Spectral sum rule for time delay in $\mathbb{R}^2$

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A local spectral sum rule for nonrelativistic scattering in two dimensions is derived for the potential class $v \in L^{4/3}(\mathbb{R}^2)$. The sum rule relates the integral over all scattering energies of the trace of the time-delay operator for a finite region $\Sigma \subset \mathbb{R}^2$ to the contributions in Σ of the pure point and singularly continuous spectra.

I. INTRODUCTION

Spectral sum rules involving the time delay for a region Σ of finite volume and the bound-state density for the same region were derived in Ref. 1 in the context of classical scattering. Here we rigorously derive the quantum mechanical counterpart of these local sum rules in two Euclidean dimensions (see Theorem 4).

We consider the quantum mechanics of a single spinless particle in two dimensions. The state space is a Hilbert space $\mathscr{H} \equiv L^2(\mathbb{R}^2)$, in which K_0 denotes the self-adjoint extension of $-\Delta$ describing the free Hamiltonian of the particle (with $\hbar^2 = 2m = 1$). We shall assume that the potential v, describing the interaction, is a measurable function in $L^{4/3}(\mathbb{R}^2)$.

The total Hamiltonian $H = K_0 + v$ will be defined by the quadratic form method[2] and we write $\mathscr{H}_{ac}(H)$ and $\mathscr{H}_s(H)$ for the absolutely continuous and singular spectral subspaces respectively for the self-adjoint operator H. Also R_z, $\rho(H)$, and E [R_z^0, $\rho(K_0)$, and E^0, respectively] will denote the resolvent, the resolvent set, and the spectral measure, respectively, for H (for K_0). The symbols $\mathscr{B}$, $\mathscr{B}_0$, $\mathscr{B}_2$, and $\mathscr{B}_1$ denote the linear spaces of all bounded, compact, Hilbert–Schmidt, and trace-class operators in $\mathscr{H}$ with $\|\cdot\|$, $\|\cdot\|_2$, and $\|\cdot\|_1$ denoting the operator, Hilbert–Schmidt, and trace norms, respectively. We also set $\mathscr{B}_4 = \{A \in \mathscr{B}_0 | A^* A \in \mathscr{B}_2\}$. Then one has $\mathscr{B}_1 \subseteq \mathscr{B}_2 \subsetneq \mathscr{B}_4 \subsetneq \mathscr{B}_0 \subsetneq \mathscr{B}$. We shall use the factorization scheme $u(x) = |v(x)|^{1/2}$, $w(x) = \operatorname{sgn} v(x) u(x)$, so that $u, w \in L^{8/3}(\mathbb{R}^2)$. The first theorem collects the results relating to the definition of H.

Theorem 1: Let $v \in L^{4/3}(\mathbb{R}^2)$.

(a) For every $\chi^2 > 0$, $u(K_0 + \chi^2)^{-1/2}$ and $w(K_0 + \chi^2)^{-1/2}$ belong to $\mathscr{B}_4$.

(b) The total Hamiltonian $H = K_0 + v$, defined as a quadratic form on $D(K_0^{1/2})$, the domain of $K_0^{1/2}$, can be extended as the quadratic form of a self-adjoint operator, also denoted by H, which is bounded below. Also, $D(|H|^{1/2}) = D(K_0^{1/2})$.

(c) For every $z \in \mathbb{C} - \{0\}$, the integral kernel $A(z)(x,y) \equiv u(x) R_z^0(x,y) w(y)$ defines a $\mathscr{B}_2$ operator, also denoted $A(z)$, which is $\mathscr{B}_2$ holomorphic in the open upper- and lower-half planes separately.

(d) $\|A(z)\|_2 \to 0$ as $|z| \to \infty$, and $A(z)$ has boundary values in $\mathscr{B}_2$ norm as $z \to \lambda \pm i0$, uniformly for λ in every closed subset of $\mathbb{R} - \{0\}$.

(e) For $z \in \rho(H) \cap \rho(K_0)$, $[1 + A(z)]^{-1} \in \mathscr{B}$ and one has the second resolvent equation

$$R_z - R_z^0 = -R_z^0 w [1 + A(z)]^{-1} u R_z^0. \tag{1}$$

Furthermore, the function $z \to [1 + A(z)]^{-1}$ is $\mathscr{B}$ holomorphic in the open upper- and lower-half planes.

Since many of the calculations are standard we only sketch the proof.

Proof: The Green's function for the free Hamiltonian is $R_z^0(x,y) = (i/4) H_0^{(1)}(\sqrt{z}|x-y|)$, where $H_0^{(1)}$ is the Hankel function of the first kind, and where we have chosen the branch of the square root so that $\operatorname{Im}\sqrt{z} > 0$. Using the bound

$$|H_0^{(1)}(\alpha)| < c_0 |\alpha|^{-1/2} e^{-\operatorname{Im}\alpha}, \tag{2}$$

for all $\alpha \in \mathbb{C} - \{0\}$ with $\operatorname{Im}\alpha > 0$ (see Ref. 3, pp. 962 and 963), we have that for $z \in \mathbb{C} - \{0\}$,

$$\|A(z)\|_2 = \frac{1}{16} \int\int dx\, dy |u(x)|^2 |H_0^{(1)}(\sqrt{z}|x-y|)|^2 |w(y)|^2$$
$$< \frac{c_0^2}{16|z|^{1/2}} \int\int dx\, dy \frac{|v(x)|\,|v(y)|}{|x-y|} < \frac{c\|v\|_{4/3}^2}{|z|^{1/2}}, \tag{3}$$

by an application of the Sobolev inequality[4] in $\mathbb{R}^2$. This proves (a) and parts of (c) and (d). The $\mathscr{B}_2$ holomorphy of $A(z)$ follows by writing

$$A(z) = u(K_0 + \chi^2)^{-1/2} [I + (z + \chi^2) R_z^0] \times (w(K_0 + \chi^2)^{-1/2})^*,$$

and observing that while the middle factor is clearly $\mathscr{B}$ holomorphic, each of the other two are $\mathscr{B}_4$.

Part (b) follows from (a) on using standard results on quadratic forms.[2,5,6] The existence of boundary values uniformly in λ is the consequence of an application of the dominated convergence theorem and the estimate (3). The resolvent equation (1) can be established as in Refs. 2 or 7. $\square$

Scattering theory for such a system can be developed along standard lines and the next theorem summarizes the results.

Theorem 2: Let $v \in L^{4/3}(\mathbb{R}^2)$. Set $\mathscr{E} = \{0\} \cup \{\lambda \in \mathbb{R} \setminus \{0\} | I + A(\lambda + i0) \text{ or } I + A(\lambda - i0) \text{ is not } 1\text{–}1\}$.

