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ON UNIQUENESS OF BAYESIAN THREE-DECISION PLANS BY ATTRIBUTES

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SUMMARY. For Bayesian three-decision ASR plans by attributes uniqueness of optimal solution has been established by using a two-point prior distribution for incoming lot quality and assuming that the expected decision loss is a monotonically decreasing function of acceptance decision number with falling rate of decrease and the point of intersection of regret functions is an increasing function of acceptance decision number. Both of these assumptions are posed as open conjectures. It is pointed out that numerical results support the truth of both the conjectures.

1. Introduction

A wide variety of Bayesian three-decision plans were developed in Pandey (1984). The numerical computations of the optimal plans yielded unique solution in each of these cases. These unique Bayesian plans are tabulated in the above work for the cases : (*i*) two-point prior restricted and unrestricted Bayes solution (*ii*) three-point prior unrestricted Bayes solution and (*iii*) beta prior unrestricted Bayes solution. An attempt is made to establish, analytically, the uniqueness of the optimal solution.

In this paper we consider the case of two-point prior restricted Bayes solution for three-decision ASR (accept-screen-reject) plan to show the uniqueness of the optimal solution. A complete rigorous proof for uniqueness is provided under two assumptions— (1) expected decision loss is monotonically decreasing function of acceptance decision number with falling rate of decrease and (2) point of intersection of regret functions is increasing function of acceptance decision number. Although, both the assumptions have been found to be true in practice on the basis of numerical results, it has not been possible to establish their truth analytically. In view of this they are posed as open conjectures.

To facilitate discussion on the uniqueness, necessary background theoretical details are also given in Sections 2 and 3. The uniqueness of Bayesian solution for other three-decision plans are attempted on similar lines and are omitted.

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Key words and phrases : Bayesian three-decision restricted plan, expected decision loss, uniqueness of Bayes solution, acceptance decision number.

2. Three-decision restricted Bayesian ASR plan

Assume that the incoming lot quality p follows a prior distribution with density $w(p)$. For the triplet (n, c_1, c_2) defining a three-decision plan the three-decision corresponds to the values of the decision variable x as follows :

Decision	Value of x
1	$0 \leqslant x \leqslant c_1$
2	$c_1 < x \leqslant c_2$
3	$c_2 < x \leqslant n$

... (2.1)

If the three terminal decisions 1, 2 and 3 are acceptance, screening or rejection of the lot respectively, we call the plan as three-decision ASR plan.

Let $k_a(p)$, $k_t(p)$ and $k_r(p)$ be the cost associated with the decision 1, 2 and 3 respectively and $k_s(p)$ be the cost of inspection when p is the incoming lot quality. It is assumed that for lots free from defectives the costs $k_a(p)$, $k_t(p)$ and $k_r(p)$ are in increasing order whereas for lots with 100% defectives they are in decreasing order. Also, the cost of inspection is assumed to be more than the minimum unavoidable decision cost $k_m(p)$. We shall write

$$k = \int_0^1 k\,(p)\, dW(p). \qquad \text{... (2.2)}$$

Assuming the simplest form of the prior distribution $W(p)$ for p as two point prior with values p' and p'' with relative frequencies w_1 and w_2 respectively $w_1+w_2 = 1$, the regret or loss function corresponding to the three-decision is given by

$$R(N, n, c_1, c_2) = \begin{cases} N, & 0 \leqslant N \leqslant n \\ n+(N-n)\, G(n, c_1, c_2), & n < N \end{cases} \qquad \text{... (2.3)}$$

where $G(n, c_1, c_2)$ denotes the expected decision loss and is a complicated function given by (2.4).

$$\begin{aligned} G(n, c_1, c_2) = {} & \lambda_{11}\,[1-B(c_1\,;\, n(c_1), p')] \\ & +\lambda_{12}\, B(c_1\,;\, n(c_1), p'') \\ & +\lambda_{21}\,[1-B(c_2(c_1)\,;\, n(c_1), p')] \\ & +\lambda_{22}\, B(c_2(c_1)\,;\, n(c_1), p'') \end{aligned} \qquad \text{... (2.4)}$$

where λ_{ij}, $i, j = 1, 2$ are constants defined in Pandey (1984) as follows :

