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THE NONCENTRAL BIVARIATE CHI DISTRIBUTION*

MARAKATHA KRISHNAN†

1. Introduction. The generalized Rayleigh or noncentral χ-distribution occurs in problems on radar and noise [13], [6]. Its moments are given by Park [12]. The two-dimensional generalized Rayleigh distribution has known probability density (see Miller [8]–[10]). Krishnaiah, Hagis and Steinberg [4] consider properties of the unbiased or central bivariate χ-distribution. This paper extends the results to the noncentral case. While [4] and [9] treat correlated variables with zero means or uncorrelated variables with nonzero means, this paper attempts a treatment of correlated variables with nonzero means. Here, the moments, conditional distribution of the noncentral bivariate χ and the distributions of the sum and ratio of two correlated noncentral χ-variates are derived. Some applications are cited.

2. The noncentral bivariate χ-distribution. Let $X_j = (X_{1j}, X_{2j})$, $j = 1, 2, \cdots, n$, be a set of n independent random vectors, each of them being distributed as a bivariate normal with a common covariance matrix Σ and means given by $E(X_j) = \mathbf{\mu}_j = (\mu_{1j}, \mu_{2j})$, where

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

is positive definite.

Let

$$U = \left[\sum_{j=1}^{n} \frac{X_{1j}^2}{\sigma_1^2}\right]^{1/2}, \quad V = \left[\sum_{j=1}^{n} \frac{X_{2j}^2}{\sigma_2^2}\right]^{1/2}. \tag{2.1}$$

Then the joint distribution of U and V is the central or noncentral bivariate χ-distribution with n degrees of freedom according as $\mu_{ij} = 0$ or $\mu_{ij} \neq 0$ for at least one (i, j), $i = 1, 2$ and $j = 1, 2, \cdots, n$.

In the sequel, we assume that $\sigma_1 = \sigma_2 = \sigma$ and $\mu_{1j} = \mu_{2j} = \mu_j$, $j = 1, \cdots, n$. Now, making the transformations

$$U = \frac{r}{\sigma}, \quad V = \frac{s}{\sigma}, \quad \lambda = \sum_{j=1}^{n} \frac{\mu_j^2}{\sigma^2} = \left(\frac{a}{\sigma}\right)^2, \quad \Sigma = M_2 \tag{2.2}$$

in Miller's result [8, p. 32], the joint distribution of U and V has density function

$$\begin{aligned} f(u, v) = Cuv \exp\left[-\frac{u^2 + v^2}{2(1 - \rho^2)}\right] \\ \cdot \sum_{k=0}^{\infty} DI_\nu\left(\frac{\rho uv}{1 - \rho^2}\right) I_\nu\left(\frac{u\sqrt{\lambda}}{1 + \rho}\right) I_\nu\left(\frac{v\sqrt{\lambda}}{1 + \rho}\right), \end{aligned} \tag{2.3}$$

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† Indian Statistical Institute, Madras Center, India, and University of Ottawa, Ottawa, Quebec.

where

$$C = \exp\left(\frac{-\lambda}{1+\rho}\right)\frac{[2(1+\rho)^2/\lambda\rho]^{n/2-1}}{1-\rho^2},$$

$$D = \Gamma\left(\frac{n}{2}-1\right)\left(\frac{n}{2}+k-1\right)\binom{n+k-3}{n-3}, \tag{2.4}$$

$$\nu = \frac{n}{2}+k-1,$$

and the modified Bessel function of the first kind and order ν in the variable z (see [1, p. 5]) is given by:

$$I_\nu(z) = \exp\left(\frac{-i\nu\pi}{2}\right)J_\nu(zi) = \sum_{m=0}^{\infty}\frac{(z/2)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)}, \tag{2.5}$$

where $J_\nu(z)$ is the Bessel function of the first kind and order ν in the variable z. We note that D in (2.4) is defined for $n = 1, 2$ with the convention that the binomial coefficients $\binom{n-3}{n-3} = 1$, $\binom{n-2}{n-3} = n-2$, even when $n = 1, 2$. So $D = \Gamma(n/2)$ when $k = 0$, and $D = n\Gamma(n/2)$ when $k = 1$.