(a) $\mathscr{E}$ is a closed and bounded set of Lebesgue measure 0.

(b) u is K_0 bounded and w is H bounded. Furthermore uE_Δ^0 and wE_Δ are K_0 and H smooth, respectively (see Refs.

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and 8 for the definition of smoothness), where Δ is any half-open interval in $R - \mathscr{E}$.

(c) The wave operators

$$\Omega_{\pm} = \text{s-lim}_{t\to\pm\infty} e^{+itH} e^{-itK_0},$$

and

$$\Omega'_{\pm} E_{R-\mathscr{E}} = \text{s-lim}_{t\to\pm\infty} e^{itK_0} e^{-itH} E_{R-\mathscr{E}}$$

exist. The scattering system defined by the pair (H, K_0) is asymptotically complete, i.e.,

$$\text{Range}\,\Omega_+ = \text{Range}\,\Omega_- = \mathscr{H}_{ac}(H) = E_{R-\mathscr{E}}\,\mathscr{H},$$

and $\mathscr{H}_s(H) \subseteq E_{\mathscr{E}}\,\mathscr{H}$.

Sketch of the proof: As in Ref. 5, p. 364, the observation that $\|A(\lambda \pm i0)\|_2 \to 0$ as $|\lambda| \to \infty$ and an application of the analytic Fredholm theorem gives us (a). For (b), we use the resolvent equation (1) and note that $\|[I + A(\lambda \pm i\eta)]^{-1}\|$ is bounded, uniformly for $\lambda \in \Delta$ and $0 < \eta < 1$. The part (c) then follows by an application of Kato–Lavine theory (Proposition 9.16 in Ref. 5). □

II. TIME DELAY AND A TRACE THEOREM

Following the reasoning in Ref. 9 we see that the expression

$$\tau(f,\Sigma) = \int_{-\infty}^{\infty} (f, e^{itK_0}[\Omega_+^* P_\Sigma \Omega_+ - P_\Sigma] e^{-itK_0} f)\,dt$$

formally describes the time delay in the state $f \in \mathscr{H}$ for the region $\Sigma \subseteq R^2$, where we have written P_Σ for the orthogonal projection defined by multiplication with the characteristic function χ_Σ.

Let $\mathscr{H}_0 = L^2(T)$, with $(\cdot,\cdot)_0$ denoting the inner product and where T is the unit circle embedded in R^2, and let $\mathscr{U}_0$: $L^2(R^2) \to L^2([0,\infty), \mathscr{H}_0)$ be the spectral transformation (see Ref. 5 for details) for the free Hamiltonian K_0 so that $(\mathscr{U}_0 K_0 f)_\lambda = \lambda(\mathscr{U}_0 f)_\lambda$ for a.a. $\lambda \in [0,\infty)$ and for $f \in D(K_0)$. Then one has the following theorem describing the properties of τ (see Theorem 2 in Ref. 9).

Theorem 3: Let K_0 and H be as described in Theorem 1, and let Σ be a measurable subset of R^2 with finite Lebesgue measure, i.e., $|\Sigma| < \infty$. Then we have the following.

(a) $P_\Sigma R_z^0$ and $P_\Sigma R_z$ are both $\mathscr{B}_2$ operators for every $z \in \rho(H) \cap \rho(K_0)$.

(b) Set $\mathscr{D}_0 = \{ f \in \mathscr{H} \mid \lambda \to \|(\mathscr{U}_0 f)_\lambda\|_0$ is a bounded function of bounded support in $[0,\infty)\}$. Then $\mathscr{D}_0$ is dense in $\mathscr{H}$ and there exists a unique measurable family $Q(\lambda,\Sigma)$ of trace-class operators in $\mathscr{H}_0$, interpreted as the energy-shell time-delay operator, such that

$$\tau(f,\Sigma) = \int_0^\infty ((\mathscr{U}_0 f)_\lambda, Q(\lambda,\Sigma)(\mathscr{U}_0 f)_\lambda)_0\,d\lambda,$$

for every $f \in \mathscr{D}_0$.

(c) Denoting $q(\lambda,\Sigma) = \text{tr}_0\,Q(\lambda,\Sigma)$, the trace of $Q(\lambda,\Sigma)$ in $\mathscr{H}_0$, one has furthermore that

$$\int_0^\infty \frac{|q(\lambda,\Sigma)|}{\lambda^2+1}\,d\lambda < \infty, \tag{4}$$

$$\int_0^\infty \frac{q(\lambda,\Sigma)}{|\lambda - z|^2}\,d\lambda = 2\pi\,\text{tr}\,R_z^{0*}[\Omega_+^* P_\Sigma \Omega_+ - P_\Sigma] R_z^0, \tag{5}$$

and

$$\frac{1}{2\pi}\int_0^\infty q(\lambda,\Sigma)\,\text{Im}\,\frac{1}{\lambda - z}\,d\lambda = \text{tr}\,P_\Sigma\,\text{Im}\,(R_z E_{ac} - R_z^0) P_\Sigma, \tag{6}$$

for every $z \in \rho(H) \cap \rho(K_0)$.

The function $q(\lambda,\Sigma)$ is interpreted as the average time-delay function of energy λ for the region Σ.

Proof: Since $|\Sigma| < \infty$, it is easy to see that $P_\Sigma R_z^0 \in \mathscr{B}_2$. This combined with (1) proves that $P_\Sigma R_z \in \mathscr{B}_2$. Part (b) is proved as in Ref. 9 by using the intertwining relation and noting that

$$R_z^{0*}[\Omega_+^* P_\Sigma \Omega_+ - P_\Sigma] R_z^0 = \Omega_+^* R_z^* P_\Sigma R_z \Omega_+ - R_z^{0*} P_\Sigma R_z^0$$

is a $\mathscr{B}_1$ operator. Equations (4) and (5) are consequences of this as in Ref. 9.

Using the cyclicity of the trace, the asymptotic completeness of Ω_+, and the resolvent equation, we write

$$\begin{aligned}
\text{tr}\,R_z^{0*}[\Omega_+^* P_\Sigma \Omega_+ - P_\Sigma] R_z^0
&= \text{tr}\,R_z^* P_\Sigma R_z \Omega_+ \Omega_+^* - \text{tr}\,R_z^{0*} P_\Sigma R_z^0 \\
&= \text{tr}\,P_\Sigma (R_z E_{ac} R_{\bar z} - R_z^0 R_{\bar z}^0) P_\Sigma \\
&= (z - \bar z)^{-1}\,\text{tr}\,P_\Sigma [(R_z - R_{\bar z}) E_{ac} - (R_z^0 - R_{\bar z}^0)] P_\Sigma,
\end{aligned}$$

which leads to (6). □

III. SUM RULE

A spectral sum rule for the time delay $q(\cdot,\Sigma)$ is derived in this section. It is convenient to introduce a standard notation[6] for the Fourier transform that maps $L^q(R^n)$ $(1 < q < 2)$ into its conjugate space $L^p(R^n)$ $(p^{-1} + q^{-1} = 1)$. The Fourier image of an element $f \in L^q(R^n)$ will be denoted by $\hat f \in L^p(R^n)$. With this notation our main result may be stated as follows.

