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A new definition of shape similarity

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Abstract: Two reference points of a region are defined which do not depend on the position, size and orientation of the region. Reference points are used to get borders on the basis of which the shape distance and shape similarity are defined.

Key words: Region, shape, least squares, major axis, reference points, directional codes, border, shape distance, shape similarity.

1. Introduction

There are two different approaches for shape analysis. One describes a shape in terms of scalar measurements and the other does it through structural descriptions. Pavlidis (1978, 1980) made two reviews of algorithms for shape analysis.

In the present paper a shape is described on the basis of its structural features using certain chain codes. The description is information preserving in the sense that it is possible to reconstruct any reasonable approximation of the shape from the descriptor. In Section 2 two reference points on the border of a region are defined which are invariant under translation, dilation and rotation of the region. From these two reference points some strings of directional codes describing the border (clockwise) are extracted in Section 3. The distance between two shapes is defined in terms of these strings. Computational techniques and results are given in Section 4.

2. Shape and its reference points

Definition 2.1. A *region* is a closed, bounded and connected subset of the Euclidean plane $\mathbb{R}^2$ such that its complement is also connected.

That is, no holes are permitted inside a region. Thus the boundary of a region is closed and describes the region uniquely. Let $\mathscr{R}$ be the set of all regions. We shall write a region A as $\{(x_i, y_i): i \in I\}$ where I is an uncountable index set.

Definition 2.2. R is an *equivalence relation* in $\mathscr{R}$ such that for A, B belonging to $\mathscr{R}$, $(A, B) \in R$ if the region A can be obtained from the region B through translation, dilation and rotation.

Definition 2.3. A *shape* is defined to be an equivalence class generated by R in $\mathscr{R}$. That is, two regions A and B have the same shape if and only if $(A, B) \in R$.

Clearly, all circles (also all squares) have the same shape. But the shape of an ellipse (or a rectangle) depends on the ratio of the lengths of its two axes.

Now, in order to extract features from a region we shall use the least squares method. The classical least squares method is as follows: Let (x_i, y_i), $i = 1, 2, \ldots, n$ be n points in $\mathbb{R}^2$. Suppose y is to be predicted on the basis of x. Then the squared error of any straight line $y = a + bx$ with respect to the n points is $\sum_{i=1}^{n} (y_i - a - bx_i)^2$. The straight line with the minimum squared error is $y = \hat{a} + \hat{b}x$ where

$$\hat{b} = \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \bar{x}\bar{y}}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2}, \quad \hat{a} = \bar{y} - \hat{b}x,$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Now if this least squares method is applied to the points (x_i, y_i) of a region, it can be seen that the slope of the best linear fit $y = \hat{a} + \hat{b}x$ remains unchanged if the input region is translated or dilated. But its relative slope may change if the region is rotated.

This drawback can be overcome by changing the definition of the squared error, that is, by taking the shortest distance between (x_i, y_i) and the line. Suppose, (r_i, θ_i) is the polar representation of (x_i, y_i), that is, $x_i = r_i \cos\theta_i$, $y_i = r_i \sin\theta_i$ where r_i is a nonnegative real number and θ belongs to $[0, 2\pi)$. Now, the shortest (perpendicular) distance from (x_i, y_i) to a straigth line $y = x\tan\theta + c$ is the absolute value of $\{c\cos\theta + r_i \sin(\theta - \theta_i)\}$, where c is a real number and θ belongs to $[0, \pi)$. So, the new squared error is

$$f(\theta, c) = \sum_{i=1}^{n} \{c\cos\theta + r_i \sin(\theta - \theta_i)\}^2. \tag{1}$$

The best linear fit in this case (that is, the line that minimizes $f(\theta, c)$) passes through $(\bar{x}, \bar{y})$ and makes an angle $\hat{\theta}$ with the x-axis where $\hat{\theta}$ satisfies

$$\tan 2\hat{\theta} = \frac{2\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2 - \sum_{i=1}^{n}(y_i - \bar{y})^2}, \tag{2}$$

$$\cos 2\hat{\theta}\left\{\sum_{i=1}^{n}(x_i - \bar{x})^2 - \sum_{i=1}^{n}(y_i - \bar{y})^2\right\} + $$

$$+ 2\sin 2\hat{\theta}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) > 0.$$

Now we shall consider the best linear fit for the points belonging to a region. For example, Q_1Q_2 is the best line for the region (indicated only by its border) in Fig. 1.

