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Quaternionic Representations of Compact Metric Groups

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Representations of compact metric groups in Hilbert spaces over the quaternions are studied. A generalization of the Peter–Weyl theorem is formulated and proved. The problem of finding all the irreducible quaternionic representations of an arbitrary compact metric group is solved, and a rule is given for computing the "Q-characters" of all the irreducible quaternionic representations once the characters of all the irreducible complex representations are known. For the Abelian case, it is shown that every irreducible quaternionic representation is equivalent to a complex representation and hence one dimensional. An example is given of a non-Abelian group whose irreducible quaternionic representations are all one dimensional.

I. INTRODUCTION

It is well known (see, e.g., Birkhoff and von Neumann,[1] Yang,[2] Mackey,[3] Michel[4]) that the lattice of closed linear manifolds of a quaternionic Hilbert space is a possible candidate for the logic of propositions (see Varadarajan[5]) of a quantum mechanical system, and that there is nothing canonical about the (classical) choice of the complex number system for the development of quantum mechanics. But, in spite of the wide-spread knowledge of this fact, very little work has been done toward setting up a theory of quaternionic quantum mechanics apart from the fundamental work[6-9] of Finkelstein, Jauch, Speiser, and Schiminovitch. We hope that our present work is of some help in this context, as the theory of group representations is indispensable for the exposition of quantum mechanics and compact metric groups are an important special case.

II. PRELIMINARY IDEAS

We present this section in some detail as our orientation differs from that of Finkelstein *et al.*

Let Q denote the division ring of real quaternions. We denote an arbitrary element q of Q by $q = q_0 + q_1 i + q_2 j + q_3 k$, where q_0, q_1, q_2, q_3 are real. We identify the reals with the set of all quaternions q with $q_1 = q_2 = q_3 = 0$ and the complex numbers with the set of all quaternions q with $q_2 = q_3 = 0$. Every $q \in Q$ may be written in the form $\alpha + \beta j$, where α and β are complex. We denote by q^* the conjugate of the quaternion q.

1. Vector Spaces

By a vector space over Q (to be called a Q-space) we always mean a left-vector space over Q. A Q-Banach space is a complete normed Q-space. If X is a topological space, we denote by $C_Q(X)$ the Q-Banach space of all bounded quaternion-valued continuous functions on X with the supremum norm.

An inner product on a Q-space V is a quaternion-valued function on V × V, denoted by $(\ldots)$, with the properties:

(i) $(x, y) = (y, x)^*$,

(ii) $(px + p'x', y) = p(x, y) + p'(x', y)$,

(iii) $(x, x) \geq 0, = 0$ if and only if $x = 0$,

where $x, x', y \in$ V, and $p, p' \in Q$. From (i) and (ii) we have

$$(x, py + p'y') = (x, y)p^* + (x, y')p'^*.$$

It is easy to prove that, on an inner product Q-space, $\|x\| = (x, x)^{\frac{1}{2}}$ defines a norm.[7] A Q-space V is called a Q-Hilbert space if there exists an inner product on V such that the induced norm makes V a complete normed Q-space. The concepts of orthogonality, basis, etc., for Q-Hilbert spaces are defined in the usual way. In what follows H denotes a Q-Hilbert space.

An operator on H is a bounded linear transformation of H into itself. An automorphism of H is a bijective operator on H. For every automorphism A, there exists an unique automorphism A^{-1} such that $AA^{-1} = A^{-1}A = I$. The set of all automorphisms is a group in a natural way.

[1] G. Birkhoff and J. von Neumann, Ann. Math. 37, 823 (1936).
[2] C. N. Yang, in *Proceedings of the Seventh Annual Rochester Conference* (Interscience Publishers, Inc., New York, 1957), p. IX-26.
[3] G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics* (W. A. Benjamin, Inc., New York, 1963), p. 73.
[4] L. Michel, *Invariance in Quantum Mechanics and Group Extension, Group-Theoretical Concepts and Methods in Elementary Particles* (Gordon and Breach Science Publishers, Inc. New York, 1964), p. 148.
[5] V. S. Varadarajan, Indian Statistical Institute preprint (1965), p. 207.
[6] D. Finkelstein, J. M. Jauch, and D. Speiser, "Notes on Quaternion Quantum Mechanics I, II, and III", CERN (1959).
[7] D. Finkelstein, J. M. Jauch, S. Schiminovitch, and D. Speiser, J. Math. Phys. 3, 207 (1962).
[8] D. Finkelstein, J. M. Jauch, S. Schiminovitch, and D. Speiser, J. Math. Phys. 4, 788 (1963).
[9] D. Finkelstein, J. M. Jauch, and D. Speiser, J. Math. Phys. 4, 136 (1963).