$$\begin{aligned}\lambda_{11} &= w_1\,[k_t(p')-k_a(p')]/(k_s-k_m)\\ \lambda_{12} &= w_2\,[k_a(p'')-k_t(p'')]/(k_s-k_m)\\ \lambda_{21} &= w_1\,[k_r(p')-k_t(p')]/(k_s-k_m)\\ \lambda_{22} &= w_2\,[k_t(p'')-k_r(p'')]/(k_s-k_m)\end{aligned} \quad \ldots \quad (2.5)$$

$$k_m = \int_0^{p_u} k_a(p)dW(p) + \int_{p_u}^{p_v} k_t(p)dW(p) + \int_{p_v}^{1} k_r(p)dW(p) \quad \ldots \quad (2.6)$$

It is desired that the Bayesian three-decision plan should satisfy certain restrictions on the probability of misclassification. Let p_1 and p_2, $p_1 < p_2$ denote the levels of incoming lot quality such that a lot of quality $p_1(p_2)$ if correctly classified should be screened (rejected). Let $\beta_1(\beta_2)$ be the probabilities of misclassification of a lot of quality $p_1(p_2)$ resulting in acceptance (acceptance or screening) of the lot under the plan (n, c_1, c_2) i.e.

$$B(c_1\,;\,n, p_1) = \beta_1 \quad \ldots \quad (2.7)$$

and

$$B(c_2\,;\,n, p_2) = \beta_2 \quad \ldots \quad (2.8)$$

where $0 \leqslant \beta_1, \beta_2 \leqslant 1$ and (2.7) and (2.8) are satisfied as closely as possible treating n, c_1 and c_2 as integers.

Let S denotes the set of plans satisfying (2.7) and (2.8) for given values of p_1, p_2, β_1 and β_2. For a plan in S if any one of the triplet (n, c_1, c_2) is fixed the remaining two parameters can be uniquely obtained. In view of this, a plan in S can be indexed according to acceptance decision number c_1 alone and denoted as $S(c_1)$ and the corresponding regret for lot of N as $R(N, c_1)$. Thus,

$$S = \{(n, c_1, c_2) : B(c_1\,;\,n, p_1) = \beta_1,\ B(c_2\,;\,n, p_2) = \beta_2\}. \quad \ldots \quad (2.9)$$

A restricted Bayesian three-decision (RBT) plan (n^0, c_1^0, c_2^0) can be defined as

$$S(c_1^0) = \{S(c_1) : R(N, c_1^0) = \inf_{S(c_1)\in S} R(N, c_1)\} \quad \ldots \quad (2.10)$$

3. Determination of RBT plan

For a fixed N, the value of c_1 minimising $R(N, c_1)$ is determined from the inequality

$$\Delta R(N, c_1-1) \leqslant 0 < \Delta R\,(N, c_1) \quad \ldots \quad (3.1)$$

To obtain the bounds for the lot size for which the plan (n, c_1, c_2) satisfying (3.1) is the optimal plan we shall define N_{c_1} from $\Delta R(N, c_1)$ as follows :

$$\begin{aligned}N_{c_1} &= n(c_1)+[m-\mu_{11}-\mu_{21}+\mu_{11}\,B(c_1+1\,;\,n(c_1+1), p')\\ &\quad -\mu_{12}\,B(c_1+1\,;\,n(c_1+1), p'')+\mu_{21}\,B(c_2(c_1+1)\,;\,n(c_1+1), p')\\ &\quad -B(c_2(c_1+1)\,;\,n(c_1+1), p'')]\,\Delta\,n(c_1)/U(c_1)\end{aligned} \quad \ldots \quad (3.2)$$

where $m = 1/\lambda_{22}$, $\mu_{ij} = m\lambda_{ij'}/\lambda_{ij} > 0$, $i, j = 1, 2$ and

$$U(c_1) = [\mu_{11}\,\Delta\, B(c_1\,;\, n(c_1), p') - \mu_{12}\,\Delta\, B(c_1\,;\, n(c_1), p'')$$
$$+\mu_{21}\,\Delta\, B(c_2(c_1)\,;\, n(c_1), p') - \mu_{22}\,\Delta\, B(c_2(c_1)\,;\, n(c_1), p'')]. \quad \ldots \quad (3.3)$$