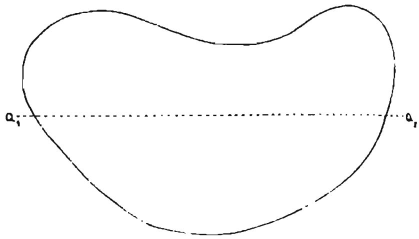

Fig. 1. Q_1Q_2 is the major axis.

Notations: Let $\mathscr{A}_1$ be the set of all regions $A = \{(x_i, y_i): i \in I\}$ such that both numerator and denominator in (2) are zero.

Note that circles and squares belong to $\mathscr{A}_1$. Let $\mathscr{A}^* = \mathscr{A} - \mathscr{A}_1$. Suppose, $A = \{(x_i, y_i): i \in I\}$.

Definition 2.4. Let $A \in \mathscr{A}$. Any straight line passing through its centre $(\bar{x}, \bar{y})$ is called an *axis* of the region A.

Definition 2.5. Let $A \in \mathscr{A}^*$. The axis of the region A which makes an angle θ with the x-axis is defined as the *major axis* of the region A (Q_1Q_2 in Fig. 1) where θ satisfies

$$\tan 2\theta = \frac{2\sum_{i \in I}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i \in I}(x_i - \bar{x})^2 - \sum_{i \in I}(y_i - \bar{y})^2},$$

$$\cos 2\theta\left\{\sum_{i \in I}(x_i - \bar{x})^2 - \sum_{i \in I}(y_i - \bar{y})^2\right\} + $$

$$+ 2\sin 2\theta\sum_{i \in I}(x_i - \bar{x})(y_i - \bar{y}) > 0.$$

It can be shown that

(i) For a region belonging to $\mathscr{A}_1$ all the axes are equally good with respect to $f(\theta, c)$ defined in (1).

(ii) For a region belonging to $\mathscr{A}^*$ the major axis is unique.

(iii) The major axis of a region belonging to $\mathscr{A}^*$ is invariant under translation, dilation and rotation of the region.

(iv) If a region belonging to $\mathscr{A}^*$ is symmetric its major axis either coincides with or is perpendicular to the axis of symmetry.

Definition 2.6. For a region belonging to $\mathscr{R}^*$ the two farthest points among the points of intersection of the major axis with the border of the region are called the *reference points* of the region (Q_1 and Q_2 in Fig. 1).

It can be seen that the reference points of a region belonging to $\mathscr{R}^*$ are unique and do not depend upon the position, size and orientation of the region.

3. Feature extraction

An approximated kind of border is extracted from a region in this section. This is done in terms of certain directional codes which will be defined now. The directional codes $1, 2, \ldots, 8$ are shown in Fig. 2. Each pair of adjacent directions makes an angle of 45° in between them. Now this definition of directional codes can be extended. Let d be any real number belonging to $(0, 8] = S_8$. Note that $i-1 < d \leq i$ for some $i \in \{1, 2, \ldots, 8\} = I_8$. Then the directional code d defines a direction that makes an angle of $(i-d)45°$ with direction i on the anticlockwise side (Fig. 3). Thus there is a one-to-one correspondence between S_8 and the set of all directions in $\mathbb{R}^2$. From now on unless otherwise mentioned directional codes will mean elements of S_8.

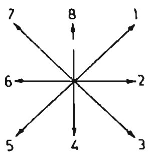

Fig. 2. Directional codes.

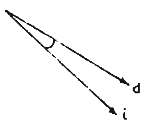

Fig. 3. Angle between d and i is $(i-d)$ 45°.

We shall now describe extraction of features from a region. Let there be n points on the border $P_0, P_1, \ldots, P_{n-1}, P_n = P_0$ such that the arc length between any two consecutive points is the same. Let the directional code d_j denote the direction $\overrightarrow{P_{j-1}P_j}$. Then the string of directional codes $\mathbf{d} = \{d_j\}_{j=1}^n$ is called an approximated border of the region which depends on the starting point P_0 but not on the position or size of the region. This approximated border can be made arbitrarily close to the original border of the region by increasing the value of n.