When $n = 1$, (2.3) can be further simplified to yield

$$\begin{aligned} f(u, v) = {} & \exp\left[-\frac{u^2+v^2}{2(1-\rho^2)} - \frac{\lambda}{1+\rho}\right] \\ & \cdot\left[\exp\left(\frac{\rho uv}{1-\rho^2}\right)\cosh\left(\frac{(u+v)\sqrt{\lambda}}{1+\rho}\right)\right. \\ & \left.+ \exp\left(\frac{-\rho uv}{1-\rho^2}\right)\cosh\left(\frac{(u-v)\sqrt{\lambda}}{1+\rho}\right)\right]\Big/[\pi\sqrt{1-\rho^2}]. \end{aligned} \tag{2.6}$$

The above noncentral bivariate χ density function for 1 degree of freedom can be verified to be the sum of bivariate normal densities:

$$f(x_1, x_2) + f(-x_1, x_2) + f(x_1, -x_2) + f(-x_1, -x_2),$$

where (x_1, x_2) follow a bivariate normal with means $(\sqrt{\lambda}, \sqrt{\lambda})$ and covariance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

The density for the central bivariate χ-distribution ($\lambda = 0$) has been obtained in [4, (2.1)].

3. Moments of the noncentral bivariate χ-distribution.

$$E(U^r V^s) = \int_0^\infty\int_0^\infty f(u, v)u^r v^s\,du\,dv$$

exists when r, s are real numbers exceeding $-n$.

Expanding $I_\nu(\rho uv/(1-\rho^2))$ in (2.3) in an infinite series, and integrating term by term, we have

$$E(U^rV^s) = C\sum_{k=0}^{\infty} D \sum_{m=0}^{\infty}\left(\frac{\rho}{2(1-\rho^2)}\right)^{\nu+2m}$$
$$\cdot\left[\int_0^{\infty}\exp\left(-\frac{u^2}{2(1-\rho^2)}\right) I_\nu\left(\frac{u\sqrt{\lambda}}{1+\rho}\right)\cdot u^{\nu+2m+r+1}\,du\right]$$
$$\cdot\left[\int_0^{\infty}\exp\left(-\frac{v^2}{2(1-\rho^2)}\right) I_\nu\left(\frac{v\sqrt{\lambda}}{1+\rho}\right)\cdot v^{\nu+2m+s+1}\,dv\right]\Big/ [m!\,\Gamma(\nu+m+1)].$$

Now (see [1, p. 50]),

$$\Gamma(\nu+1)\int_0^{\infty} J_\nu(at)\cdot\exp(-b^2t^2)t^{c-1}\,dt$$
$$(3.1)\qquad = \frac{1}{2}b^{-c}\Gamma\left(\frac{\nu+c}{2}\right)\left(\frac{a}{2b}\right)^{\nu}\cdot {}_1F_1\left(\frac{\nu+c}{2},\nu+1;-\frac{a^2}{4b^2}\right),$$
$$\operatorname{Re}(\nu+c)>0,\quad \operatorname{Re} b^2>0,$$

where

$${}_1F_1(a,b;x) = \frac{\Gamma(b)}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{\Gamma(a+r)x^r}{\Gamma(b+r)r!}.$$

Transforming the Bessel I's to the Bessel J's, evaluating the integrals for the new arguments and simplifying, we have

$$E(U^rV^s) = (1-\rho^2)^{(n+r+s)/2}\cdot 2^{(r+s)/2}\cdot\exp\left(-\frac{\lambda}{1+\rho}\right)$$
$$\cdot\sum_{k=0}^{\infty} D\,\frac{[\lambda\rho(1-\rho)/(2(1+\rho))]^k}{\Gamma^2(n/2+k)}$$
$$(3.2)\qquad \cdot\sum_{m=0}^{\infty}\frac{\rho^{2m}\Gamma(n/2+r/2+k+m)\cdot\Gamma(n/2+s/2+k+m)}{\Gamma(n/2+k+m)\cdot m!}$$
$$\cdot\,{}_1F_1\left(\frac{n+r}{2}+k+m,\frac{n}{2}+k;\frac{\lambda(1-\rho)}{2(1+\rho)}\right)$$
$$\cdot\,{}_1F_1\left(\frac{n+s}{2}+k+m,\frac{n}{2}+k;\frac{\lambda(1-\rho)}{2(1+\rho)}\right).$$

