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ON THE EQUIVALENCE OF TWO APPROACHES TO SIMPLICITY IN THE ANALYSIS OF BLOCK DESIGNS AND SOME RELATED RESULTS

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Abstract: Two seemingly different approaches to *simplicity* in the analysis of connected block designs, and their relationship to the concepts of *balance* are discussed.

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1. Introduction and two concepts of 'simplicity'

1.1. First approach

For ready reference, we take up the following results from Tocher (1952), Caliński (1971), Jones (1954), Saha (1976), Puri and Nigam (1975a, 1975b), and Kageyama (1974), regarding the more familiar approach to simplicity. Throughout we consider only connected block designs.

(i) The reduced normal equations $C\hat{\tau}=Q$ with the side restriction $r'\hat{\tau}=0$ are equivalent to $(C+rr'/n)\hat{\tau}=Q$, i.e., $\Omega^{-1}\hat{\tau}=Q$, where

$$\Omega^{-1}=C+rr'/n=r^{\delta}(I-M_0) \tag{1}$$

is positive definite with

$$M_0=r^{-\delta}Nk^{-\delta}N'-1r'/n, \tag{2}$$

where $r'=(r_1,r_2,\ldots,r_v)$ is the vector of replications, r^{δ} is the diagonal matrix $\mathrm{Diag}(r_1,r_2,\ldots,r_v)$, $r^{-\delta}$ being the inverse of r^{δ}, N is the $v\times b$ treatment-block incidence matrix, $k'=(k_1,k_2,\ldots,k_b)$ is the vector of block-sizes, 1 is a vector of unities of appropriate order, and $n=1'k=1'r$. Further,

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$$V(\lambda'\hat{\tau}) = \sigma^2 \lambda' \Omega \lambda, \tag{3}$$

whatever λ satisfying $\lambda' \mathbf{1} = 0$.

(ii) To achieve 'simplicity' in the analysis, it is demanded that the computation of Ω be simple. By that it is meant that Ω be of the form $(I_v + (1-\mu)^{-1} M_0) r^{-\delta}$, where I_v is the $v \times v$ unit matrix. This is so iff the design is a C-design (Saha (1976)), i.e., iff

$$M_0^2 = \mu M_0, \quad 0 \le \mu < 1. \tag{4}$$

(iii) Recall that a design is efficiency-balanced (EB) iff

$$M_0 = \mu(I_v - \mathbf{1} r'/n), \tag{5}$$

so that M_0 has a (unique) eigenvalue μ of multiplicity $(v-1)$.

(iv) Similarly a design is variance-balanced (VB) iff (Rao (1958))

$$C = \varrho(I_v - \mathbf{1}\mathbf{1}'/v), \tag{6}$$

where ϱ (>0) is a constant.

1.2. Second approach

Also, as for example in Kiefer (1958), the reduced normal equations $C\hat{\tau} = Q$ are equivalently written as $(PCP')P\hat{\tau} = PQ$ where

$$(\mathbf{1}/\sqrt{v} | P')$$

is an orthogonal matrix. Writing $P\tau = \eta$, and observing that PCP' is p.d., one obtains

$$\hat{\eta} = (PCP')^{-1} PQ, \tag{7}$$

and that, for any contrast $\lambda'\tau$ with $\lambda'\mathbf{1} = 0$,

$$V(\lambda'\hat{\tau}) = V(\delta'\hat{\eta}) = \sigma^2 \delta'(PCP')^{-1}\delta, \tag{8}$$

where $\delta = P\lambda$.

It is well known that (3) holds iff (8) holds for all λ satisfying $\lambda'\mathbf{1} = 0$. To achieve 'simplicity' in this case, one would demand that the computation of $(PCP')^{-1}$ be 'simple'. Let $\Sigma = (PCP')^{-1}$, so that $\Sigma^{-1} = PCP' = Pr^{\delta}(I_v - M_0 - \mathbf{1}r'/n)P'$. We then demand that Σ be of the form

$$\Sigma = P[I_v + (1-\mu)^{-1} M_0 - \mathbf{1}r'/n] r^{-\delta} P' = P[I_v + (1-\mu)^{-1} M_0] r^{-\delta} P', \tag{9}$$

since $P\mathbf{1} = \mathbf{0}$.

Theorem 1. *We have*

$$M_0^2 = \mu M_0 \tag{10}$$

iff Σ *is of the form in* (9).