Now, let Q_1 and Q_2 be the two reference points of a region. Suppose, $\mathbf{d}_k = \{d_{kj}\}_{j=1}^n$ is the approximated border of the region starting from Q_k for $k = 1, 2$. The set of these two strings $\boldsymbol{d} = \{\mathbf{d}_1, \mathbf{d}_2\}$ is the feature extracted from the region. Note that this feature $\boldsymbol{d}$ is information preserving and is invariant under translation and dilation. Suppose, F_n is the set of all possible feature values of $\boldsymbol{d}$ (that is, from all regions belonging to $\mathscr{R}^*$). Thus, for any fixed value of n, we have described a mapping g from $\mathscr{R}^*$ to F_n.

In the rest of this section we shall establish some relations between the feature values of $\boldsymbol{d}$ and shapes and then develop a concept of distance between shapes on the basis of the feature $\boldsymbol{d}$.

Notation: Let T_n be the set of all possible strings of directional codes of length n. Let $\mathbf{d}_1 = \{d_{1j}\}_{j=1}^n$ and $\mathbf{d}_2 = \{d_{2j}\}_{j=1}^n$ be two elements of T_n. We write $\mathbf{d}_2 = \mathbf{d}_1 + a$ for a real number a if $d_{2j} = d_{1j} + a \pmod 8$ for all j. In that case we say that $\mathbf{d}_2$ is a rotation of $\mathbf{d}_1$.

Definition 3.1. Let $\boldsymbol{d}^1 = \{\mathbf{d}_1^1, \mathbf{d}_2^1\}$ and $\boldsymbol{d}^2 = \{\mathbf{d}_1^2, \mathbf{d}_2^2\}$ be two elements of F_n. We say that $\boldsymbol{d}^2$ is a *rotation* of $\boldsymbol{d}^1$ if any one of the following conditions holds.

(i) $\mathbf{d}_1^2$ is a rotation of $\mathbf{d}_1^1$ and $\mathbf{d}_2^2$ is a rotation of $\mathbf{d}_2^1$,

(ii) $\mathbf{d}_1^2$ is a rotation of $\mathbf{d}_2^1$ and $\mathbf{d}_2^2$ is a rotation of $\mathbf{d}_1^1$.

We write $(\boldsymbol{d}^1, \boldsymbol{d}^2) \in R_n^*$ if $\boldsymbol{d}^2$ is a rotation of $\boldsymbol{d}^1$ where R_n^* is a relation in F_n.

It can be seen that the relation R_n^* is reflexive, symmetric and transitive. Hence R_n^* is an equivalence relation in F_n. Let $\mathscr{F}_n$ be the set of all equivalence classes generated by R_n^* in F_n. Now, R_n^* induces an equivalence relation in $\mathscr{R}^*$ in the following way: We have a function g from $\mathscr{A}^*$ to F_n. Let R'_n be a relation in $\mathscr{A}^*$ such that for A, B belonging to $\mathscr{A}^*$, $(A, B) \in R'_n$ if $(g(A), g(B)) \in R^*_n$. Clearly, R'_n is an equivalence relation in $\mathscr{A}^*$. Let $\mathscr{S}_n$ be the set of all equivalence classes generated by R'_n in $\mathscr{A}^*$. Note that each equivalence class generated by R'_n includes one or more equivalence classes generated by R in $\mathscr{A}$. If $(A, B) \in R'_n$ we say that the two regions A and B have the same shape at the level n. That is, each equivalence class generated by R'_n contains regions whose shapes resemble one another closely. And the degree of this closeness increases as the value of n increases. In fact, each such equivalence class tends to a single shape as n goes to infinity.

Let R' be a relation in $\mathscr{A}^*$ such that for A, B belonging to $\mathscr{A}^*$ $(A, B) \in R'$ if $(A, B) \in R'_n$ for all positive integers n. Clearly, R' is an equivalence relation in $\mathscr{A}^*$. Note that each equivalence class generated by R' is exactly one equivalence class generated by R in $\mathscr{A}$ and hence corresponds to a unique shape. The set of all equivalence classes generated by R' is a proper subset of the set of equivalence classes generated by R, the difference being the equivalence classes obtained from the regions belonging to $\mathscr{A}_1$.