Theorem 4 (Spectral Sum Rule): (i) Suppose $v \in L^{4/3}(R^2)$ and let Σ be a measurable subset of R^2 with finite Lebesgue measure, i.e., $|\Sigma| < \infty$.

(ii) Assume, furthermore, that $\hat v \hat\chi_\Sigma \in L^1(R^2)$. Then the function $q(\cdot,\Sigma)$: $[0,\infty) \to R$ has a finite improper integral

$$\int_0^\infty q(\lambda,\Sigma)\,d\lambda = \lim_{a\to\infty} \int_0^a q(\lambda,\Sigma)\,d\lambda,$$

which satisfies

$$\int_0^\infty q(\lambda,\Sigma)\,d\lambda = -2\pi\,\text{tr}\,P_\Sigma E_s P_\Sigma - \frac{1}{2}\int_\Sigma v(x)\,dx. \tag{7}$$

Theorem 4 is demonstrated by breaking the proof into three propositions. The basic idea is to apply Cauchy's integral theorem to the holomorphic function $z \to \text{tr}\,P_\Sigma (R_z - R_z^0) P_\Sigma + \text{tr}\,P_\Sigma R_z^0 v R_z^0 P_\Sigma$ on a suitable contour in $\rho(H) \cap \rho(K_0)$. Proposition 5 determines the real axis contribution of $\text{tr}\,P_\Sigma (R_z - R_z^0) P_\Sigma$. The second factor proportional to v is the Born term and its real axis contribution is found in Proposition 6. Finally the large radius contribution of both terms to the Cauchy integral is described in Proposition 9. In Propositions 5, 6, and 9 the set Σ is defined to be a measurable subset of R^2.

Before proceeding to these propositions it is helpful to identify the region in z (the complex energy plane) where Born dominance prevails. Let Π be the canonically cut plane composed of the complex plane with the non-negative reals removed. Theorem 1 (d) shows that $A(z)$, $z\in\Pi$, has $\mathscr{B}_2$-norm continuous extensions to the real axis from either above or below. For positive reals these two extensions are different. Take Π_c to be the closure of the canonically cut plane which maintains the distinction between the two possible boundary values along the positive real axis. The large z bound for $\|A(z)\|_2$ allows the following definition of $\Lambda_\theta < \infty$.

Definition: For each $\theta\in(0,1)$, let Λ_θ be the infinum of the set

$$\{\Lambda \in R^+ \mid \|A(z)\|_2 < \theta < 1, \quad \forall z \in \Pi_c \text{ with } |z| > \Lambda\}.$$

In the Born dominant region of Π_c, i.e., $|z| > \Lambda_\theta$, it is evident that $[1 + A(z)]^{-1}$ is a bounded operator on $\mathscr{H}$ and has norm bound $\|[1 + A(z)]^{-1}\| < (1-\theta)^{-1}$. Thus for each $\theta\in(0,1)$, $\mathscr{E}$ is contained in $[-\Lambda_\theta, \Lambda_\theta]$. Our first proposition describes the behavior of $\operatorname{tr} P_\Sigma \operatorname{Im}[R_{\lambda+i\delta} - R^0_{\lambda+i\delta}] P_\Sigma$ on the finite intervals of the real axis that contain $\mathscr{E}$.

Proposition 5: Suppose $v \in L^{4/3}(\mathbb{R}^2)$ and $|\Sigma| < \infty$. For every finite interval $(a,b) \supset [-\Lambda_\theta, \Lambda_\theta] \supset \mathscr{E}$, $1 > \theta > 0$,

$$\lim_{\delta\to 0^+} \int_a^b d\lambda \operatorname{tr} P_\Sigma \operatorname{Im}[R_{\lambda+i\delta} - R^0_{\lambda+i\delta}] P_\Sigma$$
$$= \frac{1}{2}\int_0^b q(\lambda,\Sigma)\, d\lambda + \pi \operatorname{tr} P_\Sigma E_s P_\Sigma. \tag{8}$$

Proof: Take $\delta > 0$. Theorem 2 (a) and the resolvent equation (1) for R_z implies that $P_\Sigma \operatorname{Im} R_{\lambda+i\delta} P_\Sigma \in \mathscr{B}_1$. The spectral decomposition of $\mathscr{H} = \mathscr{H}_{ac} \oplus \mathscr{H}_s$ (with the associated orthogonal projectors E_{ac} and E_s) leads to

$\operatorname{tr} P_\Sigma \operatorname{Im} R_{\lambda+i\delta} P_\Sigma$

$$= \operatorname{tr} P_\Sigma \operatorname{Im} R_{\lambda+i\delta} E_{ac} P_\Sigma + \operatorname{tr} P_\Sigma \operatorname{Im} R_{\lambda+i\delta} E_s P_\Sigma.$$

Thus (for $\delta > 0$) the left-hand-side integral in (8) is the sum $I_{ac} + I_s$, where

$$I_{ac}(\delta) = \int_a^b d\lambda \operatorname{tr} P_\Sigma \operatorname{Im}[R_{\lambda+i\delta} E_{ac} - R^0_{\lambda+i\delta}] P_\Sigma,$$

$$I_s(\delta) = \int_a^b d\lambda \operatorname{tr} P_\Sigma \operatorname{Im} R_{\lambda+i\delta} E_s P_\Sigma.$$

First, consider the $\delta\to 0^+$ limit of $I_{ac}(\delta)$. Theorem 3, Eq. (6), gives us the representation

$$I_{ac}(\delta) = \frac{1}{2\pi}\int_a^b d\lambda \int_0^\infty d\mu \frac{\delta}{(\mu-\lambda)^2 + \delta^2} q(\mu,\Sigma). \tag{9}$$

The elementary $d\lambda$ integral can be written in either of two equivalent forms:

$$\int_a^b d\lambda \frac{\delta}{(\mu-\lambda)^2+\delta^2} = \tan^{-1}\frac{b-\mu}{\delta} + \tan^{-1}\frac{\mu-a}{\delta}$$
$$= \tan^{-1}\frac{\delta(b-a)}{(\mu-\mu_+)(\mu-\mu_-)}. \tag{10}$$

