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BALANCED ARRAYS AND WEIGHING DESIGNS[1]

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Summary

Dey (1971), Saha (1975), Kageyama & Saha (1983) and others have shown how optimum chemical balance weighing designs can be constructed from the incidence matrices of balanced incomplete block (BIB) designs. In this paper, it is shown that weighing designs can be constructed from some suitably chosen two-symbol balanced arrays of strength two, which need not always be incidence matrices of BIB designs. The findings lead us to construct new optimum chemical balance weighing designs from incidence matrices of BIB designs.

Key words: Balanced array; Chemical balance weighing design; BIB design; Incidence matrix; Optimum

1. Introduction

From an examination of the existing results on connections between incomplete block designs (which are always equireplicated and equiblock-sized) and weighing designs, it is clear that the parameter k (being the block size) has no role to play in the weighing design set up. This leads us to think of the use of balanced arrays, pairwise balanced designs and others, as a weighing design. Some utility of pairwise balanced designs for the construction of optimum chemical balanced weighing designs is found in Kageyama & Saha (1983). We here call a chemical balance weighing design to be optimum if it estimates each of the weights with minimum variance. In this case, the $(v \times b)$ weighing design matrix W for v objects in b weighings satisfies $WW' = bI_v$, where I_v is the identity matrix of order v. For standard terms and definitions a reference may be made to Raghavarao (1971).

In Section 2, we obtain the basic results for the construction of optimum chemical balance weighing designs from suitably chosen balanced arrays of two symbols and strength two. In Section 3, we use them to construct new optimum weighing designs from the incidence

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matrices of BIB designs. In the course of these investigations we make some observations on results of Dey (1971), given in the last section. It is realized that optimum chemical balance weighing designs and two-symbol orthogonal arrays of strength two coexist.

2. Basic Results

A balanced array of strength t with two symbols, v constraints, b runs and index set $\{\mu_0, \mu_1, \ldots, \mu_t\}$ is a $(v \times b)$ matrix $\mathbf{B}$ whose elements are the two symbols (0 and 1, say) such that, in every $(t \times b)$ submatrix $\mathbf{B}_0$ of $\mathbf{B}$, every t-vector (i.e., a vector with t elements) $\boldsymbol{\alpha}$ of weight i ($i = 0, 1, \ldots, t$; the weight of $\boldsymbol{\alpha}$ is the number of 1's in it) appears as a column of $\mathbf{B}_0$ exactly μ_i times. We denote it by $BA[v, b, 2, t; \mu_0, \mu_1, \ldots, \mu_t]$. In particular, if $\mu_i = \mu$ for all i, the array is called an orthogonal array. It is known that $b = \sum_{i=0}^{t} \binom{t}{i} \mu_i$ for a balanced array.

Let $\mathbf{B} = B(0, 1)$ be a $BA[v, b, 2, 2; \mu_0, \mu_1, \mu_2]$, and let $W = W(v, b)$ denote an optimum chemical balance weighing design for v objects in b weighings. Clearly $\mathbf{B}$ and $\mathbf{W}$ are $(v \times b)$ matrices. So, $\mathbf{W}$ has ± 1 as its elements and $\mathbf{W}\mathbf{W}' = b\mathbf{I}_v$. We use $\mathbf{J}_{m \times n}$ and $\mathbf{O}_{m \times n}$ for $(m \times n)$ matrices with all entries equal to 1 and 0 (zero), respectively. The following two basic results can now be easily proved by considering the meaning of parameters μ_0, μ_1 and μ_2.

Theorem 1. *The existence of a balanced array* $\mathbf{B}$ *with parameters* $(v, b, 2, 2; \mu_0, \mu_1, \mu_2)$ *satisfying* $\mu_0 + \mu_2 \leqq 2\mu_1$, *or equivalently* $b \leqq 4\mu_1$, *implies the existence of an optimum chemical balance weighing design* $\mathbf{W}$ *for* v *objects in* $4\mu_1$ *weighings, where* $\mathbf{W}$ *is given by*

$$\mathbf{W} = [\mathbf{B}^* : \mathbf{J}_{v \times p}], \quad p = 2\mu_1 - \mu_0 - \mu_2$$

and $\mathbf{B}^* = \mathbf{B}(-1, 1)$ *obtained from* $\mathbf{B} = \mathbf{B}(0, 1)$ *by replacing* 0's *by* -1's.

Sketch of proof. It follows that $WW' = B^*B^{*\prime} + pJ_{v \times v}$. In this case, from the definition of $B^* = B(-1, 1)$, it holds that diagonal elements of $B^*B^{*\prime}$ are all $\mu_0 + 2\mu_1 + \mu_2$ $(= b)$ and off-diagonal elements are all $\mu_0 - 2\mu_1 + \mu_2$. This shows that $WW' = 4\mu_1 I_v$, which completes the proof.