Clearly,

$$\Delta\, R(N, c_1) = U(c_1)\,(N_{c_1} - N)/m$$

and

$$\Delta\, R(N, c_1 - 1) = U(c_1 - 1)\,(N_{c_1 - 1} - N)/m, \qquad m > 0.$$

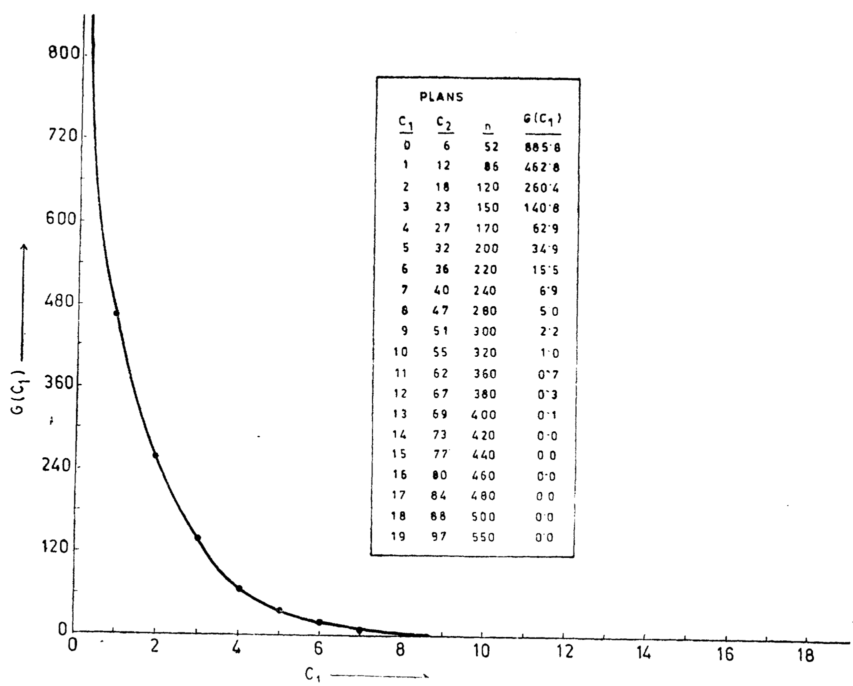
The function $U(c_1)$ is related to $G(c_1) = G(n, c_1, c_2)$ defined by (2.4) and is used subsequently as $U(c_1)/m = -\Delta\, G(c_1)$.

Although, it has not been possible to study the monotonocity of $G(c_1)$ analytically, extensive computations show that it is a monotonically decreasing function of c_1 with falling rate of decrease i.e., $\Delta\, G(c_1) < 0$ and $\Delta^2\, G(c_1) > 0$ as in Fig. 1 and, hence, that $U(c_1) > 0$ for all values of c_1. Further, numerical results show that $G(c_1) < 1$ for all the values of c_1.

Therefore, it follows from (3.1) that the plan (n, c_1, c_2) is optimal for the lot size N if

$$N_{c_1 - 1} \leqslant N < N_{c_1} \quad \ldots \quad (3.4)$$

For fixed c_1 and λ_{ij}'s note that $R(N, c_1)$ as defined in (2.3) is always an increasing linear function of N.



C_1	C_2	n	$G(C_1)$
0	6	52	885·8
1	12	86	462·8
2	18	120	260·4
3	23	150	140·8
4	27	170	62·9
5	32	200	34·9
6	36	220	15·5
7	40	240	6·9
8	47	280	5 0
9	51	300	2·2
10	55	320	1·0
11	62	360	0·7
12	67	380	0·3
13	69	400	0·1
14	73	420	0·0
15	77	440	0 0
16	80	460	0·0
17	84	480	0 0
18	88	500	0·0
19	97	550	0·0

Fig. 1. Expected decision loss (standardised) as a function of c_1 for Bayesian plan with double binomial as a prior distribution $p' = 0.01$, $p'' = 0.15$ and $w_1 = 0.93$, $w_2 = 1 - w_1$ and $G(c_1)$ in the units of 10^{-3} where $p_1 = 0.05$, $p_2 = 0.10$, $\beta_1 = 0.07$, $\beta_2 = 0.10$.

Consider two plans—plan 1 : (n_1, c_1', c_2') and plan 2 : (n_2, c_1'', c_2'') and let (N_1, N_1') be the range of values of N where plan 1 is optimal and (N_2, N_2') be the range of values where plan 2 is optimal, according to (3.4).

