for the apparently less influential person may have a larger number of supporters under his few friends, whereas the apparently most dominating person may have fewer supporters under his many friends. Such situations as locating the most powerful person in sociograms [3] are studied with the help of graphs and their associated matrices [1]. With every individual, a measure denoting the direct and indirect influence of kth order is associated, from which a limit measure over all orders is finally derived.

In this paper the problem dealt with is the problem of finding such a person (the talented) in the society who has the greatest power to influence and simultaneously has the least weakness to be influenced by (or to be led by or to be swayed away by, for example, money). In other words, in tournaments one may be interested in locating the really talented man in the sense that he has won over the largest number of opponents but simultaneously he has been defeated by only a few opponents. The solution of this problem from this balanced point of view is given here using graph theoretic tools.

A measure is obtained to indicate the depth of weakness for each individual and is used with the measure of power to influence, after standardization to rank the individuals with respect to the blending of the two dual characteristics.

2. *Problem of the Talented*

Let X represent a set of people, represented on the figure by thick points. Suppose R is a psychological or sociological aspect studied on some or all of the pairs (x, y) of the people. For simplicity, without any loss of generality let the relation R stand for "influencing." If x influences y, we shall join x to y by a line with an arrow on it directed from x to y. If y also influences x, then we shall have another arc from y to x; but if y cannot, then a second arc joins x to y. Thus, in case of tournaments, if x wins over y, we shall join x to y by two distinct arcs. If a draw between x and y results, we have two oppositely oriented arcs joining x and y. This set of arcs is denoted by U. The whole figure is a graph G: (X, U), sometimes known as a sociogram.

As an example, which we shall be using throughout, we may consider the following sociogram between five persons x_1 , x_2 , x_3 , x_4 , x_5 , where every person specifies the other persons whom he can influence. A loop is added at each vertex x_i . For simplicity the two distinct arcs are united into one preserving the arrows.

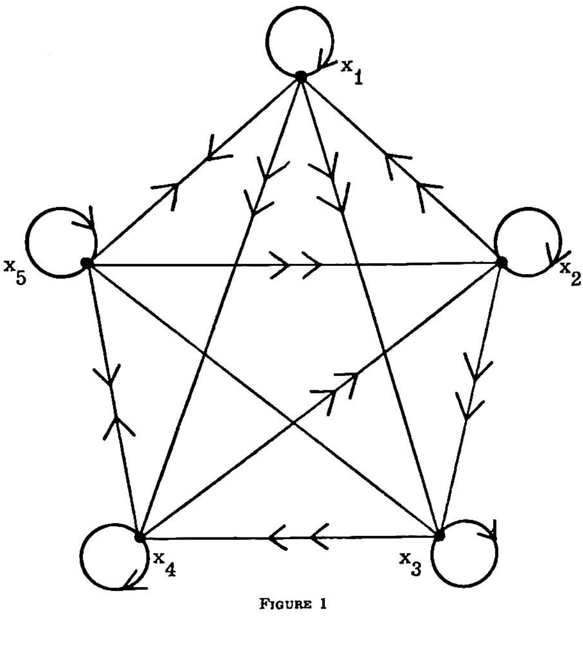

FIGURE 1

The procedure of locating "the talented person" is illustrated in this example. Before proceeding further, a few notions are introduced in the following section.

3. *Matrices Associated with Graphs*

Given a graph $G:(X, U)$, matrices which illustrate the incidence relations stated by graphs are well known [1]. A matrix $A = [a_{ij}]$, where a_{ij} denotes the number of arcs joining x_i to x_j, is known as the matrix associated with the graph. In fact, many properties of this matrix are known, the details of which one may find in [1]. The element $p_j^i(k)$ of kth power A^k of A denotes the number of paths of length k going from x_i to x_j. This is the matrix used to define the power of an individual to influence.

Now we shall use a matrix B, where the element b_{ij} denotes the number of arcs coming into i from j, to define a measure which indicates the weakness of an individual to be influenced by. Obviously B is the transpose of A.

The associated matrices A and B of the figure are given below.

$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix} \end{array} \qquad B = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 & 1 \end{bmatrix} \end{array}$$

It is obvious that the general element $q_j^i(k)$ of kth power B^k denotes the number of paths of length k that come into x_i from x_j.

4. *Iterated Power and Weakness*

The number

$$p^i(k) = p_1^i(k) + p_2^i(k) + \cdots + p_n^i(k)$$

is called the iterated power of order k of person x_i $(i = 1, 2, \cdots, n)$ to influence.