The elementary theory of Q-Hilbert spaces can now be developed as in the complex case. We note in particular that, for every operator A on H, there exists a unique operator A^* on H such that $(Ax, y) = (x, A^*y)$ for all $x, y \in$ H. A^* is called the adjoint of A. An operator A on H is called Hermitian if $A = A^*$ and unitary if $AA^* = A^*A = I$.

The spectral theory of Hermitian operators in Q-Hilbert spaces parallels the theory in the complex case.[6,7]

Let now V be a finite-dimensional Q-space. (Note that V may be endowed with a Q-Hilbert space structure.) Given a basis $(e_1, \cdots, e_n)$ of V, every linear transformation A on V has a matrix representation (a_{rs}), defined by

$$Ae_s = \sum_r a_{rs} e_r .$$

If A and B are two linear transformations with matrices (a_{rs}) and (b_{rs}), respectively, then the matrix of AB is given by (c_{rs}), where

$$c_{rs} = \sum_t b_{ts} a_{rt} .$$

Observe that our rule for matrix multiplication differs from the usual rule for matrices over a field.

If A has the matrix (a_{rs}) with respect to an orthonormal basis (e_r), then $a_{rs} = (Ae_s, e_r)$. The matrix of A^* with respect to the same basis is then (b_{rs}), where $b_{rs} = (A^*e_s, e_r) = a^*_{sr}$. If A is Hermitian, then $A = A^*$ and hence $a_{rs} = a^*_{sr}$. If A is unitary, $A^*A = AA^* = I$ and hence

$$\sum_t a^*_{ts} a_{rt} = \delta_{rs} = \sum_t a_{ts} a^*_{tr} .$$

We note here that, if A has the matrix (a_{rs}) with respect to a basis (e_r), then

$$\operatorname{Re}(\operatorname{tr} A) = \operatorname{Re}\big(\sum_r a_{rr}\big)$$

is defined independently of the basis (e_r).

2. The Symplectic Picture

It is convenient for our purposes to restate the usual definition[10] in geometric language.

If V is a Q-space, then the additive group of V can be considered as a C-space (i.e., a vector space over the complex numbers). This we denote by V^C and call the symplectic picture of V. If $(e_1, \cdots, e_n)$ is a basis for V, then $(e_1, \cdots, e_n, je_1, \cdots, je_n)$ is a basis for V^C. Hence V^C is of dimension $2n$. A linear transformation A on V is also a linear transformation on V^C. This we denote by A^C. If the matrix of A with respect to the basis $(e_1, \cdots, e_n)$ is $A_1 + A_2 j$, where A_1 and A_2 are complex matrices, then the matrix of A^C with respect to the basis $(e_1, \cdots, e_n, je_1, \cdots, je_n)$ is

$$\begin{vmatrix} A_1 & A_2 \\ -\bar{A}_2 & \bar{A}_1 \end{vmatrix},$$

where $\bar{\alpha}$ denotes the complex conjugate of the complex number α and $\bar{B} = (\bar{b}_{rs})$ if B is the complex matrix (b_{rs}).

3. Integration Theory

Let (X, Σ, μ) be a measure space. We always identify functions which differ only on μ-null sets. A quaternion-valued measurable function

$$f(x) = f_0(x) + f_1(x)i + f_2(x)j + f_3(x)k$$

on X, where f_r [$r = 0, 1, 2, 3$, are real-valued (measurable) functions on X] is said to be integrable with respect to μ if and only if f_0, f_1, f_2, f_3 are integrable with respect to μ. If f is integrable, the integral of f with respect to μ is defined as

$$\int f\,d\mu = \int f_0\,d\mu + \left(\int f_1\,d\mu\right)i + \left(\int f_2\,d\mu\right)j + \left(\int f_3\,d\mu\right)k.$$

The following properties of the integral are easily verified ($q \in Q$ is arbitrary):

(i) $\int (f + g)\,d\mu = \int f\,d\mu + \int g\,d\mu,$

(ii) $\int (pfq)\,d\mu = p\left(\int f\,d\mu\right)q,$

(iii) $\left(\int f\,d\mu\right)^* = \int f^*\,d\mu,$

(iv) $\left|\int f\,d\mu\right| \le \int |f|\,d\mu.$

The only nontrivial relation is (iv). This may be proved by a slight modification of Cramér's proof[11] for the complex case.