Since, in practice, we have to deal with finite n, we shall consider only R^*_n, R'_n etc.

We shall now develop a definition of distance between two equivalence classes belonging to $\mathscr{F}_n$ (which in turn defines a distance function on $\mathscr{S}_n$).

Definition 3.2. D is a distance defined on T_n such that for two strings $\mathbf{d}_1 = \{d_{1j}\}_{j=1}^n$ and $\mathbf{d}_2 = \{d_{2j}\}_{j=1}^n$ belonging to T_n,

$$D(\mathbf{d}_1, \mathbf{d}_2) = \sum_{j=1}^{n} \min\{|d_{1j} - d_{2j}|, 8 - |d_{1j} - d_{2j}|\}. \tag{3}$$

It can be seen that D is a metric on T_n. Note that $D(\mathbf{d}_1, \mathbf{d}_2) = D(\mathbf{d}_1 + a, \mathbf{d}_2 + a)$ for any real number a.

Definition 3.3. D' is a distance defined on T_n such that for two strings $\mathbf{d}_1$ and $\mathbf{d}_2$ belonging to T_n

$$D'(\mathbf{d}_1, \mathbf{d}_2) = \inf_a D(\mathbf{d}_1, \mathbf{d}_2 + a).$$

D' has the following properties:

(i) $D'(\mathbf{d}_1, \mathbf{d}_2) \geq 0$,

(ii) $D'(\mathbf{d}_1, \mathbf{d}_2) = 0$ if and only if $\mathbf{d}_2$ is a rotation of $\mathbf{d}_1$,

(iii) $D'(\mathbf{d}_1, \mathbf{d}_2) = D'(\mathbf{d}_2, \mathbf{d}_1)$,

(iv) $D'(\mathbf{d}_1, \mathbf{d}_2) + D'(\mathbf{d}_2, \mathbf{d}_3) \geq D'(\mathbf{d}_1, \mathbf{d}_3)$.

Definition 3.4. D^* is a distance defined on $\mathscr{F}_n$ such that for any two equivalence classes E_1 and E_2 belonging to $\mathscr{F}_n$,

$$D^*(E_1, E_2) = \operatorname{Min}\{D'(\mathbf{d}_1^1, \mathbf{d}_1^2) + D'(\mathbf{d}_2^1, \mathbf{d}_2^2), D'(\mathbf{d}_1^1, \mathbf{d}_2^2) + D'(\mathbf{d}_2^1, \mathbf{d}_1^2)\}$$

where $\boldsymbol{d}^1 = \{\mathbf{d}_1^1, \mathbf{d}_2^1\}$ is an element of E_1 and $\boldsymbol{d}^2 = \{\mathbf{d}_1^2, \mathbf{d}_2^2\}$ is an element of E_2.

It can be seen that this definition of $D^*(E_1, E_2)$ is unambiguous, that is, if $\boldsymbol{e}^1 = \{\mathbf{e}_1^1, \mathbf{e}_2^1\}$ is any element of E_1 and $\boldsymbol{e}^2 = \{\mathbf{e}_1^2, \mathbf{e}_2^2\}$ is any element of E_2, then

$$\begin{aligned}&\operatorname{Min}\{D'(\mathbf{d}_1^1, \mathbf{d}_1^2) + D'(\mathbf{d}_2^1, \mathbf{d}_2^2), D'(\mathbf{d}_1^1, \mathbf{d}_2^2) + D'(\mathbf{d}_2^1, \mathbf{d}_1^2)\} = \\ &= \operatorname{Min}\{D'(\mathbf{e}_1^1, \mathbf{e}_1^2) + D'(\mathbf{e}_2^1, \mathbf{e}_2^2), D'(\mathbf{e}_1^1, \mathbf{e}_2^2) + D'(\mathbf{e}_2^1, \mathbf{e}_1^2)\}.\end{aligned}$$

It can be proved that D^* is a metric on $\mathscr{F}_n$.

Now, there is a one-to-one correspondence between $\mathscr{F}_n$ and $\mathscr{S}_n$. Thus D^* also gives the distance between any two elements belonging to $\mathscr{S}_n$. For finite values of n an element of $\mathscr{S}_n$ means a set of different shapes. The number of such different shapes comprising one single element of $\mathscr{S}_n$ reduces an n increases. In fact this number goes to unity as n goes to infinity. Thus asymptotically D^* gives the distance between shapes such that for any two different shapes the value of D^* will be greater than zero. So, in order to get more accurate results, greater values of n are taken.