Proof. The *if* part is trivial. To prove the *only if* part we take Σ as in (9) and consider the identity $I_{v-1} = \Sigma\Sigma^{-1}$, which leads to

$$P\left(\frac{M_0\mu - M_0^2}{1-\mu}\right)P' = 0,$$

or,

$$M_0\mu - M_0^2 = \begin{bmatrix} a_1 & a_2 & \dots & a_v \\ a_1 & a_2 & & a_v \\ \vdots & & & \vdots \\ a_1 & a_2 & \dots & a_v \end{bmatrix}$$

with $\sum_{i=1}^{v} a_i = 0$, since $M_0 1 = 0$. Now, using the fact that $r'M_0 = 0'$, one gets $a_1 = a_2 = \dots = a_v = 0$, and hence $M_0^2 = \mu M_0$.

Note 1. Theorem 1 can be alternatively proved as follows. Since $(PCP')^{-1} = PDP'$ is equivalent to $CDC = C$, demanding (9) is equivalent to demanding that $[I_v + (1-\mu)^{-1}M_0]r^{-\delta}$ be a g-inverse of C, which holds iff $C[I_v + (1-\mu)^{-1}M_0]r^{-\delta}$ is idempotent of rank $(v-1)$, or equivalent iff, $C[I_v + (1-\mu)^{-1}M_0]r^{-\delta} + r1'/n$ is idempotent of rank v, i.e., I_v. Thus (9) holds iff

$$C[I_v + (1-\mu)^{-1}M_0]r^{-\delta} + r1'/n = I_v. \tag{11}$$

(10) now follows from (11) on substitution of C by $r^{\delta}(I_v - M_0 - 1r'/n)$.

Note 2. Thus the two approaches to simplicity lead to the same condition, namely $M_0^2 = \mu M_0$, first derived by Caliński (1971), who, however, did perhaps not realise that this condition is necessary as well for a block design to possess the desired implication of simplicity in its analysis.

2. Balanced designs and some related results

Let C^- denote a g-inverse (g.i.) of C.

Theorem 2. *A block design is VB iff $C^- \propto I_v$ for some g.i. C^- of C. Also, a block design is EB iff $C^- \propto r^{-\delta}$ for some g.i. C^- of C. (Here the block design need not be connected.)*

Proof. (i) A design is VB iff $(\lambda' C^- \lambda)/(\lambda'\lambda)$ is a constant for every λ from the row-space of C, whatever C^- be used. This, however, implies $C^- \propto I_v$ for some C^-.

(ii) A design is EB iff $(\lambda' C^- \lambda)/(\lambda' r^{-\delta}\lambda)$ is a constant for all λ in the row-space of C, for all g-inverses C^-. This also leads to the condition that $C^- \propto r^{-\delta}$ for at least one g-inverse C^- of C.

Since C^- has such simple forms as I_v and $r^{-\delta}$ in VB and EB designs respectively, the analysis turns out to be extremely simple.

Using the fact (already used in Note 1) that the condition $\{AGA=A\}$ is equivalent to $\{AG$ is idempotent, and of rank equal to rank $A\}$ or $\{GA$ is idempotent, and of rank equal to rank $A\}$, we can characterize the C-matrices of VB and EB block designs in the following manner.

Corollary 2.1. *A block design is VB iff $C=\theta L$, where L is idempotent. And, a block design is EB iff $r^{-\delta}C=\theta L$, where L is idempotent.*

It is easy to see that $C=\theta L$ iff the positive roots of C are all equal. When the block design is connected we can determine L more specifically, and this leads to the following.

Corollary 2.2. *A connected block design is* (i) *VB iff $C=\theta(I_v-\mathbf{1}\mathbf{1}'/v)$, and* (ii) *EB iff $C=\theta(r^{\delta}-rr'/n)$.*

Proof. (i) Since $C\mathbf{1}=\mathbf{0}$, we must have $L\mathbf{1}=\mathbf{0}$, and $\mathbf{1}'L=\mathbf{0}'$. Therefore, $L+\mathbf{1}\mathbf{1}'/v$ is idempotent of rank $(v-1)+1=v$. Hence $L+\mathbf{1}\mathbf{1}'/v=I_v$, from which the result follows.

(ii) Since $r'r^{-\delta}C=\mathbf{1}'C=\mathbf{0}'$, and $r^{-\delta}C\mathbf{1}=\mathbf{0}$, we must have $L\mathbf{1}=\mathbf{0}$ and $r'L=\mathbf{0}'$. When the rank of C is $(v-1)$ the matrix $(L+\mathbf{1}r'/n)$ is idempotent and of rank $(v-1)+1=v$, which implies that $L=I_v-\mathbf{1}r'/n$. Hence the result.

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