We define $L^2_Q(X)$ as the set of all quaternion-valued measurable functions f such that $|f|^2$ is integrable with respect to μ. It follows that $f \in L^2_Q(X)$ implies that $f^* \in L^2_Q(X)$. If we define for f and g in $L^2_Q(X)$ $(f, g) = \int fg^*\,d\mu$ then $L^2_Q(X)$ becomes a Q-Hilbert space with $(\cdot,\cdot)$ as inner product.

If $f, g \in L^2_Q(X)$ and $\int f^*g\,d\mu = 0$, we say that f and g are left orthogonal. If f and g are also orthogonal, we say that f and g are bothways orthogonal.

[10] C. Chevalley, *Theory of Lie Groups, I* (Princeton University Press, Princeton, New Jersey, 1946), p. 18.

[11] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, New Jersey, 1946), p. 65.

We note that if $f \in L_Q^2(X)$ and $p \in Q$, then $fp \in L_Q^2(X)$. If f and g are left orthogonal then fp and gq are left orthogonal for any $p, q \in Q$.

III. Q-REPRESENTATIONS

In what follows, we denote by G a compact metric group and by μ the unique normalized Haar measure on Σ, the class of Borel sets of G.

Let H be a separable Q-Hilbert space and A(H) the group of automorphisms of H. By a Q-representation[9] A of G in H we mean a homomorphism $g \to A_g$ from G to A(H) such that $g \to A_g x$ from G to H is continuous for every fixed $x \in$ H. The Q-representation A is called unitary if A_g is unitary for every $g \in G$. An example of a Q-representation of G in $L_Q^2(G)$ is the right regular representation. This is, in fact, unitary.

When H is finite-dimensional, we may, on occasion, regard the A_g as matrices with respect to some fixed basis of H.

The notions of equivalence, irreducibility, etc., of Q-representations are defined in the usual way.[12]

We now state some basic theorems. The departure from the complex case is only slight and so we omit the proofs.

Theorem 1: Any Q-representation A of G in H is equivalent to a unitary Q-representation.

Theorem 2: Every unitary Q-representation of G is a direct sum of irreducible unitary Q-representations of G. Every irreducible Q-representation of G is finite-dimensional.

The irreducible unitary Q-representations of G split up into equivalence classes in a natural way. We shall index these equivalence classes by α. (It follows from our analysis that the set of all α's is countable.) Let n_α be the dimension of any irreducible Q-representation of type α.

Consider now a unitary Q-representation of G in H. Let

$$\mathsf{H} = \bigoplus_i S_i$$

be a direct sum decomposition of H into irreducible subspaces and let the irreducible subspaces S_i of type α be indexed by a set of cardinality c_α. We call c_α the multiplicity of type α in the decomposition

$$\mathsf{H} = \bigoplus_i S_i.$$

[12] G. W. Mackey, "Theory of Group Representations," Lecture Notes, The University of Chicago (1955), p. 3.

Theorem 3: In any decomposition of H into irreducible subspaces the same types occur with the same multiplicities.

Schur's Lemma[9]*:* Let H_1 and H_2 be two finite-dimensional Q-spaces. Let (A_α) and (B_β) be irreducible collections of linear transformations on H_1 and H_2, respectively. If M is any linear transformation from H_1 to H_2 such that $(B_\beta M) = (MA_\alpha)$, then M is either 0 or an isomorphism.

Corollary 1: If U and V be two inequivalent irreducible unitary Q-representations of G in Q-Hilbert spaces H_1 and H_2, respectively, then

$$\int (V_g M U_g^{-1} x, y)\, dg = 0, \quad x \in \mathsf{H}_1, \quad y \in \mathsf{H}_2$$

for any linear transformation M from H_1 to H_2.

Corollary 2: Let U be an irreducible unitary Q-representation of G in a Q-Hilbert space H of dimension n. Then for any Hermitian operator M of H into itself

$$\int (U_g M U_g^{-1} x, y)\, dg = \frac{\mathrm{Re}\,(\mathrm{tr}\, M)}{n}(x, y).$$

Remark: Note that with our geometric approach Corollary 2 may be proved directly without invoking the *ersatz* determinant used by Finkelstein *et al.*[9]

IV. ORTHOGONALITY RELATIONS AND THE PETER–WEYL THEOREM

We now begin an analysis of the irreducible (and hence finite-dimensional) Q-representations of a compact metric group G.

