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THE MATRIX EQUATION $AXB+CXD=E$*

SUJIT KUMAR MITRA†

Abstract. Canonical representation of a singular pencil given in Gantmacher [1] and a theorem of Kučera [3] for a special case are used to solve the matrix equation $AXB+CXD=E$.

1. Introduction. Let $\mathscr{C}^{m\times n}$ denote the vector space of complex matrices of order $m\times n$. For given matrices $A, C\in\mathscr{C}^{p\times m}$, $B, D\in\mathscr{C}^{n\times q}$ and $E\in\mathscr{C}^{p\times q}$ we consider the problem of determining the solutions $X\in\mathscr{C}^{m\times n}$ of the matrix equation

$$AXB+CXD=E. \tag{1}$$

Equations of this type occur in the MINQUE theory of estimating covariance components in a covariance components model (Rao [7]) and are likely to occur elsewhere. A special case of this equation where B and C are identity matrices has been extensively studied in literature. See for example the books by Gantmacher [2], Pease [6] and Lancaster [4]. A comprehensive review of work done in this area appears in Lancaster [5]. A more recent account is given in Kučera [3]. Our approach follows that of Kučera.

A basic method is to express (1) in an equivalent vector form as follows. The column string of X(cs X) is the column vector obtained by writing the columns of X one below the other in the natural order. Let Γ denote the matrix of order $pq\times mn$

$$\Gamma=B'\otimes A+D'\otimes C$$

where $\otimes$ denotes Kronecker product, and prime on a matrix represents its transpose, then (1) is equivalent to the equation

$$\Gamma \operatorname{cs} X=\operatorname{cs} E \tag{1'}$$

the consistency of which can be examined by standard methods and a general solution obtained in terms of a generalized inverse Γ^- of Γ (Rao and Mitra [8]). It is of interest to examine if (1) could be solved by a method which does not explicitly require generalized inversion of Γ, usually a matrix of large order.

2. Regular and singular pencils. Consider the pencil $A+\lambda C$ determined by a pair of matrices A and C of the same order. The pencil is regular if A and C are square and the determinant $|A+\lambda C|$ is not identically equal to zero. The pencil is called singular otherwise.

Let F_r and T_r denote respectively the matrices obtained by deleting the first row and the last row of an identity matrix of order $(r+1)$. Write

$$L_r(\lambda)=F_r+\lambda T_r.$$

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† Indian Statistical Institute, New Dehli 110029, India.

THEOREM 2.1 (Gantmacher [1]). *Given a singular pencil* $A+\lambda C$ *there exists a pair of nonsingular matrices* P *and* Q *such that*

$$P(A+\lambda C)Q=\operatorname{diag}(0, L_{r_1}(\lambda), \cdots, L_{r_a}(\lambda), L'_{s_1}(\lambda), \cdots, L'_{s_c}(\lambda), A_0+\lambda C_0), \tag{2}$$

where 0 *is the null matrix of order* $p_1\times m_1$ *and* $A_0+\lambda C_0$ *is a regular pencil of order* $v\times v$. *If* A *and* C *are of order* $p\times m$.

$$p=p_1+\sum r_i+\sum s_j+c+v, \qquad m=m_1+\sum r_i+\sum s_j+a+v.$$

3. The equation $AXB+CXD=E$.

3.1. The case where the pencils $A+\lambda C$ and $B+\lambda D$ regular. Since $|A+\lambda C|$ and $|B+\lambda D|$ are not identically equal to zero, $|A+\lambda C|$ being a polynomial in λ of degree $\leqq m$ can vanish at most at m distinct values of λ and $|B+\lambda D|$ similarly can vanish at most at n distinct values of λ. Let e be a scalar such that

$$|-eA+C|\neq 0, \qquad |B+eD|\neq 0.$$

Observe that

$$\begin{aligned} &AXB+CXD=E \\ &\Leftrightarrow AX(B+eD)+(C-eA)XD=E \\ &\Leftrightarrow (C-eA)^{-1}AX+XD(B+eD)^{-1}=(C-eA)^{-1}E(B+eD)^{-1}. \end{aligned} \tag{3}$$

The last equation is of the same type as considered by Kučera [3] and others and can be solved accordingly.

3.2. The general case. In the general case let matrices P, Q, R, S be determined as in Theorem 2.1 so that

$$PAQ=\operatorname{diag}(0, F_{r_1}, \cdots, F_{r_a}, F'_{s_1}, \cdots, F'_{s_c}, A_0)=\tilde{A},$$
$$PCQ=\operatorname{diag}(0, T_{r_1}, \cdots, T_{r_a}, T'_{s_1}, \cdots, T'_{s_c}, C_0)=\tilde{C},$$

where 0 is a null matrix of order $p_1\times m_1$, A_0 and C_0 are or order $v\times v$ each,

$$RBS=\operatorname{diag}(0, F_{t_1}, \cdots, F_{t_b}, F'_{u_1}, \cdots, F'_{u_d}, B_0)=\tilde{B},$$
$$RDS=\operatorname{diag}(0, T_{t_1}, \cdots, T_{t_b}, T'_{u_1}, \cdots, T'_{u_d}, D_0)=\tilde{D},$$

where 0 is the null matrix of order $n_1\times q_1$, B_0 and D_0 are of order $w\times w$ each. Put $PES=F$, $X=QYR$. Then (1) is equivalent to the equation

$$\tilde{A}Y\tilde{B}+\tilde{C}Y\tilde{D}=F. \tag{4}$$

Since the coefficient matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are block diagonal, (4) can be solved by solving equations of smaller order in which the diagonal blocks appear as coefficient matrices. Most of these subequations, each one of which fixes separate portions of the Y matrix, involve matrices of the type F_r, T_r or their transposes as coefficient matrices. These equations can be algebraically solved without much difficulty. Of the remaining equations there is exactly one which involves the matrices A_0, C_0, B_0, D_0 of the regular pencils $A_0+\lambda C$, $B_0+\lambda D_0$. This again can be solved as in § 3.1. The rest involves both matrices of the type F_n, T_r or their transposes and the matrices of one or the other of the two regular pencils. These equations can first be simplified taking advantage of the invertibility of A_0+eC_0 or of B_0+eD_0 for some scalar e, on the same lines as indicated in § 3.1 and then algebraically solved after simplification.

4. An open problem. The method discussed here does not extend itself to the situation where the left-hand side of equation (1) has one or more additional terms of the same structure as AXB and CXD. Solution of such an equation remains an unsolved problem.

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