4. Computation and results

For the computation of the directional codes in $\boldsymbol{d}$, we do not explicitly find the points P_j on the border. First, the border is extracted as a string of the directional codes belonging to I_8, say, $\{d_i\}$ starting from each of the points Q_k, $k = 1, 2$. Note that all d_i's do not have the same length. In fact, the odd directional codes 1,3,5,7 have length $\sqrt{2}r$ and the even directional codes 2,4,6,8 have length

x where x is the size of the square pixel of the input. To get rid of this inequality in lengths we modify the string of directional codes by replacing d_i by 7 consecutive d_i's if d_i is odd and by 5 consecutive d_i's if d_i is even. The lengths of the new d_i's can be considered the same since $\frac{7}{5}$ is very close to $\sqrt{2}$. But the size of the new string $\{d_i\}$, say N, will be larger (5 to 7 times the old one). Suppose, n is the desirable size of the final string. Let $N=nr$. Now for each $j=1,2,\ldots,n$ a sort of average directional code d_j^r is obtained from the set of directional codes $\{d_i : (j-1)r+1 \le i \le jr\}$. In fact, d_j^r is taken as the direction $\overrightarrow{PQ}$ where P is the starting point of $d_{(j-1)r+1}$ and Q is the end point of d_{jr}. This average string $\{d_j^r\}_{j=1}^{n}$ is obtained starting from each of Q_1 and Q_2.

Suppose, $\mathbf{d}_k = \{d_{kj}^r\}_{j=1}^{n}$ is the average string for Q_k, $k=1,2$. Thus $d=\{\mathbf{d}_1, \mathbf{d}_2\}$ is computed.

The computational aspect of D^* involves computation of D and D'. But the computation of D' on the basis of the present definition of D is difficult since $D(\mathbf{d}_1, \mathbf{d}_2 + a)$ is not in general differentiable with respect to a. We shall now propose an alternative definition of D which retains all the desirable properties of the earlier D and which is differentiable with respect to a. Changing D defined in (3), we write

$$D(\mathbf{d}_1, \mathbf{d}_2) = \sum_{j=1}^{n} \sin^2 \tfrac{1}{8}\pi(d_{1j} - d_{2j}). \tag{4}$$

For fixed $\mathbf{d}_1$ and $\mathbf{d}_2$, the function

$$f(a) = D(\mathbf{d}_1, \mathbf{d}_2 + a) = \sum_{j=1}^{n} \sin^2 \tfrac{1}{8}\pi(d_{1j} - d_{2j} - a)$$

is differentiable with respect to a and attains the minimum/maximum at

$$a = \frac{4}{\pi} \tan^{-1} \left\{ \frac{\sum_{j=1}^{n} \sin\theta_j}{\sum_{j=1}^{n} \cos\theta_j} \right\} \tag{5}$$

where $\theta_j = \frac{1}{4}\pi(d_{1j} - d_{2j})$. Let a_1 belonging to $[0,4)$ satisfy (5). Then $a_2 = a_1 + 4$ also satisfies (5). Now, for only one of a_1 and a_2 the second derivative

$$\frac{\mathrm{d}^2 f(a)}{\mathrm{d}a^2} = (\tfrac{1}{8}\pi)^2 \left\{ \sin \tfrac{1}{4}\pi a \sum_{j=1}^{n} \sin\theta_j + \cos \tfrac{1}{4}\pi a \sum_{j=1}^{n} \cos\theta_j \right\} > 0. \tag{6}$$

That value of a, say $\hat{a}$, is obtained (which minimizes $f(a)$). So the value of D' is also obtained as $f(\hat{a})$. Now to compute the value of D^*, four values of D', namely, $D'(\mathbf{d}_1^1, \mathbf{d}_1^2)$, $D'(\mathbf{d}_2^1, \mathbf{d}_2^2)$, $D'(\mathbf{d}_2^1, \mathbf{d}_1^2)$, $D'(\mathbf{d}_1^1, \mathbf{d}_2^2)$ are to be computed.