Let A be an irreducible Q-representation of G in H of dimension n and let $[a_{rs}(g)]$ be the matrix of A_g with respect to an orthonormal basis (e_r). The function $a_{rs}(g) = (A_g e_r, e_s)$ is a continuous function on G for every r, s; i.e., the matrix entries $[a_{rs}(.)]$ of A with respect to an orthonormal basis are continuous. It follows that the matrix entries of A with respect to any basis of H are continuous, i.e., are elements of $\mathsf{C}_Q(G)$ and hence of $L_Q^2(G)$.

We know that (see Theorem 24 in Ref. 13) in the complex case the matrix entries of two inequivalent irreducible unitary representations are orthogonal. A similar result holds in the quaternionic case. To see this, let U and V be inequivalent irreducible unitary Q-representations acting on Q-Hilbert spaces H and K, respectively, and let $u_{rs}(g)$ [respectively $v_{rs}(g)$] be the matrix entries of $U_g(V_g)$ with respect to the orthonormal basis $(e_r)[(f_r)]$. If M: H $\to$ K is the linear

[13] L. Pontrjagin, *Topological Groups* (Princeton University Press, Princeton, New Jersey, 1958).

transformation defined by $Me_t = f_w$, $Me_s = 0$ if $s \neq t$, then by Corollary 1 to Schur's Lemma

$$0 = \int (V_g M U_g^{-1} e_r, f_v)\, dg = \int u_{rt}^*(g) v_{vw}(g)\, dg,$$

and also by the invariance of the integral,

$$= \int (V_g^{-1} M U_g e_r, f_v)\, dg = \int u_{tr}(g) v_{wv}^*(g)\, dg.$$

In words, every matrix entry of U is bothways orthogonal to every matrix entry of V.

To study the orthogonal relations between the matrix entries of a single representation U, let $M: \mathsf{H} \to \mathsf{H}$ be the linear transformation defined by $Me_t = e_w$ and $Me_s = 0$ if $s \neq t$. Then we have, as above,

$$\int (U_g M U_g^{-1} e_r, e_s)\, dg = \int u_{rt}^*(g) u_{sw}(g)\, dg = \int u_{tr}(g) u_{ws}^*(g)\, dg. \quad \text{(A)}$$

In case $r = s$ and $t = w$, M is Hermitian with $\mathrm{Re}\,(\mathrm{tr}\, M) = 1$ and so, from Corollary 2 to Schur's Lemma, it follows that

$$\int |u_{ws}(g)|^2\, dg = \frac{1}{n} \quad \text{for all } w, s.$$

Further, Eq. (A) shows that the n^2 matrix entries are mutually orthogonal if and only if they are mutually left orthogonal. However, in contrast to the complex case, it is not necessary that they be orthogonal, as the following example shows.

Example 1: Let G be the symmetric group of degree 3. The elements of G are

$$g_0 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix},\quad g_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix},\quad g_2 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix},\quad g_4 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix},\quad g_5 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}.$$

Let $I = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, $J = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ and define

$$U_0 = I, \quad U_1 = [(i + j)/\sqrt{2}]J,$$

$$U_2 = [(\sqrt{3} - 1)i/2\sqrt{2} - (\sqrt{3} + 1)j/2\sqrt{2}]J,$$

$$U_3 = [(-1 + \sqrt{3}k)/2]I, \quad U_4 = [(-1 - \sqrt{3}k)/2]I,$$

$$U_5 = [-(\sqrt{3} + 1)i/2\sqrt{2} + (\sqrt{3} - 1)j/2\sqrt{2}]J.$$

Then the representation $g_t \to U_t$ is unitary and, moreover, irreducible, because the only vector sent into a multiple of itself by all the U_t is the null vector. Since in each matrix the two elements in the principal diagonal are equal, two of the matrix entries are identical.

Let $A_\alpha = [a_{rs}(g)]$ be any irreducible Q-representation of G of type α. Define

$$\mathsf{F}_\alpha = \mathrm{Span}\,[a_{rs}(g)q : 1 \le r, s \le n_\alpha, q \in Q].$$

It is easy to check that F_α depends only on the type α of the representation and not on the particular representation chosen. We call F_α the space of matrix entries of type α. Since every element of the generating set of F_α is a (real) linear combination of the $4n_\alpha^2$ elements of the type

$$a_{rs}(g),\ a_{rs}(g)i,\ a_{rs}(g)j,\ a_{rs}(g)k, \quad 1 \le r, s \le n_\alpha,$$

F_α is a closed linear manifold of $\mathsf{L}_Q^2(G)$ of dimension at most $4n_\alpha^2$ (see also Theorem 11, this paper).