Theorem 2. *The existence of a balanced array* $\mathbf{B} = \mathbf{B}(0, 1)$ *with parameters* $(v, b, 2, 2; \mu_0, \mu_1, \mu_2)$ *satisfying the condition that* $\max\{\mu_0, \mu_1, \mu_2\} = \mu_1$, *implies the existence of an orthogonal array* $\mathbf{A}$ *with parameters* $(v^* = v, b^* = 4\mu_1, 2, 2; \mu^* = \mu_1)$, *where the array* $\mathbf{A}$ *is given by*

$$\mathbf{A} = [\mathbf{B} : \mathbf{O}_{v \times p_0} : \mathbf{J}_{v \times p_1}], \quad p_0 = \mu_1 - \mu_0, \quad p_1 = \mu_1 - \mu_2.$$

This theorem yields immediately the following.

Corollary 1. *The existence of a balanced array as in Theorem 2 implies the existence of an optimum chemical balance weighing design* $\mathbf{W}$ *for* $v+1$ *objects in* $4\mu_1$ *weighings, where*

$$\mathbf{W} = \begin{bmatrix} \mathbf{A}^* \\ \hdashline \mathbf{J}_{1\times b^*} \end{bmatrix},$$

and $\mathbf{A}^* = \mathbf{A}$ *with* 0's *replaced by* -1's.

Remark. (i) Since $\max\{\mu_0, \mu_1, \mu_2\} = \mu_1$ implies $\mu_0 + \mu_2 \leqq 2\mu_1$, we see that the existence of a $BA[v, b, 2, 2; \mu_0, \mu_1, \mu_2]$ such that $\max\{\mu_0, \mu_1, \mu_2\} = \mu_1$, implies the existence of two optimum chemical balance weighing designs, one for v objects in $4\mu_1$ weighings (by Theorem 1), and the other for $v+1$ objects in $4\mu_1$ weighings (by Theorem 2). In these resulting designs, it should be noted that the number of weighings is the same, but the number of objects to be estimated is different by one. Finally, note (Rafter & Seiden (1974)) that, for $v \geqq 3$, $\mu_1 \leqq \mu_0 + \mu_2$ holds. (ii) For applications of balanced arrays satisfying the above restriction, refer to the next section.

Example. Consider a $BA[v=4, b=7, 2, 2; \mu_0 = 2, \mu_1 = 2, \mu_2 = 1]$ whose array is given by

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{matrix}$$

Then this array yields two optimum chemical balance weighing designs, one for 4 objects in 8 weighings, and the other for 5 objects in 8 weighings.

3. Optimum Weighing Designs from BIB Designs

Let $\mathbf{B} = \mathbf{B}(0, 1)$ be the $(v \times b)$ incidence matrix of a BIB design (BIBD) with parameters v, b, r, k and λ. In this case, it is known (Saha (1975)) that the existence of such a $\mathbf{B}(0, 1)$ satisfying $b \leqq 4(r-\lambda)$ implies the existence of an optimum chemical balance weighing design $\mathbf{W} = [\mathbf{B}(-1, 1) : \mathbf{J}_{v\times p}]$ with $p = 4(r-\lambda) - b$, for v objects in $4(r-\lambda)$ weighings. We shall now show that some of these $\mathbf{B}(0, 1)$'s also yield optimum chemical balance weighing designs for $v+1$ objects as well.

Theorem 3. *The existence of a* $BIBD(v, b, r, k, \lambda)$ *with* $v = 2k-1$, $2k$, *or* $2k+1$, *implies the existence of an orthogonal array* $\mathbf{A}(v^*, b^*, 2, 2; \mu^*)$ *with* $v^* = v$, $b^* = 4(r-\lambda)$ *and* $\mu^* = r - \lambda$, *which again implies the existence of an optimum chemical balance weighing design* $\mathbf{W}(v+1, b^*)$. *When* $\mathbf{B} = \mathbf{B}(0, 1)$ *is the incidence matrix of the*

BIBD(v, b, r, k, λ), *the orthogonal array* $\mathbf{A}$ *is given by*

(i) $\mathbf{A}=[\mathbf{B}:\mathbf{O}_{v\times p}]$, $p=\mu_1-\mu_0$, if $v=2k-1$;

(ii) $\mathbf{A}=[\mathbf{B}:\mathbf{O}_{v\times p}:\mathbf{J}_{v\times p}]$, $p=\mu_1-\mu_2=\mu_1-\mu_0$, if $v=2k$;

(iii) $\mathbf{A}=[\mathbf{B}:\mathbf{J}_{v\times p}]$, $p=\mu_1-\mu_2$, if $v=2k+1$;

and $\mathbf{W}$ *is given by* $\mathbf{A}^*=\begin{bmatrix}\mathbf{A}\\ \hdashline \mathbf{J}_{1\times b^*}\end{bmatrix}$ *with* 0's *replaced by* -1's.