The number $p^i(1)$ denotes a measure of the number of the persons whom the individual can influence directly. For this example it is simply the sum of elements of the ith row.

$$p(1) = [p^1(1)\ p^2(1)\ p^3(1)\ p^4(1)\ p^5(1)] = [6\ 4\ 6\ 4\ 5].$$

This is the first-order iterated power vector. Similarly, for $k = 2$ we have

$$p(2) = [31\ 22\ 28\ 17\ 23],$$

and for $k = 3$,

$$p(3) = [144\ 112\ 130\ 84\ 115].$$

(The second-order iterated power vector is obtained by multiplying the first-order iterated power vector by each row vector of A, and, in general, $p^i(k)$ is obtained by taking the inner product of $p(k-1)$ with the ith row of A.) Thus at the first instance individuals x_1 and x_3 are equally influential but later on we see that x_1 has more power to influence when indirect influence over others is taken into account.

4.1 *Definition*

The number

$$q^i(k) = q_1^i(k) + q_2^i(k) + \cdots + q_n^i(k)$$

is called "the iterated weakness of order k of person x_i to be influenced by."

We have the iterated weakness vectors

$$\text{for} \quad k = 1: \quad q(1) = [4\ 6\ 4\ 6\ 5];$$

$$\text{for} \quad k = 2: \quad q(2) = [21\ 32\ 18\ 27];$$

$$\text{for} \quad k = 3: \quad q(3) = [108\ 150\ 92\ 128\ 107].$$

($q^i(k)$ is obtained by taking the inner product of $q(k-1)$ with the ith row of B.)

Thus at the first instance we observe that x_1 and x_3 are the least influenced; in other words, x_1 and x_3 have the least weakness. But when we consider measures of weakness of the persons that influence x_1 and x_3, we are led to the second-order weakness vector $q(2)$ which shows that x_3 is the least influenced. Thus, under the weakness characteristic x_3 is the best to choose, as we may observe that further iterations do not alter the rankings.

Let us consider the vectors $p(k)$ and $q(k)$. The iterated power-weakness ratio vector $r(k)$ is defined to be the vector

$$r(k) = [r^1(k)\ r^2(k)\ \cdots\ r^n(k)],$$

where $r^i(k) = p^i(k)/q^i(k)$.

Naturally the one who possesses the largest power-weakness ratio of order k is the talented person at the kth-stage considerations.

Thus we are led to define the power-weakness vector R as the limit of the vector $r(k)$ as $k \to \infty$. Then the individual with the largest power-weakness ratio may be selected as the most talented.

The limiting procedure may be done in another way also, but it results in the same vector.

The power vector

$$\pi = [\pi^1\ \pi^2\ \cdots\ \pi^n]$$

may be defined as the limit as $k \to \infty$ of

$$\pi(k) = [\pi^1(k)\ \pi^2(k)\ \cdots\ \pi^n(k)],$$

where

$$\pi^i(k) = \frac{p^i(k)}{p^1(k) + p^2(k) + \cdots + p^n(k)};$$

the weakness vector

$$\chi = [\chi^1 \chi^2 \cdots \chi^n]$$

may be defined as the limit as $k \to \infty$ of

$$\chi(k) = [\chi^1(k) \chi^2(k) \cdots \chi^n(k)],$$

where

$$\chi^i(k) = \frac{q^i(k)}{q^1(k) + q^2(k) + \cdots + q^n(k)};$$

and the power-weakness ratio vector R as the vector

$$[r^1 r^2 \cdots r^n],$$

where

$$r^i = \pi^i/\chi^i \qquad (i = 1, 2, \cdots, n).$$

But it is easy to see that this is nothing but the power-weakness vector which we have already defined.

In the example given, the limit of the power vector is observed to be

$$[0.9614 \quad 0.7741 \quad 0.9470 \quad 0.6197 \quad 0.8237]$$

and the limit of the weakness vector to be

$$[0.7491 \quad 1.0000 \quad 0.6593 \quad 0.9295 \quad 0.7881];$$

thus the power-weakness ratio vector is found to be

$$[1.2833 \quad 0.7741 \quad 1.4363 \quad 0.6667 \quad 1.0444].$$

Hence the most talented person is x_3, and the ranks of the five persons x_1, x_2, x_3, x_4, and x_5 are 2, 4, 1, 5, and 3, respectively.

The power-weakness ratio can never contain an infinite element since we have added loops to the graph at each vertex.

REFERENCES

[1] Berge, C. *Theorie des graphes et ses applications.* Paris: Dunod, 1958.

[2] Harary, F. and Norman, R. Z. *Graph theory as a mathematical model in social science.* Ann Arbor: Univ. Michigan, 1953.

[3] Moreno, J. L. *Who shall survive?* (2nd ed.) Beacon, N. Y.: Beacon House, 1953.