The following theorem generalizes the Peter–Weyl theorem to the quaternionic case.

Theorem 4: The subspaces F_α and F_β are bothways orthogonal if $\alpha \neq \beta$. If $\sum_\alpha \mathsf{F}_\alpha$ denotes the set of finite sums of elements of $\bigcup_\alpha \mathsf{F}_\alpha$, where α ranges over all types and $\overline{\sum_\alpha \mathsf{F}_\alpha}$ the uniform closure of $\sum_\alpha \mathsf{F}_\alpha$, then

$$\overline{\sum_\alpha \mathsf{F}_\alpha} = \mathsf{C}_Q(G) \quad \text{and} \quad \bigoplus_\alpha \mathsf{F}_\alpha = \mathsf{L}_Q^2(G).$$

Proof: Let $[u_{rs}(g)]$, $[v_{rs}(g)]$ be unitary representations of types α and β respectively. For any $p, q \in Q$

$$\int [u_{rs}(g)p][v_{tw}(g)q]^*\, dg = pq^* \int [(pq^*)^{-1} u_{rs}(g) pq^*] v_{tw}^*(g)\, dg = 0,$$

by the orthogonality relations proved earlier, since, for any quaternion q, the representation $[q^{-1}u_{rs}(g)q]$ is equivalent to $[u_{rs}(g)]$. Since the elements of F_α and F_β are linear combinations of elements of the form $[u_{rs}(g)p]$ and $[v_{tw}(g)q]$, respectively, we have shown that F_α and F_β are orthogonal. To prove that F_α and F_β are left orthogonal, it is enough to show that, for $p, q \in Q$, $pu_{rs}(g)$ and $qv_{tw}(g)$ are left orthogonal. But

$$\int [pu_{rs}(g)]^* [qv_{tw}(g)]\, dg = \int u_{rs}^*(g)[p^* q v_{tw}(g)(p^* q)^{-1}]\, dg (p^* q) = 0.$$

For the second part, let us denote by Δ the set of all real functions arising from all possible real representations of G. Then (Ref. 13, p. 119) the finite real linear combinations of elements of Δ are dense in $C_R(G)$, the Banach space of real-valued continuous functions on G. It follows that finite quaternion linear combinations of elements of Δ are dense in $C_Q(G)$.

Therefore, to prove that

$$\sum_\alpha F_\alpha = C_Q(G),$$

it is enough to show that every function in Δ is a linear combination (and hence a finite sum) of functions in $\cup_\alpha F_\alpha$. But since every real representation A is equivalent to a direct sum of irreducible Q-representations and since the matrix entries of irreducible Q-representations belong to $\cup_\alpha F_\alpha$, it follows that every matrix entry of A and hence every element of Δ is a linear combination of elements of $\cup_\alpha F_\alpha$.

Since $C_Q(G)$ is dense in $L^2_Q(G)$ and uniform convergence implies L^2-convergence and since the F_α are mutually orthogonal subspaces of $L^2_Q(G)$, we have

$$L^2_Q(G) = \bigoplus_\alpha F_\alpha .$$

Corollary: There exists at most a countable number of inequivalent irreducible Q-representations of G.

Proof: $L^2_Q(G)$ is separable.

The following theorem (cf. Ref. 13, p. 120) may now be proved exactly as in the complex case.

Theorem 5: We select one representative from each equivalence class of irreducible Q-representations of G and denote them by

$$U^{(1)}, \cdots, U^{(n)}, \cdots.$$

Then for every element $g \in G$ distinct from the identity, there exists an n such that $U^{(n)}_g$ is not the identity transformation.