For plan 1, $R(N)$ increases when N rises from N_1 to N_1' and for plan 2 it increases when N increases from N_2 to N_2'. Let $N_1 < N_2 < N_1' < N_2'$; then (N_2, N_1') is the range of overlap in N. We shall now examine the question as to which of the two plans—plan 1 or plan 2—should be preferred in (N_2, N_1').

Since $N_1 < N_2 < N_1' < N_2'$ and it is given that

$$\min\{R(N, n_1, c_1', c_2'), R(N, n_2, c_1'', c_2'')\} = \begin{cases} R(N, n_1, c_1', c_2') \text{ for } N \in M_1 \\ R(N, n_2, c_1'', c_2'') \text{ for } N \in M_2 \end{cases}$$

where $M_1 = \{N \,;\, N_1 \leqslant N \leqslant N_1'\}$ and $M_2 = \{N \,;\, N_2 \leqslant N \leqslant N_2'\}$ and further $R(N)$ is increasing linear function of N, the $R(N)$ function for plan 1 and plan 2 must intersect at some point in (N_2, N_1'), the range of overlap (Figure 2). At the point of intersection in (N_2, N_1') the values of $R(N)$ for the two plans must be equal i.e.,

$$R(N, n_1, c_1', c_2') = R(N, n_2, c_1'', c_2'') \qquad \dots \quad (3.5)$$

which gives the expression for $N(1, 2)$ the point of intersection. For example, $N(c_1, c_1+1)$ the point of intersection of $R(N, c_1)$ and $R(N, c_1+1)$ is given by

$$N(c_1, c_1+1) = \frac{n(c_1+1)[1-G(c_1+1)]-n(c_1)[1-G(c_1)]}{-\Delta\, G(c_1)} \qquad \dots \quad (3.6)$$

Thus

$$R(N, n_1, c_1', c_2') \lesseqgtr R(N, n_2, c_1'', c_2'') \text{ according as } N \lesseqgtr N_{12} \qquad \dots \quad (3.7)$$

and hence in (N_2, N_{12}) we should prefer plan 1 to plan 2 and in (N_{12}, N_1') we should prefer plan 2 to plan 1 where $N_{12} = N(c_1, c_1+1)$.

For any c_1 the plan $S(c_1) \in S$ is optimal for lot range $N_{c_1-1} \leqslant N < N_{c_1}$ as stated by (3.4). The function $R(N, c_1^0)$ is a concave function of N according to (2.3).

Writing

$$N_{c_1} = n(c_1) + \frac{1-G(c_1+1)}{-\Delta\, G(c_1)}\, \Delta n(c_1) \qquad \dots \quad (3.8)$$

we note that $N_{c_1} > n(c_1)$. N_{c_1} is an increasing function of $n(c_1)$. Hence, as stated earlier, for increasing values of c_1 of various optimal plans in S, the corresponding lot size ranges would be moving to the right, possibly overlapping according to (3.7).