If $2\delta < b-a$ the roots $\mu_\pm$ are real, given by $2\mu_\pm = b + a \pm [(b-a)^2 - 4\delta^2]^{1/2}$, and always fall inside (a, b). Specifically, $\mu_+ = b - \epsilon_+$ and $\mu_- = a + \epsilon_-$, where $\epsilon_\pm \to 0^+$ as $\delta\to 0$. The inequality $|\tan^{-1}\mu| < |\mu|$ and estimate (4) suffices to establish that the double integral in (9) is absolutely convergent. Fubini's theorem allows a change of integration order whereby (9) becomes

$$I_{ac}(\delta) = \frac{1}{2\pi}\int_0^\infty d\mu\, q(\mu,\Sigma)$$
$$\times\left[\tan^{-1}\frac{b-\mu}{\delta} + \tan^{-1}\frac{\mu-a}{\delta}\right]. \tag{11}$$

Treating the $\mu > 2b$ and the $\mu < 2b$ contributions to integral (11) separately leads to the construction of a δ-independent $L^1(d\mu)$ majorant. For $0 < \delta < 1$ and $\mu > 2b$ a majorizing function is $|q(\mu,\Sigma)|(b-a)[(\mu-b)(\mu-a)]^{-1}$, whereas for $0<\mu<2b$ the bounding function is $\pi|q(\mu,\Sigma)|$. Theorem 3, estimate (4), confirms that this majorant is $L^1(d\mu)$. Dominated convergence now applies to (11) yielding

$$\lim_{\delta\to 0^+} I_{ac}(\delta) = \frac{1}{2}\int_0^b d\mu\, q(\mu,\Sigma). \tag{12}$$

It remains to investigate the limit of $I_s(\delta)$. A useful intermediate result is the following. Suppose $\{C_n\}$ is a sequence of operators in $\mathscr{B}$ converging strongly to C. If $A, B\in\mathscr{B}_2$, then

$$\lim_{n\to\infty} \operatorname{tr} AC_n B = \operatorname{tr} ACB. \tag{13}$$

(See Ref. 5, Lemma 8.23.)

Recall $\sigma_s(H) \subseteq \mathscr{E} \subset [a, b]$. Since $E_{[a,b]} P_\Sigma \in \mathscr{B}_2$, it follows that $E_s P_\Sigma \in \mathscr{B}_2$. The function $[a,b] \ni \lambda \to \operatorname{Im} R_{\lambda+i\delta} \in \mathscr{B}$ is $\mathscr{B}$-norm continuous (for $\delta > 0$) and has a $\mathscr{B}$-valued strong Riemann integral on $[a,b]$. Likewise the map $[a,b] \ni \lambda \to P_\Sigma E_s \operatorname{Im} R_{\lambda+i\delta} E_s P_\Sigma \in \mathscr{B}_1$ is $\mathscr{B}_1$-norm continuous and so $\lambda \to \operatorname{tr} P_\Sigma E_s \operatorname{Im} R_{\lambda+i\delta} E_s P_\Sigma$ has an ordinary Riemann integral on $[a,b]$. By the definition of these two integrals, the linearity of the trace, and (13) it follows that

$$I_s(\delta) = \operatorname{tr} P_\Sigma E_s \left[\int_a^b d\lambda \operatorname{Im} R_{\lambda+i\delta}\right] E_s P_\Sigma. \tag{14}$$

Neither a nor b are eigenvalues of H. The strong Riemann integral of $\operatorname{Im} R_{\lambda+i\delta}$ gives the standard result (Ref. 5, p. 360)

$$\text{s-}\lim_{\delta\to 0^+} \int_a^b d\lambda \operatorname{Im} R_{\lambda+i\delta} = \pi E_{[a,b]}. \tag{15}$$

A second application of (13) together with (15) controls the $\delta\to 0^+$ limit of (14),

$$\lim_{\delta\to 0^+} I_s(\delta) = \pi \operatorname{tr} P_\Sigma E_s E_{[a,b]} E_s P_\Sigma = \pi \operatorname{tr} P_\Sigma E_s P_\Sigma. \quad \square$$

Proposition 6: Let $v \in L^{4/3}(\mathbb{R}^2)$ and suppose that (i) $|\Sigma| < \infty$, and (ii) $\bar v \chi_\Sigma \in L^1(\mathbb{R}^2)$. Then for $a < -\Lambda_\theta$ with $\theta\in(0,1)$,

$$\lim_{b\to\infty}\lim_{\delta\to 0^+} I(b,\delta)$$
$$\equiv \lim_{b\to\infty}\lim_{\delta\to 0^+}\int_a^b d\lambda \operatorname{tr} P_\Sigma \operatorname{Im}(R^0_{\lambda+i\delta} v R^0_{\lambda+i\delta}) P_\Sigma$$
$$= \frac{1}{4}\int_\Sigma v(x)\,dx.$$

Proof: Note that as in the proof of Theorem 1, Eq. (3), we have $\|P_\Sigma R^0_{\lambda\pm i\delta} u\|_2 < c|\lambda|^{-1/4}$ (c independent of λ and δ).

$\mathcal{V}$ tr P_Σ Im $(R^0_{\lambda+i\delta} v R^0_{\lambda+i\delta}) P_\Sigma$

$\to$ tr $(P_\Sigma R^0_{\lambda+i0} v R^0_{\lambda+i0} P_\Sigma - P_\Sigma R^0_{\lambda-i0} v R^0_{\lambda-i0} P_\Sigma)$,

for every $\lambda \neq 0$ as $\delta \to 0^+$, and that $P_\Sigma R^0_{\lambda+i0} v R^0_{\lambda+i0} P_\Sigma = P_\Sigma R^0_{\lambda-i0} v R^0_{\lambda-i0} P_\Sigma$, for $\lambda < 0$. Thus by an application of the dominated convergence theorem,

$$2i\, I(b,0^+) = 2i \lim_{\delta \to 0^+} I(b,\delta)$$

$$= \int_0^b d\lambda \,\mathrm{tr}\,(P_\Sigma R^0_{\lambda+i0} v R^0_{\lambda+i0} P_\Sigma - P_\Sigma R^0_{\lambda-i0} v R^0_{\lambda-i0} P_\Sigma). \quad (16)$$

Upon writing

$$P_\Sigma R^0_{\lambda+i0} v R^0_{\lambda+i0} P_\Sigma - P_\Sigma R^0_{\lambda-i0} v R^0_{\lambda-i0} P_\Sigma$$

$$= P_\Sigma R^0_{\lambda+i0} v (R^0_{\lambda+i0} - R^0_{\lambda-i0}) P_\Sigma + P_\Sigma (R^0_{\lambda+i0} - R^0_{\lambda-i0}) v R^0_{\lambda-i0} P_\Sigma,$$

and observing that the trace of the product of two $\mathscr{B}_2$ operators can be evaluated as the iterated integral of the associated L^2 kernels (see Ref. 10, p. 524), one has that the integrand in (16) is

$$\int dx \int dy\, \chi_\Sigma(x) [R^0_{\lambda+i0} + R^0_{\lambda-i0}](x,y) \times v(y) [R^0_{\lambda+i0} - R^0_{\lambda-i0}](x,y) \chi_\Sigma(x),$$

where we have also used the fact that $R^0_{\lambda \pm i0}(x,y) = R^0_{\lambda \pm i0}(y,x)$.