V. Q-CHARACTERS

Let $A_g = [a_{rs}(g)]$ be a Q-representation of G of degree n. Define

$$X(A_g) = \mathrm{Re}\,[\sum_r a_{rr}(g)].$$

Then it is easy to see that if A and B are equivalent Q-representations, then $X(A_g) = X(B_g)$. In this way we may associate with every equivalence class of Q-representations a real-valued function $X(g)$ which we call (see also Finkelstein, Jauch, and Speiser[9]) its Q-character (to distinguish it from the usual definition of the character of a complex representation which we call the C-character). We denote by $X_\alpha(g)$ the Q-character of any irreducible Q-representation of type α. Note that if A_g is of type α, then

$$X_\alpha(g) = \tfrac{1}{4} \sum [a_{rr}(g) + i a_{rr}(g) i^* + j a_{rr}(g) j^* + k a_{rr}(g) k^*] \in F_\alpha .$$

Thus we have the following theorem.

Theorem 6: Two irreducible Q-representations are equivalent if and only if they have the same Q-character. Moreover, Q-characters of inequivalent irreducible Q-representations are orthogonal.

VI. CLASSIFICATION OF IRREDUCIBLE Q-REPRESENTATIONS

We now proceed to study the inter-relations between the irreducible Q-representations and the irreducible C-representations of G. Let B be an irreducible C-representation of G and $\bar{B}$ its contragredient.[14] Recall that (if χ denotes the complex character) $\chi(\bar{B}_g) = \overline{\chi(B_g)}$. B satisfies exactly one of the following three conditions[15,16]:

(a) B is not equivalent to $\bar{B}$.

(b) There exists a matrix M such that $M = M^T$ (the transpose of M) and $MB_gM^{-1} = \bar{B}_g$ for all $g \in G$.

(c) There exists a matrix M such that $M = -M^T$ and $MB_gM^{-1} = \bar{B}_g$ for all $g \in G$. We say (cf. Ref. 16) that B is nonreal, potentially real or pseudoreal according as it satisfies (a), (b), or (c).

Note that every C-matrix representation B may be considered to be a Q-matrix representation since we have identified the complex field with a fixed subfield of the quaternions. However, even if B is irreducible as a C-representation, it need not be irreducible as a Q-representation. The following theorem[9] gives a necessary and sufficient condition.

Theorem 7: An irreducible C-representation B is an irreducible Q-representation if and only if B is not pseudoreal. If B is pseudoreal, then B decomposes over Q into the direct sum of two equivalent irreducible Q-representations.

Consider now an irreducible Q-representation A of G. We say that A is (i) of class R if it is equivalent to a real representation, (ii) of class C if it is equivalent to a C-representation but not equivalent

[14] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1931), p. 123.

[15] G. Frobenius and I. Schur, Sitzber. Akad. Wiss. Berlin Kl. Phys. Math. 186 (1906).

[16] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), p. 285 *et seq.*

to any real representation, and (iii) of class Q if it is neither of class R nor of class C. The following three theorems establish correspondences between the various classes of irreducible Q-representations and C-representations.

Theorem 8: A Q-representation is of class R if and only if it is equivalent to a potentially real representation. Two potentially real representations are Q-inequivalent if and only if they are C-inequivalent.

Proof: Since a C-representation is potentially real if and only if it is equivalent to a real representation,[14] the first part follows. For the second part, we have only to note that the C-character of a potentially real representation is real and hence equal to its Q-character.

Theorem 9: A Q-representation is of class C if and only if it is equivalent to a nonreal representation. Two nonreal representations B and C are Q-inequivalent if and only if B is C-inequivalent to both C and $\bar{C}$.

Proof: If A be a Q-representation of class C, Q-equivalent to a C-representation B, then it is clear that B cannot be potentially real. Also, since B is Q-irreducible, B cannot be pseudoreal by Theorem 7. Hence B must be nonreal. To prove the converse, we have only to show that a nonreal representation B cannot be Q-equivalent to a potentially real representation D. But this is evident, since $X(B_g) = \frac{1}{2}[\chi(B_g) + \chi(\bar{B}_g)]$ is orthogonal to $X(D_g) = \chi(D_g)$, using the classical orthogonality relations.

If B and C are Q-inequivalent, then $X(B_g)$ is not equal to $X(C_g)$ and hence $\chi(B_g)$ is not equal to either $\chi(C_g)$ or $\chi(\bar{C}_g)$, i.e., B is C-inequivalent to both C and $\bar{C}$. Conversely, if B is C-inequivalent to both C and $\bar{C}$, then $X(B_g) = \frac{1}{2}[\chi(B_g) + \chi(\bar{B}_g)]$ is orthogonal to $X(C_g) = \frac{1}{2}[\chi(C_g) + \chi(\bar{C}_g)]$ and hence B and C are Q-inequivalent.