For the new definition of D in (4) the upper bound of D' is $\frac{1}{2}n$ (assume n is even), and hence the upper bound of D^* is n. Thus, D^* can be normalized and we can get a similarity measure between two shapes as

$$\mu = 1 - D^*/n \quad \text{where } 0 \le \mu \le 1.$$

We shall now see to what degree an arbitrary shape is similar to the circular shape. Though strictly speaking a circle does not belong to $\mathscr{S}^*$, its feature value d can be defined. Let $\mathbf{d}_1^c = \{d_{1j}\}_{j=1}^{n}$ where $d_{1j} = x + j\alpha_n$ (mod 8) and $\alpha_n = 8/n$ and $\mathbf{d}_2^c = \mathbf{d}_1^c + 4$. For any number x belonging to $(0,8]$, $d_x^c = \{\mathbf{d}_1^c, \mathbf{d}_2^c\}$ describes a circle at the level n (d_x^c is in fact a regular polygon with n sides). Now, let $d = \{\mathbf{d}_1, \mathbf{d}_2\}$ be the feature value of an arbitrary shape. Let $d \in E_1$ and $d_x^c \in E_2$. So, $D^*(E_1, E_2) = D'(\mathbf{d}_1, \mathbf{d}_1^c) + D'(\mathbf{d}_2, \mathbf{d}_2^c)$ because $D'(\mathbf{d}_1, \mathbf{d}_2) = D'(\mathbf{d}_1, \mathbf{d}_2 + a)$ for any $\mathbf{d}_1, \mathbf{d}_2$ and a. In fact,

Table 1

$\mathbf{d}_1 =$	7.55	8.00	0.78	1.16	2.00	2.22	2.45	2.47
	2.22	2.00	1.38	1.22	1.55	2.48	3.35	3.78
	4.22	4.45	4.52	5.00	5.12	5.31	5.35	5.74
	5.78	6.22	6.65	6.57	6.85	7.15	7.15	7.19
$\mathbf{d}_2 =$	4.45	4.75	5.00	5.15	5.48	5.48	5.65	6.16
	6.28	6.29	7.00	7.13	7.00	7.15	7.45	7.78
	0.25	1.00	1.69	2.00	2.45	2.45	2.45	2.00
	1.78	1.28	1.35	2.00	2.81	3.59	4.12	4.29

$D'(\mathbf{d}_1, \mathbf{d}_1^c) = 1.5712$ $D'(\mathbf{d}_3, \mathbf{d}_1^c) = 1.5502$

$D^*(E_1, E_2)$ = The distance between the circular shape and the shape of the region in Fig. 1
= 3.1214

$\mu(E_1, E_2)$ = Shape similarity measure
= 0.9025

Ratio of the side lengths of a rectangle	Shape similarity measure between a rectangle and a circle (μ)
1 1.25	0.9196
1 1.50	0.9015
1 2.00	0.8920
1 3.00	0.8388
1 4.00	0.8246

$D^*(E_1, E_2) = D'(\mathbf{d}_1, \mathbf{d}_1^c) + D'(\mathbf{d}_2, \mathbf{d}_1^c)$. The similarity measure between the circular and an arbitrary shape is $1 - D^*(E_1, E_2)/n$.

For the region in Fig. 1, the values of $\mathbf{d}_1$, $\mathbf{d}_2$, $D'(\mathbf{d}_1, \mathbf{d}_1^c)$, $D'(\mathbf{d}_2, \mathbf{d}_1^c)$, $D^*(E_1, E_2)$ and μ are given in Table 1 where the values of the shape similarity measures between rectangles and a circle are also given.

The computation of the two reference points of a region involves all the points of the region and hence is costly. This can be avoided by considering only the border points of a region. The major axis defined on the basis of the border points has all the earlier invariant properties.

The distance D^* can also be defined in terms of area. Suppose A and B are two regions with the same area. Then the distance between the two corresponding shapes is defined as

$$D^* = \frac{\text{The area of } (A - B) \cup (B - A)}{\text{Twice the area of } A}. \tag{7}$$

D^* is normalised, that is, $0 \leq D^* < 1$.

The definition of the major axis can be extended for gray level pictures where the gray level is the weight of a pixel. In that case the definition of D^* in (7) can also be extended.

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