Note that $R^0_{\lambda+i0}(x,y) = (i/4)\, H_0^{(1)}(\sqrt{\lambda}\,|x-y|)$ for $\lambda > 0$ and then the choice of the branch of $\sqrt{z}$ leads to $R^0_{\lambda-i0}(x,y) = -(i/4)\, H_0^{(2)}(\sqrt{\lambda}\,|x-y|)$, so that

$$[R^0_{\lambda+i0} + R^0_{\lambda-i0}](x,y) = -\tfrac{1}{2} N_0(\sqrt{\lambda}\,|x-y|)$$

and

$$[R^0_{\lambda+i0} - R^0_{\lambda-i0}](x,y) = \frac{i}{2} J_0(\sqrt{\lambda}\,|x-y|),$$

where J_0 and N_0 are the Bessel and Neumann functions of order 0. Thus

$$2i\, I(b,0^+) = -\frac{i}{4} \int_0^b d\lambda \int dx\, \chi_\Sigma(x) \int dy\, J_0(\sqrt{\lambda}\,|x-y|) \times N_0(\sqrt{\lambda}\,|x-y|) v(y).$$

Denoting $s_\lambda(x) \equiv s_\lambda(|x|) = -(i/4)\, J_0(\sqrt{\lambda}\,|x|)\, N_0(\sqrt{\lambda}\,|x|)$ for $\lambda > 0$, we can rewrite this as

$$2i\, I(b,0^+) = \int_0^b d\lambda \int dx\, \chi_\Sigma(x) \int dy\, s_\lambda(x-y) v(y)$$

$$= \int_0^b d\lambda \int dx\, s_\lambda(x) (\chi_\Sigma * v)(x), \quad (17)$$

where we have written $(\chi_\Sigma * v)(x) = \int \chi_\Sigma(x+y) v(y)\, dy$ and also noted that the above integral converges absolutely by the estimate $(|s_\lambda(x)| < c|x|^{-1})$, and by an application of the Sobolev inequality so that Fubini's theorem can be used.

From Ref. 3, p. 673, formula (6), we note that the improper Riemann Fourier transform of s_λ exists, i.e.,

$$\frac{1}{2\pi} \int_{|x|<n} e^{-ik\cdot x} s_\lambda(x)\, dx = -\frac{i}{4} \int_0^n J_0(|k|\, r) J_0(\sqrt{\lambda}\, r) N_0(\sqrt{\lambda}\, r) r\, dr$$

converges pointwise for $0 < |k| \neq 2\sqrt{\lambda}$ as $n \to \infty$ to a function $\tilde{s}_\lambda$ with

$$\tilde{s}_\lambda(k) = \frac{i}{4} \begin{cases} 0, & \text{if } 0 < |k| < 2\sqrt{\lambda}, \\ (2/\pi)|k|^{-1}(k^2 - 4\lambda)^{-1/2}, & \text{if } 2\sqrt{\lambda} < |k| < \infty. \end{cases} \quad (18)$$

It is clear that $\tilde{s}_\lambda \in L^p(\mathbb{R}^2)$ with $1 < p < 2$ and thus by the Hausdorff–Young theorem (Ref. 6, p. 11) its inverse Fourier transform $\mathscr{F}^{-1} \tilde{s}_\lambda \in L^q(\mathbb{R}^2)$ with $p^{-1} + q^{-1} = 1$ and furthermore $\mathscr{F}^{-1}$ is a continuous linear map from $L^p(\mathbb{R}^2)$ into $L^q(\mathbb{R}^2)$. Also since convergence in the L^q norm implies convergence pointwise almost everywhere for a subsequence (Ref. 4, p. 18) we conclude that the improper Riemann inverse Fourier transform of $\tilde{s}_\lambda$,

$$\lim_{N \to \infty} \frac{1}{2\pi} \int_{|k|<N} e^{ik\cdot x}\, \tilde{s}_\lambda(k)\, dk,$$

if it exists, equals $(\mathscr{F}^{-1} \tilde{s}_\lambda)(x)$ a.e. That this improper Riemann integral exists and is equal to $s_\lambda(x)$ is the formula (6) of Ref. 3, p. 682. Therefore, by Lemma 8, Eq. (17) reduces to

$$2i\, I(b,0^+) = \int_0^b d\lambda \int \tilde{s}_\lambda(k)\, \widetilde{\chi_\Sigma * v}(k)\, dk$$

$$= 2\pi \int_0^b d\lambda \int \tilde{s}_\lambda(k)\, \tilde{\chi}_\Sigma(k) \tilde{v}(-k)\, dk, \quad (19)$$

where we have observed that since $\chi_\Sigma \in L^1(\mathbb{R}^2)$, $(\widetilde{\chi_\Sigma * v})(k) = 2\pi\, \tilde{\chi}_\Sigma(k) \tilde{v}(-k)$.

An elementary integration shows that

$$\widetilde{S}_b(k) \equiv 2\pi \int_0^b \tilde{s}_\lambda(k)\, d\lambda$$

$$= \frac{i}{2} \begin{cases} 1, & \text{if } 0 < |k| < 2\sqrt{b}, \\ 1 - (1 - 4b/k^2)^{1/2}, & \text{if } 2\sqrt{b} < |k|. \end{cases}$$

Since $|\tilde{s}_\lambda(k)| = -i\, \tilde{s}_\lambda(k)$, it follows that

$$\int_0^b |\tilde{s}_\lambda(k)|\, d\lambda = -i \int_0^b \tilde{s}_\lambda(k)\, dk = -\frac{i}{2\pi} \widetilde{S}_b(k) < (4\pi)^{-1}$$

for all $|k| > 0$, and recalling the hypothesis $\tilde{\chi}_\Sigma \tilde{v} \in L^1$, we can apply Fubini's theorem to (19) and obtain

$$2i\, I(b,0^+) = \int \widetilde{S}_b(k)\, \tilde{\chi}_\Sigma(k) \tilde{v}(-k)\, dk. \quad (20)$$