We now turn our attention to pseudoreal representations. If B is one such, then by Theorem 7, $B = B^1 \odot B^2$ where B^1 and B^2 are equivalent irreducible Q-representations. Since $\chi(B)$ is real, $X(B^1_g) = \frac{1}{2}\chi(B_g)$ and hence the equivalence class of B^1 is uniquely determined by B. We call any member of this equivalence class a Q-representation induced by B.

Theorem 10: A Q-representation A is of class Q if and only if it is induced by a pseudoreal representation. Two pseudoreal representations are C-inequivalent if and only if their induced Q-representations are Q-inequivalent.

Proof: Let the Q-representation of class Q of dimension n act on the Q-space V. We may assume that A is unitary. Then $g \to A^C_g$ is a unitary C-representation of G in V^C.

We first prove that $g \to A^C_g$ is irreducible. If it is not, let $(e_1, \cdots, e_r)$ be a basis in V^C of some invariant subspace S for A^C. Since A^C is unitary, by replacing S by $S^\perp$ if necessary, we may assume that $r \leq n$. The Q-subspace spanned by $(e_1, \cdots, e_r)$ in V is then invariant under A. Since A is irreducible, we can conclude that $r = n$. But then the matrix of A_g with respect to $(e_1, \cdots, e_n)$ is the matrix of A^C_g restricted to S with respect to $(e_1, \cdots, e_n)$ which is complex—a contradiction since A is of class Q. Hence A^C is irreducible.

We show next that A^C is pseudoreal. If A_g has the matrix $A^1_g + A^2_g j$ (where A^1_g and A^2_g are complex) with respect to some basis in V, then with respect to the corresponding basis in V^C, A^C_g has the matrix

$$\begin{vmatrix} A^1_g & A^2_g \\ -\bar{A}^2_g & \bar{A}^1_g \end{vmatrix}$$

Since A^C_g is unitary, $\bar{A}^C_g$ has the matrix

$$\begin{vmatrix} \bar{A}^1_g & \bar{A}^2_g \\ -A^2_g & A^1_g \end{vmatrix}$$

The matrix

$$M = \begin{vmatrix} 0 & -I \\ I & 0 \end{vmatrix}$$

has the properties $M = -M^T$ and $MA^C_gM^{-1} = \bar{A}^C_g$, i.e., A^C is pseudoreal.

Since the equality $X(A_g) = \frac{1}{2}\chi(A^C_g)$ is evident by looking at the matrices of A_g and A^C_g, we conclude that A is induced by A^C.

Conversely, if B is a pseudoreal representation inducing the Q-representation A, then A has to be of class Q. For, if not, we may assume, by what has been proved so far, that A is either a potentially real or a nonreal representation. In either case $\chi(A)$ is orthogonal to $\chi(B) = 2X(A) = 2 \operatorname{Re}[\chi(A)]$—a contradiction.

The second part is proved by a comparison of characters.

To sum up, the situation is as follows: There is a one-to-one correspondence between the equivalence classes of potentially real (respectively pseudoreal) representations and the equivalence classes of Q-representations of class R (class Q). There is a one-to-one correspondence between pairs of equivalence classes of nonreal representations, each pair consisting of the equivalence classes of a representation and its contragredient, and the equivalence classes of Q-representations of class C.

This leads us to the following rule for the computation of irreducible Q-characters. Recall that an irreducible C-representation with character χ is nonreal, potentially real or pseudoreal according as

$$\int \chi(g^2)\,dg = 0 \qquad (1)$$

$$= +1 \qquad (2)$$

or

$$= -1. \qquad (3)$$

Rule: Every real irreducible C-character $\chi(g)$ determines an irreducible Q-character $X(g) = \chi(g)$ or $\frac{1}{2}\chi(g)$ according as χ satisfies (2) or (3). Every nonreal irreducible C-character $\chi(g)$ determines an irreducible Q-character $X(g) = \mathrm{Re}\,[\chi(g)]$. All the irreducible Q-characters are obtained in this way.

In the complex case, a C-character χ is irreducible if and only if its (L^2-) norm is unity. For the quaternionic case, we may show that the square of the norm of an irreducible Q-character is 1, $\frac{1}{2}$, or $\frac{1}{4}$ according as the corresponding representation is of class R, C, or Q. This does not in general give us a criterion for deciding the irreducibility of an arbitrary finite-dimensional Q-representation, but if the square of the norm of its Q-character is $\frac{1}{4}$, we can conclude that the representation is irreducible and is of class Q.