By summation on m after expanding the two confluent hypergeometric functions, we get an alternative form for the terms after the summation sign:

$$\sum_{k=0}^{\infty} D\,\frac{\rho^k}{\Gamma(n/2+k)}$$
$$(3.3)\qquad \cdot\sum_{i,j=0}^{\infty}\Gamma\left(\frac{n+r}{2}+k+i\right)\Gamma\left(\frac{n+s}{2}+k+j\right)$$
$$\cdot\frac{{}_2F_1\left(\frac{n+r}{2}+k+i,\frac{n+s}{2}+k+j;\frac{n}{2}+k;\rho^2\right)\cdot\left\{\frac{\lambda(1-\rho)}{2(1+\rho)}\right\}^{i+j+k}}{\Gamma(n/2+k+i)\Gamma(n/2+k+j)i!\,j!},$$

where

$$ {}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\Gamma(b+r)x^r}{\Gamma(c+r)r!}. $$

4. Conditional distribution of U given V: moments. The conditional distribution of U given V has frequency function

$$ f(u \mid v) = \frac{f(u, v)}{\int_0^{\infty} f(u, v)\, du}. $$

Since

$$ \int_0^{\infty} f(u, v)\, du = f(v) = \exp\left(-\frac{v^2}{2} - \frac{\lambda}{2}\right) \sqrt{\lambda} \left(\frac{v}{\sqrt{\lambda}}\right)^{n/2} I_{n/2-1}(v\sqrt{\lambda}) $$

(see [8, p. 28]),

$$ \begin{aligned} f(u \mid v) = \exp&\left[\frac{-\lambda(1-\rho)}{2(1+\rho)} - \frac{(u^2 + v^2\rho^2)}{2(1-\rho^2)}\right]\left[\frac{2(1+\rho)^2}{v\rho\sqrt{\lambda}}\right]^{n/2-1} \\ &\cdot \frac{u}{(1-\rho^2)I_{n/2-1}(v\sqrt{\lambda})} \sum_{k=0}^{\infty} D \cdot I_\nu\left(\frac{\rho u v}{1-\rho^2}\right) \cdot I_\nu\left(\frac{u\sqrt{\lambda}}{1+\rho}\right) \cdot I_\nu\left(\frac{v\sqrt{\lambda}}{1+\rho}\right). \end{aligned} \tag{4.1} $$

The rth moment $E(U^r \mid V) = \int_0^{\infty} u^r f(u \mid v)\, du$. Expanding $I_\nu(\rho uv/(1-\rho^2))$ in powers of u and applying the integral formula (3.1), we obtain

$$ \begin{aligned} E(U^r \mid V) = \exp&\left[\frac{-\lambda(1-\rho)}{2(1+\rho)} - \frac{v^2\rho^2}{2(1-\rho^2)}\right] \frac{(1+\rho)^{n/2-1}(2(1-\rho^2))^{r/2}}{I_{n/2-1}(v\sqrt{\lambda})} \\ &\cdot \sum_{k=0}^{\infty} D \cdot I_\nu\left(\frac{v\sqrt{\lambda}}{1+\rho}\right) \frac{[v\rho\sqrt{\lambda}/(2(1+\rho))]^k}{\Gamma(\nu+1)} \\ &\cdot \sum_{m=0}^{\infty} \frac{[\rho^2 v^2/(2(1-\rho^2))]^m \Gamma(\nu+m+1+r/2)}{\Gamma(\nu+m+1)m!} \\ &\cdot {}_1F_1\left(\nu+m+1+\frac{r}{2}, \nu+1; \frac{\lambda(1-\rho)}{2(1+\rho)}\right). \end{aligned} \tag{4.2} $$