Note that $\widetilde{S}_b$ converges to $i/2$ pointwise for all $|k| > 0$ as $b \to \infty$ and that $|\widetilde{S}_b(k)| < \frac{1}{2}$. Therefore, we apply dominated convergence to (20) to arrive at

$$\lim_{b \to \infty} I(b,0^+) = \frac{1}{4} \int \tilde{\chi}_\Sigma(k) \tilde{v}(-k)\, dk. \quad (21)$$

Finally an application of Lemma 7 to (21) gives the required result. □

Lemma 7: Let $\psi \in L^r(\mathbb{R}^n)$ for some $r \in [1,2]$ and $f \in L^2 \cap L^t(\mathbb{R}^n)$, where $r^{-1} + t^{-1} = 1$. Assume furthermore that $\hat{\psi} \hat{f} \in L^1(\mathbb{R}^n)$. Then

$$\int \overline{\psi(x)}\, f(x)\, dx = \int \overline{\hat{\psi}(k)}\, \hat{f}(k)\, dk. \quad (22)$$

Proof: See the Appendix. □

Lemma 8: Assume $v \in L^{4/3}(\mathbb{R}^2)$ and $|\Sigma| < \infty$. Let s_λ and $\bar{s}_\lambda$ be as defined in Proposition 6. Then

$$\int s_\lambda(x)\chi_\Sigma * v(x)dx = \int \bar{s}_\lambda(k)\widetilde{\chi_\Sigma * v}(k)dk. \tag{23}$$

Proof: Set $\psi \equiv \bar{s}_\lambda$ and $f \equiv \widetilde{\chi_\Sigma * v}$ and utilize Lemma 7 with $\mathbb{R}^n = \mathbb{R}^2$ and $r = \frac{4}{3}$. As noted in Proposition 6, $\bar{s}_\lambda \in L^{4/3}$. The function f is the Fourier transform of a convolution and is proportional to the product $\tilde{\chi}_\Sigma(k)\tilde{v}(-k)$. Because $\tilde{\chi}_\Sigma \in L^2 \cap L^\infty$ and $\tilde{v} \in L^4$ we have from Hölder's inequality that $f \in L^2 \cap L^4$. It remains only to verify that the requirement $\check{\psi}\check{f} \in L^1$ is met. Both $\widetilde{\chi_\Sigma * v}$ and $\chi_\Sigma * v$ are in L^2 and thus a.e. $\check{f}(x) = \chi_\Sigma * v(-x)$. Since $\chi_\Sigma \in L^1$ and $v \in L^{4/3}$ it follows that $\check{f} \in L^{4/3}$. Finally, $s_\lambda \in L^4$, so Hölder's inequality implies $\check{\psi}\check{f} \in L^1$.

Observe that $\check{\psi}(x) = s_\lambda(-x)$ and that both $s_\lambda(x)$ and $\bar{s}_\lambda(k)$ have purely imaginary values. Thereby, it is seen that (23), with $\psi = \bar{s}_\lambda$ and $f = \widetilde{\chi_\Sigma * v}$, is equivalent to the identity (22). □

For $b > \Lambda_\theta$, define a large radius integration contour in Π by $C_\delta(b) = \{z \in \Pi \mid |z| = \sqrt{b^2 + \delta^2}$ and $|\mathrm{Im}\, z| > \delta$ if $\mathrm{Re}\, z > 0\}$. The contour integral over $C_\delta(b)$ will be taken in the conventional right-hand sense.

Proposition 9: Suppose $v \in L^{4/3}(\mathbb{R}^2)$ and $|\Sigma| < \infty$. Then

$$\lim_{b\to\infty} \lim_{\delta\to 0^+} \int_{C_\delta(b)} \mathrm{tr}\, P_\Sigma [R_z - R_z^0 + R_z^0 v R_z^0] P_\Sigma \, dz = 0. \tag{24}$$

Proof: The identities (valid for $z \in \Pi_c$, $|z| > \Lambda_\theta$, $1 > \theta > 0$)

$$[1 + A(z)]^{-1} = 1 - A(z) + A(z)^2 - A(z)^3[1 + A(z)]^{-1}$$

and $R_z^0 v R_z^0 = (R_z^0 w)(u R_z^0)$, when combined with (1), give

$$P_\Sigma [R_z - R_z^0 + R_z^0 v R_z^0] P_\Sigma = \sum_{i=1}^{3} K_i(z),$$

where

$$K_3(z) = P_\Sigma R_z^0 w A(z)^3 [1 + A(z)]^{-1} u R_z^0 P_\Sigma,$$

$$K_i(z) = (-1)^{i+1} P_\Sigma R_z^0 w [A(z)]^i u R_z^0 P_\Sigma, \quad i = 1,2.$$

Consider the K_3 contribution first. If we take the polar representation of $z = |z|\exp(i\psi)$, $\psi \in [0,2\pi]$, then $z \in C_\delta(b)$ requires $\phi < \psi < 2\pi - \phi$, where $\tan\phi = \delta/b$. Bound estimate (3) is of the form $\|A(z)\|_2 = O(|z|^{-1/4})$. Since $\chi_\Sigma \in L^{4/3}(\mathbb{R}^2)$, a similar Sobolev estimate shows that both $\|P_\Sigma R_z^0 w\|_2$ and $\|u R_z^0 P_\Sigma\|_2$ decay like $O(|z|^{-1/4})$ for large $|z|$. After noting that $\|[1 + A(z)]^{-1}\| < (1 - \theta)^{-1}$, one finds

$$\left| \int_{C_\delta(b)} \mathrm{tr}\, K_3(z) dz \right| < \frac{c^5}{(b^2 + \delta^2)^{1/8}} \left[\frac{2(\pi - \phi)}{1 - \theta} \|\chi_\Sigma\|_{4/3} \|v\|_{4/3}^4 \right], \tag{25}$$

where c is the constant arising in the Sobolev estimate. The right side of (25) vanishes in the double limit $\delta \to 0^+$, $b \to \infty$.