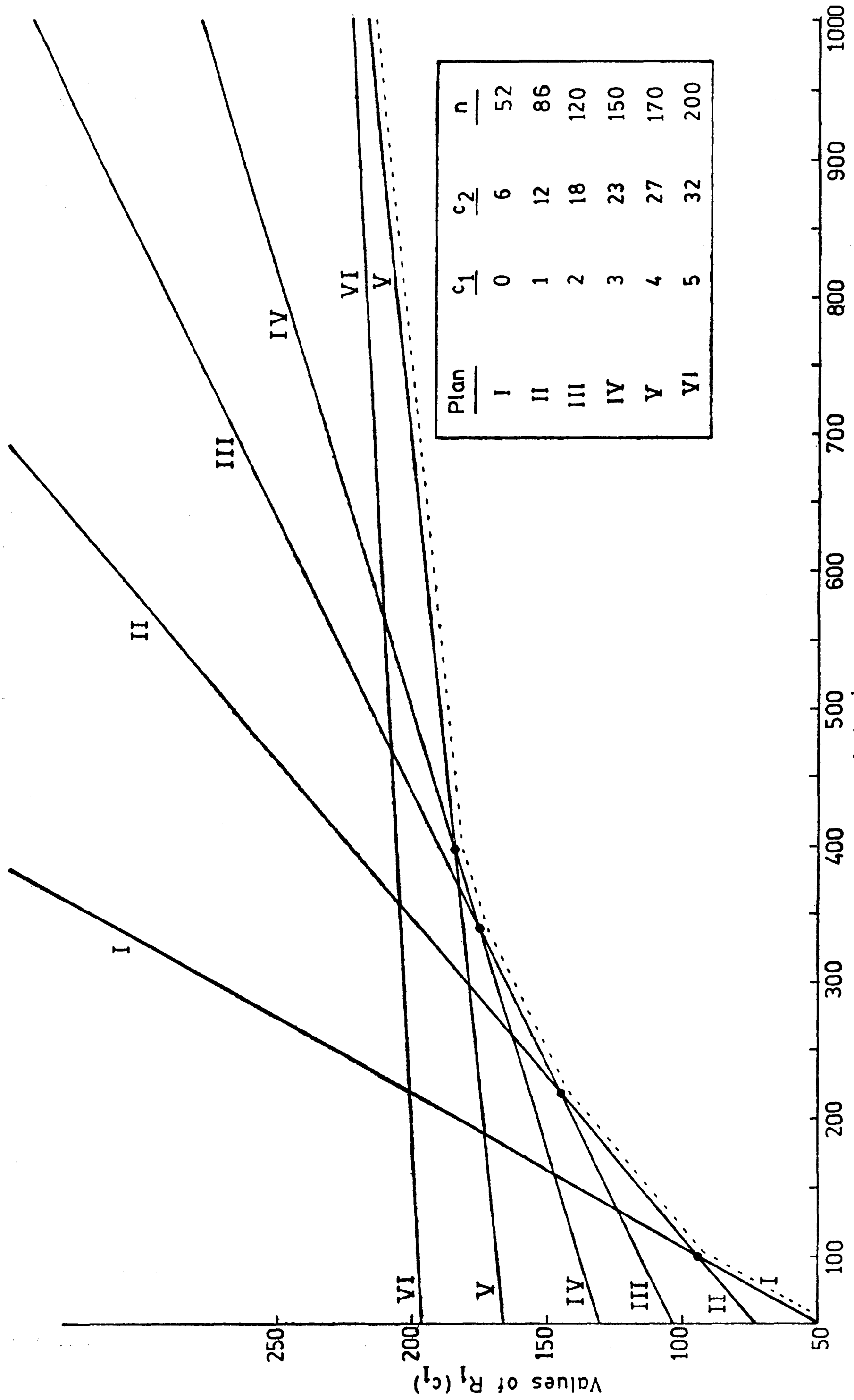


Fig.2. The values of $R_1(c_1)$ for varying lot sizes. (The parameter values are as in Fig. 1)

The optimal plans can be systematically tabulated, as indicated in Pandey (1984), as follows :

Step 1 : Take some arbitrary values of c_1 and obtain a plan, say $S(c_1) \epsilon S$ by using the fact that (2.7) and (2.8) are satisfied as closely as possible.

Step 2 : For the plan $S(c_1)$ so obtained, compute the value of N_{c_1} and N_{c_1-1} using (3.2).

Step 3 : Choose $c_1 = 0, 1, 2, 3, \ldots$ systematically and proceed as in steps $1 \rightarrow 2$ and tabulate the sampling plans and the corresponding bound for the lot sizes.

Step 4 : For two plans with overlapping N-intervals use (3.7) to select the optimal plan.

Steps 1-4 have yielded unique plans which are available in Pandey (1984). We shall devote the subsequent section to analytical uniqueness of the optimal plans.

4. Uniqueness of restricted Bayesian ASR plan

The uniqueness of optimal Bayes solution discussed in the previous sections, can be proved analytically provided—

(a) the function $G(c)$, denoting c_1 as c for simplicity of notation, as defined in (2.4) is analytically shown as a decreasing function of c with falling rate of decrease i.e., $\Delta_c\ G(c) < 0$ and $\Delta_c^2\, G(c) > 0$ and

(b) the point of intersection $N(c, c+1)$ of $R(N, c)$ and $R(N, c+1)$ as defined in (3.6) is analytically shown as an increasing function of c i.e., $\Delta_c\, N(c, c+1) > 0$.

It has not been possible to prove (a) and (b) analytically and it is noted from Hald (1960) that it has not been possible to prove (b) analytically even in the case of two-decision plans.