Since the pth moment of a χ'-variate with f degrees of freedom and noncentrality parameter L is

$$ \mu_p'(\chi'_{f,L}) = 2^{p/2}\,\Gamma\left(\frac{f}{2} + \frac{p}{2}\right) \exp\left(-\frac{L}{2}\right) {}_1F_1\left(\frac{f}{2} + \frac{p}{2}, \frac{f}{2}; \frac{L}{2}\right) \Big/ \Gamma\left(\frac{f}{2}\right) $$

(see [12]), the term $\exp[-\lambda(1-\rho)/(2(1+\rho))]/\Gamma(\nu+1)$ taken with the summation on m in (4.2) can be expressed as

$$ \sum_{m=0}^{\infty} \left[\frac{\rho v}{2\sqrt{1-\rho^2}}\right]^{2m} \mu'_{2m+r}(\chi'_{2\nu+2,\lambda(1-\rho)/(1+\rho)}). \tag{4.3} $$

When $\rho = 0$, which implies $k = 0$, $m = 0$ in (4.2) or (4.3), $E(U^r \mid V) = \mu_r'(\chi'_{n,\lambda})$. This is because U is then independent of V.

5. Distribution of the sum of two correlated χ'-variates. If we let $W = U + V$ and $V = V$ in (2.3), we get the joint frequency function for W, V:

$$f(w, v) = (w - v)vC \exp\left[-\frac{(w-v)^2 + v^2}{2(1-\rho^2)}\right] \cdot \sum_{k=0}^{\infty} D \cdot I_\nu\left(\frac{\rho v(w-v)}{1-\rho^2}\right) I_\nu\left(\frac{\sqrt{\lambda}(w-v)}{1+\rho}\right) I_\nu\left(\frac{\sqrt{\lambda}v}{1+\rho}\right).$$

Expanding the I functions and integrating out v, we get the frequency function for W:

$$f(W) = \exp\left[-\frac{w^2}{4(1-\rho^2)} - \frac{\lambda}{1+\rho}\right] \cdot \sum_{k,m,i,j=0}^{\infty} D \cdot \frac{\rho^{k+2m}(1-\rho^2)^{j-m}(\lambda/(1+\rho)^2)^{k+i+j} w^{2\nu+2m+2i+1} 2^{-(3\nu+3m+3j+2i+k+1)}}{\Gamma(\nu+m+1)\,\Gamma(\nu+i+1)\Gamma(\nu+j+1)m!i!j!}$$

$$(5.1) \qquad \cdot \sum_{r=0}^{2\nu+2m+2i+1} \left(\frac{\sqrt{1-\rho^2}}{w\sqrt{2}}\right)^r \binom{2\nu+2m+2i+1}{r} \cdot (2\nu+2m+2j+r+1)!\,Hh_{2\nu+2m+2j+r+1}\left(-\frac{w}{\sqrt{2(1-\rho^2)}}\right),$$

where $Hh_f(z) = \int_0^\infty \exp(-(x+z)^2/2)x^f/f!\,dx$ (see [2]).

If $\rho = 0$ in (5.1), that is, $k = 0$, $m = 0$, then

$$(5.2) \qquad f(w) = \exp\left(-\frac{w^2}{4} - \lambda\right) \sum_{i,j=0}^{\infty} \frac{\lambda^{i+j} w^{2i-1} 2^{-(2i+3j+3n/2-2)}}{\Gamma(n/2+i)\Gamma(n/2+j)i!\,j!} \cdot \sum_{r=0}^{n+2i-1} \left(-\frac{1}{w\sqrt{2}}\right)^r \binom{n+2-1}{r} (n+2j+r-1)!\,Hh_{n+2j+r-1}\left(-\frac{w}{\sqrt{2}}\right)$$

is the frequency function of the sum of two independent noncentral χ-variates.