Every Q-character $X(g)$ is an invariant function, i.e., $X(g) = X(hgh^{-1})$ for all $h \in G$. In contrast to the complex case, it is not in general true that the irreducible Q-characters form a basis for the subspace of invariant functions I in $L^2_Q(G)$. In Example 2 of Sec. VII, for instance, there are only five irreducible Q-characters, whereas $L^2_Q(G)$ is of dimension 8. However, since, as is easily checked, the irreducible C-characters form a basis for I, we may conclude from our analysis that the irreducible Q-characters form a basis for I if and only if every irreducible C-character is real. This happens, for instance, when $G = SO(3)$.

In passing we note that $SO(3)$ does not admit of any irreducible Q-representation of class Q, since it does not admit of any irreducible C-representation of even degree.

We conclude this section with the following result.

Theorem 11: If A is an irreducible Q-representation of type α and degree n, then the subspace F_α has dimension n^2, $2n^2$, or $4n^2$ according as A is of class R, C, or Q.

Proof: If A is of class R, we may assume that A is real and orthogonal. Since the (real-valued) matrix entries of A are then orthogonal and the reals commute with all the quaternions, F_α is of dimension n^2.

If A is of class C, then again we may take A to be complex and unitary. If A_α has the matrix $[a_{rs}(g)]$, its contragredient has the matrix $[\bar{a}_{rs}(g)]$. By definition, every element of F_α is a linear combination of elements of the form $a_{rs}(g)(\beta + \gamma j) = \beta a_{rs}(g) + \gamma j \bar{a}_{rs}(g)$, where β and γ are complex. Again using the classical orthogonal relations, we may conclude that F_α is of dimension $2n^2$.

Now, let A be in class Q. Consider A^C. By Theorems 7 and 10, there exists a matrix M such that

$$MA^C_\alpha M^{-1} = \begin{vmatrix} B_\alpha & 0 \\ 0 & C_\alpha \end{vmatrix},$$

where B and C are equivalent to A. Therefore, F_α is spanned by the right Q-multiples of the matrix entries of MA^CM^{-1} and hence of A^C. But the set of matrix entries of A^C is closed (except possibly for sign) with respect to complex conjugation and by the same method used earlier in the proof, we can conclude that F_α is spanned by the matrix entries of A^C. But, by Theorem 10 again, A^C is an irreducible C-representation. Invoking the classical orthogonal relations once more, we conclude that F_α is of dimension $4n^2$.

VII. ABELIAN GROUPS

Let now G denote a compact metric Abelian group. Since every irreducible C-representation of G is one dimensional, it follows from Theorem 10 that G does not admit of any irreducible Q-representations of class Q, i.e., every irreducible Q-representation of G is equivalent to a C-representation. It follows immediately that every irreducible Q-representation of G is one dimensional. However, in contrast to the complex case, it is not true that if every irreducible Q-representation of a compact metric group G is one dimensional, then G is Abelian, as the following example shows. We denote by G^0 the group opposite to G (i.e., the elements of G^0 are those of G and the group operation in G^0 is given by $g \cdot h = hg$).

Example 2: Let G be the quaternion group, i.e., $G = [\pm 1, \pm i, \pm j, \pm k]$. Consider G^0. We show that every irreducible Q-representation of G^0 is one dimensional.

If $q \in Q$, let R_q denote the linear transformation of the Q-space Q, given by $R_q(p) = pq$ for all p in Q.

Consider the representations:

(1) $g \to R_g$;
(2) $g \to A_2 = R_1$ for all $g \in G^0$;
(3) $g \to A_3 = R_1$ if $g = \pm 1, \pm i$,
$= R_{-1}$ otherwise;
(4) $g \to A_4 = R_1$ if $g = \pm 1, \pm j$,
$= R_{-1}$ otherwise;
(5) $g \to A_5 = R_1$ if $g = \pm 1, \pm k$,
$= R_{-1}$ otherwise.

It is easy to verify that the above five (one-dimensional and hence irreducible) Q-representations are mutually inequivalent. If F_r is the subspace in $L^2_{\mathfrak{Q}}(G^0) = Q^{(8)}$ associated with the rth-representation above, then F_1 has dimension four and each of the remaining F_r has dimension one. It follows that G^0 cannot have any irreducible Q-representation inequivalent to all the five above and in particular that G^0 does not have any Q-representation of degree greater than one.

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