The analysis of the contribution of both K_1 and K_2 to the limit in (24) is similar, so we shall restrict the discussion to the K_1 term. The operator K_1 is the product of three $\mathscr{B}$ operators, so $\mathrm{tr}\, K_1$ may be calculated as the triple iterated integral of the kernels associated with these Hilbert–Schmidt operators (Ref. 10, p. 524). Upon using estimate (2) for $R_z^0(x,y)$ and setting $\Gamma = (b^2 + \delta^2)^{1/2}$ we have, for $z \in C_\delta(b)$,

$$|\mathrm{tr}\, K_1(z)| < \frac{c}{\Gamma^{3/4}} \int dx\, \chi_\Sigma(x) \int\int dy_1\, dy_2 \times \frac{|v(y_1)v(y_2)|\, e^{-r\,\mathrm{Im}\sqrt{z}}}{|x - y_1|^{1/2}|y_1 - y_2|^{1/2}|y_2 - x|^{1/2}},$$

where $r = |x - y_1| + |y_1 - y_2| + |y_2 - x|$. Doing the $|dz|$ integral along contour $C_\delta(b)$ gives the bound

$$\int_{C_\delta(b)} |\mathrm{tr}\, K_1(z)|\, |dz| < \frac{2\pi c}{\Gamma^{1/4}} \int\int\int dx\, dy_1\, dy_2 \times \frac{\chi_\Sigma(x)|v(y_1)|\, |v(y_2)|}{r|x - y_1|^{1/2}|y_1 - y_2|^{1/2}|y_2 - x|^{1/2}}, \tag{26}$$

where the fact that the integrand is non-negative has been used to justify changing the order of integration. Clearly if the triple integral in (26) is finite, then the $b \to \infty$, $\delta \to 0^+$ limit of the K_1 term in (24) vanishes. The finiteness of this triple integral follows by the inequality $r > |x - y_1|^{1/4} |y_1 - y_2|^{1/2} |y_2 - x|^{1/4}$ together with Schwartz inequality to bound the dx integral and the Sobolev inequality to estimate the $dy_1\, dy_2$ integral. □

Proof of Theorem 4: For $z \in \rho(H) \cap \rho(K_0)$ define $\Phi(z) \in \hat{\mathscr{B}}$ by

$$\Phi(z) \equiv P_\Sigma [R_z - R_z^0 + R_z^0 v R_z^0] P_\Sigma.$$

From Theorem 1, Eq. (1) it is seen that Φ may also be represented as

$$\Phi(z) = (P_\Sigma R_z^0 w) A(z) [1 + A(z)]^{-1} (u R_z^0 P_\Sigma). \tag{27}$$

The outer two factors on the right side of (27) are $\mathscr{B}_2$ holomorphic in $\rho(K_0)$ while the inner factors are norm holomorphic in $\rho(H)$. It follows that $z \to \Phi(z)$ is trace-norm holomorphic on the domain $\rho(H) \cap \rho(K_0)$.

Select a and b such that $(a,b) \supset [-\Lambda_\theta, \Lambda_\theta]$ for some $0 < \theta < 1$. For fixed a, b, and $\delta > 0$ choose a closed contour in the canonical cut plane Π to be $C_T \equiv C_\delta(a) + C_\pm(b,\delta) + C_\delta(b)$, where $C_\delta(b)$ has been given above and

$$C_\pm(b,\delta) = \{z \in \Pi \mid z = \lambda \pm i\delta,\ \lambda \in [a,b]\},$$

$$C_\delta(a) = \{z \in \Pi \mid z = a + i\eta,\ \eta \in [-\delta,\delta]\}.$$

Define a holomorphic function on $\rho(H) \cap \rho(K_0) \subset \mathbb{C}$ by setting $h(z) = \mathrm{tr}\, \Phi(z)$. Cauchy's integral theorem asserts that the C_T contour integral of $h(z)$ vanishes. Specifically, for each $\delta > 0$,

$$i\int_{-\delta}^{\delta} h(a + i\eta)d\eta + \int_{C_\delta(b)} h(z)dz + 2i\int_a^b \mathrm{Im}\, h(\lambda + i\delta)d\lambda = 0. \tag{28}$$

Consider that $\delta \to 0^+$ limit of the first integral in (28). Use (27) to rewrite the argument of $\mathrm{tr}\, \Phi$. After applying the Sobolev inequality to estimate the $\|\cdot\|_2$ norm of $P_\Sigma R_z^0 w$, $A(z)$, and $u R_z^0 P_\Sigma$, and using $\|[1 + A(z)]^{-1}\| < (1 - \theta)^{-1}$, it shows (if $a < -\Lambda_\phi$) that $|h(a+i\eta)|$ is uniformly bounded in η. Thus the integral $\int_{-\delta}^{\delta} h(a+i\eta)d\eta$ vanishes as $\delta \to 0^+$.

Now take the $\delta \to 0^+$, $b \to \infty$ limit of identity (28). The limiting value of the middle term is determined by Proposition 9 to be zero, leaving us with

$$\lim_{b\to\infty} \lim_{\delta\to 0^+} \int_a^b \operatorname{Im} h(\lambda + i\delta) d\lambda = 0. \qquad (29)$$

Inserting the results of Propositions 5 and 6 into (29) yields

$$\lim_{b\to\infty} \frac{1}{2}\int_0^b q(\lambda,\Sigma)d\lambda + \pi \operatorname{tr} P_\Sigma E_s P_\Sigma + \frac{1}{4}\int_\Sigma v(x)\,dx = 0.$$

Here, the second and third factors are both finite. This requires that

$$\lim_{b\to\infty} \int_0^b q(\lambda,\Sigma)d\lambda$$

be finite; i.e., the improper integral of $\lambda \to q(\lambda,\Sigma)$ satisfies (7). □

IV. DISCUSSION

We conclude by making a number of remarks concerning the spectral sum rule.

(1) Consider the behavior of hypothesis (ii) in Theorem 4. The condition (ii) acts as a joint constraint on v and Σ. Given a fixed set Σ, (ii) restricts the choice of v; or given a fixed $v \in L^{4/3}$, (ii) defines an admissible class of sets $\Sigma \subset \mathbb{R}^2$. Two examples illustrate how (ii) works. For every $v \in L^{4/3}$, one can find a Σ such that (ii) is valid. Suppose Σ is a rectangle. Then $\hat{\chi}_{\text{rect}} \in L^{4/3}$, and furtheremore, since $\hat{v} \in L^4$, Hölder's inequality implies $\hat{v}\hat{\chi}_{\text{rect}} \in L^1$. On the other hand, hypothesis (ii) need not be fulfilled by all pairs (v,Σ) allowed by (i). Let Σ be a disk. Then $\hat{\chi}_{\text{disk}} \in L^{4/3+} \cap L^\infty$. In this case, if the potential class is restricted to $v \in L^{4/3} \cap L^{4/3+}$, then (ii) will be satisfied for the disk. Finally, we observe that if the potential class is further narrowed to $v \in L^{4/3} \cap L^2$, then (ii) is obeyed for all Σ with $|\Sigma| < \infty$.