6. Distribution of the ratio of two correlated χ'-variates: moments. Let $Z = U/V$. Then the density of Z is

$$f(z) = \int_0^\infty f(vz, v)v\,dv = C\sum_{k=0}^{\infty} D \int_0^\infty v^3 z \exp\left(-\frac{v^2(1+z^2)}{2(1-\rho^2)}\right) I_\nu\left(\frac{\rho v^2 z}{1-\rho^2}\right) I_\nu\left(\frac{vz\sqrt{\lambda}}{1+\rho}\right) \cdot I_\nu\left(\frac{v\sqrt{\lambda}}{1+\rho}\right) dv$$

and can be simplified to give

$$(6.1) \qquad f(z) = 2(1-\rho^2)^{n/2} \exp\left(-\frac{\lambda}{1+\rho}\right) \sum_{k=0}^{\infty} D \sum_{m,i,j=0}^{\infty} \left[\frac{\lambda(1-\rho)}{2(1+\rho)}\right]^{k+i+j} \rho^{k+2m} \cdot \frac{\Gamma(n+2k+2m+i+j)z^{n+2k+2m+2i-1}}{(1+z^2)^{n+2k+2m+i+j}\,\Gamma(n/2+k+m)\,\Gamma(n/2+k+i)\,\Gamma(n/2+k+j)\,m!\,i!\,j!}.$$

The rth moment of this distribution, $E(z^r)$, could be obtained from results in (3.2), (3.3) for $E(U^r V^s)$ with $r, s > -n$ and the substitution $s = -r$. We note that the rth moment for Z exists only for $r < n$.

7. Particular cases and related applications. We could obtain results for the central or unbiased case when $\mu_{ij} = 0$, $i = 1, 2$, $j = 1, \cdots, n$, by substituting $\lambda = 0$, and for the uncorrelated case, with $\rho = 0$. These have been considered in [4], [9].

Let

$$S_i = \sum_{i=1}^{n_i+1} (X_{ij} - \bar{X}_i)^2, \qquad \lambda_i = \sum_{i=1}^{n_i+1} \frac{(\mu_{ij} - \bar{\mu}_i)^2}{\sigma_i^2}, \qquad i = 1, 2,$$

where X_{ij} is a normal variate with $E(X_{ij}) = \mu_{ij}$, $\operatorname{var}(X_{ij}) = \sigma_i^2$, $E(X_{1j} - \mu_{1j}) \cdot (X_{2j} - \mu_{2j}) = \rho\sigma_1\sigma_2$. Then the statistic $U = S_1(n_2)/S_2(n_1)$ is used for the test of $(\sigma_1 = \sigma_2)$, assuming that $\rho = 0$ and the means are homogeneous: $\mu_{ij} = \mu$, or $\lambda_1 = \lambda_2 = 0$. U is distributed as an F with n_1, n_2 d.f. under the null hypothesis. The same test could also be used for testing the homogeneity of means $(\lambda_1 = \lambda_2 = 0)$, assuming $\sigma_1 = \sigma_2$ and $\rho = 0$. When $\rho = 0$ but the means are heterogeneous, U is distributed as F_1', a doubly noncentral F with n_1, n_2 d.f. and noncentrality parameters λ_1, λ_2 when $\sigma_1 = \sigma_2$. The distribution of F_1' and approximations to it have been considered in [5]. When further, the assumption of $\rho = 0$ is not justified, U is distributed as Z^2 when $\sigma_1 = \sigma_2$, provided $\mu_{1j} = \mu_{2j}$ or $\lambda_1 = \lambda_2$, and $n_1 = n_2$. The density for Z^2 could be obtained from that of Z in (6.1).

$\Pr(Z^2 \subset A)$ could also be thought of as the power of the test of $H_0(\sigma_1 = \sigma_2, \lambda_1 = \lambda_2 = 0, \rho = 0)$ against the wider class of alternatives $H(\sigma_1 = \sigma_2, \lambda_1 = \lambda_2 = \lambda, \rho \neq 0)$, the size of the critical region being identical with $\Pr(F \subset A)$.

From existing tables of the bivariate normal distribution [11], the corresponding bivariate χ-probabilities with 1 degree of freedom, could be obtained (see (2.6)). The distribution of the ratio of two correlated normal variables occurs in regression theory [7]. The corresponding distribution using the modulus of the variates is given by that of Z in (6.1) with $n = 1$. The bivariate χ-distribution is also used in certain multiple comparison test procedures [3].

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