Proof. It is well known that when $\mathbf{B}(0, 1)$ is the incidence matrix of a BIBD(v, b, r, k, λ), $\mathbf{B}(0, 1)$ is also a $BA[v, b, 2, 2; \mu_0, \mu_1, \mu_2]$, where

$$\mu_j=\frac{\lambda\binom{v-2}{k-j}}{\binom{v-2}{k-2}}, \quad j=0, 1, 2. \tag{3.1}$$

Now, since $\binom{n}{r} \gtreqless \binom{n}{r-1}$ according as $n-r+1 \gtreqless r$, we have

(i) $\mu_0 \gtreqless \mu_1$ according as $v \gtreqless 2k+1$; and

(ii) $\mu_1 \gtreqless \mu_2$ according as $v \gtreqless 2k-1$.

Thus, it follows that $\max\{\mu_0, \mu_1, \mu_2\}=\mu_1$ iff $\mu_1 \geqq \mu_0$ and $\mu_1 \geqq \mu_2$, that is, iff $2k-1 \leqq v \leqq 2k+1$. Hence, from Theorem 2, these observations complete the proof of the first part of the theorem. Also, note that (i) $v=2k-1$ implies $\mu_1=\mu_2$; (ii) $v=2k$ implies $\mu_0=\mu_2$; and (iii) $v=2k+1$ implies $\mu_0=\mu_1$. Therefore, the structure of A as given in Theorem 3 follows.

Remark. (i) In (3.1), $\mu_0=b-2r+\lambda$, $\mu_1=r-\lambda$ and $\mu_2=\lambda$. (ii) It also holds that (a) $v \gtreqless 2k+1$ iff $b \gtreqless 3r-2\lambda$, and (b) $v \gtreqless 2k-1$ iff $r \gtreqless 2\lambda$.

Corollary 2. *The existence of a* BIBD(v, b, r, k, λ) *satisfying* $v=2k$ *or* $2k\pm 1$ *implies the existence of two optimum chemical balance weighing designs; one for* v *objects in* $4(r-\lambda)$ *weighings and the other for* $v+1$ *objects in* $4(r-\lambda)$ *weighings.*

Remark. The complement of a BIBD$(v=2k+1)$ is a BIBD$(v=2k-1)$ and conversely. Furthermore, the value of $r-\lambda$ is invariant by the complementary operation. So, it is sufficient to consider Corollary 2 for $v=2k$ and $2k+1$ or $2k-1$.

There are a large number of BIB designs satisfying $v=2k$ or $v=2k-1$ (Takeuchi (1962), Kageyama & Saha (1983)). Incidentally, we present some series of such BIB designs (Raghavarao (1971), Sprott (1956)):

(i) $v = b = 4t+3$, $r = k = 2t+1$, $\lambda = t$ (when $4t+3$ is a prime or a prime power);

(ii) $v = 4t+1$, $b = 8t+2$, $r = 4t$, $k = 2t$, $\lambda = 2t-1$ (when $4t+1$ is a prime or a prime power);

(iii) $v = 4(t+1)$, $b = 2(4t+3)$, $r = 4t+3$, $k = 2(t+1)$, $\lambda = 2t+1$ ($4t+3$ is a prime or a prime power).

Note (Kageyama (1980)) that a BIBD $(v = 2k)$ exists (i) for even $k \leqq 132$ and (ii) for odd $k \leqq 67$. Hence, we can produce many optimum chemical balance weighing designs by the present approach.

4. Concluding Remarks

One can easily show that a necessary condition for the existence of an optimum chemical balance weighing design $W(v, b)$ is that b be a multiple of four. Furthermore, by using the same idea as in the above, one can prove the following stronger result.

Proposition. *The existence of a* $W(v, b)$ *is equivalent to the existence of an orthogonal array with parameters* $(v-1, b, 2, 2; \mu = b/4)$.

Sketch of proof. The necessity is shown after changing the first row of the $W(v, b)$ to all $+1$'s and deleting the row with replacement of -1 by 0. The sufficiency is obvious.

Thus, the whole problem of constructing a $W(v, b)$ from the incidence matrices of incomplete block designs reduces to constructing orthogonal arrays from such 0–1 matrices.

We conclude this section with a few observations on the paper of Dey (1971) referred to in Section 1. He shows that a BIBD (v, b, r, k, λ) satisfying $v \neq 2k$ and $b = 4(r-\lambda)$ is an optimum chemical balance weighing design. He also shows that a BIBD (v, b, r, k, λ) with $v = (1/2)[4k+1 \pm \sqrt{8k+1}]$ is an optimum chemical balance weighing design. Now, one can easily see that $4(r-\lambda) - b \geqq 0$ iff $v \geqq (v-2k)^2$, and the equalities are both attained at the same time. Obviously $b = 4(r-\lambda)$ implies $v \neq 2k$. Hence, the condition $v \neq 2k$ in his first result is redundant. Again $v = (1/2)[4k+1 \pm \sqrt{8k+1}]$ is a solution of the quadratic equation in $v: v = (v-2k)^2$ which is equivalent to $b = 4(r-\lambda)$, and hence his second result is the same as the first result.

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