(2) It is often desirable to separate the contributions of the point spectrum and the singularly continuous spectrum. Suppose $\{\psi_i\}$ is the family of independent $L^2(\mathbb{R}^2)$ eigenfunctions of H having eigenvalues λ_i and normalization $\|\psi_i\| = 1$. These eigenvalues always lie within the interval $[-\Lambda_\phi, \Lambda_\phi]$ and may assume negative, zero, or positive values. The family $\{\psi_i\}$ may be empty, finite, or infinite. (In particular, the assumption $v \in L^{4/3}$ is not known to rule out an infinite number of positive eigenvalues.) The spectral subspace decomposition $E_s = E_{pp} + E_{sc}$ implies

$$\operatorname{tr} P_\Sigma E_s P_\Sigma = \sum_i \int_\Sigma |\psi_i(x)|^2 dx + \operatorname{tr} P_\Sigma E_{sc} P_\Sigma.$$

(3) Various sufficient conditions on v are known to ensure the absence of the singular continuous spectrum and of the positive point spectrum of H. We quote only two representative results.

Theorem: Let $(1+|x|)^\nu v(x) \in L^{4/3}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$, $\nu > 1$. Then $\mathscr{H}_{sc}(H) = \{0\}$. Furthermore, there are a finite number of positive eigenvalues of H with finite multiplicity in every compact subset of $(0,\infty)$.

This result follows from both time-dependent Enss–Mourre theory[11] as well as from time-independent theory.[12]

Theorem: Let $(1+|x|)v(x) \in L^2(\mathbb{R}^2)$. Then H has no positive eigenvalues.

This is a specialized version of the more general results obtained by Froese *et al.*[13]

(4) For any $|\Sigma| < \infty$, Remark (1) shows the spectral sum rule identity (7) is valid for all $v \in L^{4/3} \cap L^2$. The potential class $L^{4/3} \cap L^2$ does not prohibit the appearance of zero energy resonances (see Refs. 14 and 15). For example, if one varies v in $L^{4/3} \cap L^2$ by changing the coupling constant it is possible to introduce zero energy resonances in the scattering system. However, the local spectral sum rule (7) [takes the same form (7) for all $v \in L^{4/3} \cap L^2$, and so] is structurally insensitive to the presence or absence of a zero energy resonance.

(5) Global sum rules (Levinson's theorem) obtain if $\Sigma = \mathbb{R}^2$. A result of the literature that is closely related to the spectral sum rule in Theorem 4 is the $\mathbb{R}^2$–Levinson theorem derived by Cheney.[16] Let $S(k)$: $L^2(T) \to L^2(T)$ denote the energy-shell S-matrix operator, where $|k| = \sqrt{\lambda} > 0$. Then for a potential class that prohibits (1) the singularly continuous spectrum, (2) non-negative eigenvalues, and (3) zero-energy resonances, it is found that[16]

$$i[\log \det S(0) - \log \det S(\infty)] = -2\pi N - \frac{1}{2}\int_{\mathbb{R}^2} v(x)dx,$$

where N is the number of negative energy bound states.

For scattering in $\mathbb{R}^3$ the effect of zero-energy resonances on the form of Levinson's theorem has been discussed several times.[17,18] In a notation analogous to the above, Newton[17] finds

$$\delta(0) - \lim_{k\to\infty}\left[\delta(k) + \frac{k}{4\pi}\int_{\mathbb{R}^3} v(x)dx\right] = \pi\left(N + \frac{1}{2}q\right),$$

where $\delta(k)$ is an appropriately chosen phase parametrization for the S matrix, $\ln(\det S(k)) = 2i\,\delta(k)$. The factor $q = 0$, if there are no zero-energy resonances, and $q = 1$, otherwise.

Here N is the number of zero-energy and negative-energy eigenfunctions. It is this type of zero-energy resonance modification of Levinson's global sum rule that does not occur in the local sum rule of Theorem 4.

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APPENDIX: PROOF OF LEMMA 7

Set $\phi(x) = (2\pi)^{-n/2}\exp(-x^2/2)$ and for every $\epsilon > 0$, $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ so that $\int \phi_\epsilon(x)\,dx = 1$ and $\hat{\phi}_\epsilon(k) = (2\pi)^{-n/2}\exp(-k^2\epsilon^2/2)$. Define

$$\psi_\epsilon(x) = \int \psi(x+y)\,\phi_\epsilon(y)\,dy = \int \psi(x+\epsilon y)\,\phi(y)\,dy. \qquad (30)$$

Note that $\psi_\epsilon \in L^r(\mathbb{R}^n)$ and $\|\psi_\epsilon\|_r \leq \|\psi\|_r$. Since the map $\lambda \to |\lambda|^r$ is convex on R^+ for $r \geq 1$ and since $\int \phi(y)dy = 1$, we have, by Jensen's inequality[19] and (30),

$$\|\psi - \psi_\epsilon\|_r^r = \int |\psi(x) - \psi_\epsilon(x)|^r \, dx$$

$$\leq \int dx \int |\psi(x + \epsilon y) - \psi(x)|^r \phi(y)\, dy$$

$$= \int \|(T_{\epsilon y} - I)\psi\|_r^r \, \phi(y) dy,$$

where $(T_y \psi)(x) = \psi(x + y)$.

Now $T_{\epsilon y}\psi \to \psi$ in L^r norm as $\epsilon \to 0^+$ for every y fixed. Furthermore T_y is an isometry. Therefore, by dominated convergence one has that $\|\psi - \psi_\epsilon\|_r \to 0$ as $\epsilon \to 0^+$. Since $\psi_\epsilon \in L^2(\mathbb{R}^n)$ by Young's theorem (Ref 6, p. 28), we have by Plancherel's theorem that

$$\int \overline{\psi_\epsilon(x)}\, f(x) dx = \int \overline{\hat{\psi}_\epsilon(k)}\, \hat{f}(k) dk. \tag{31}$$

The left-hand side of (31) converges to $\int \overline{\psi(x)}\, f(x) dx$ since $\|\psi_\epsilon - \psi\|_r \to 0$ as $\epsilon \to 0^+$ and since $f \in L^2(\mathbb{R}^n)$. On the other hand, $\hat{\psi}_\epsilon(k) = (2\pi)^{n/2} \hat{\psi}(k) \hat{\phi}_\epsilon(-k) = \hat{\psi}(k) e^{-k^2\epsilon^2/2} \to \hat{\psi}(k)$ pointwise and $|\hat{\psi}_\epsilon(k)| \leq |\hat{\psi}(k)|$ so that an application of the dominated convergence theorem to the right side of (31) along with the hypothesis $\hat{\psi}\hat{f} \in L^1(\mathbb{R}^n)$ leads to the original result. □

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