We pose (a) and (b) as "open conjectures". However, as mentioned earlier we have carried out extensive numerical computations and found that both the conjectures (a) and (b) are true for the range of values of c taken. Our numerical results in respect of (a) and (b) are illustrated in Figures 1 2 respectively.

In the light of the above numerical investigations if we accept (a) and (b) as true, then, the proof for uniqueness proceeds rigorously as follows :

Lemma 1. *Let* $R(N, c) = n(c)(1-G(c))+G(c)N$. *For any* $0 \leqslant c' < c'' \leqslant n$ *there exists a unique* $N_0 > 0$ *such that* $R(N_0, c') = R(N_0, c'')$. *Further, we have* $R(N, c'') > R(N, c')$ *for all* $0 \leqslant N < N_0$ *and* $R(N, c'') < R(N, c')$ *for* $N > N_0$.

Proof: It can be easily shown that the functions $f_1(x) = a_1+b_1x$ and $f_2(x) = a_2+b_2x$, $x \epsilon R^2$ for $b_1 > b_2 > 0$, $0 < a_1 < a_2$ intersect at $x_0 > 0$ and $f_1(x)$ meets $f_2(x)$ from below.

Now, for plans in S we have $n(c+k) > n(c)$ for any $k = 1, 2, \ldots$. We take $n(c'') > n(c')$ and note that $n(c'')(1-G(c'')) > n(c')(1-G(c'))$ and $G(c'') < G(c')$. The required results follow by putting $a_1 = n(c')(1-G(c'))$, $a_2 = n(c'')(1-G(c''))$, $b_1 = G(c')$ and $b_2 = G(c'')$.

Theorem 1: *Let* $c_0 > 1$ *and let* N_0 *be such that* $R(N_0, c_0) = R(N_0, c_0+1)$. *Then* $c = c_0+1$ *is the unique value which satisfies the condition* $\Delta R(N_0, c_0) \leqslant 0 < \Delta R(N_0, c_0+1)$.

Proof: By hypothesis we have $\Delta R(N_0, c_0) = 0$. We have by Lemma 1, $\Delta R(N, c_0+1) > 0$ for all $0 \leqslant N < N(c_0+1, c_0+2)$. Since $N_0 = N(c_0, c_0+1) < N(c_0+1, c_0+2)$, we have $\Delta R(N_0, c_0+1) > 0$. Consider any $c > c_0+1$. For all $N < N(c, c+1)$ we have $\Delta R(N, c) > 0$. Since $N_0 < N(c, c+1)$ we have $\Delta R(N_0, c) > 0$ for all $c > c_0+1$. Now, consider any $0 \leqslant c < c_0$. By Lemma 1, $\Delta R(N, c) < 0$ for all $N > N(c, c+1)$.

Since $N_0 = N(c_0, c_0+1) > N(c, c+1)$ we have $\Delta R(N_0, c) < 0$.

Theorem 2: *For any* $\overline{N} > 0$ *there exists an unique* c_0 *such that* $\Delta R(\overline{N}, c_0) \leqslant 0 < \Delta R(\overline{N}, c_0+1)$.

Proof: For simplicity of notation let $N_k = N(k, k+1)$. If $N_k \leqslant \overline{N} < N_{k+1}$, let $c_0 = k$. We have from the proof of the Theorem 1 for $N_k = M$, $\Delta R(M, k) = 0$ and $\Delta R(N, c) < 0$ for all $c < k$ and $N > N_c$. But $\overline{N} \geqslant N_k > N_c$ and hence $\overline{\Delta} R(N, c) < 0$. Similarly, we can show that $\Delta R(\overline{N}, c) > 0$ for all $c \geqslant k$.

This completes the proof that the solution is unique.

5. Concluding remarks

For different prior distributions and different terminal decisions the point of intersection of the regret functions and the expected decision loss have similar expressions but varying degree of complexity. Approach presented here remains basically same for other cases with some minor modifications in the proof. However, the conjectures, still form the main foundation in all the cases. In case of a continuous prior distribution uniqueness of Bayesian three-decision plans is implied analytically under certain regularity conditions as it can be seen in Pandey (1987). It is felt that it may be relatively easier to show uniqueness of solution analytically in case of Bayesian three-decision plans by variables.

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