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OPTIMALITY OF BLUE'S IN A GENERAL LINEAR MODEL WITH INCORRECT DESIGN MATRIX

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Abstract: We consider the Gauss–Markoff model $(Y, X_0\beta, \sigma^2 V)$ and provide solutions to the following problem: What is the class of all models $(Y, X\beta, \sigma^2 V)$ such that a specific linear representation/some linear representation/every linear representation of the BLUE of every estimable parametric functional $p'\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is (a) an unbiased estimator, (b) a BLUE, (c) a linear minimum bias estimator and (d) best linear minimum bias estimator of $p'\beta$ under $(Y, X\beta, \sigma^2 V)$? We also analyse the above problems, when attention is restricted to a subclass of estimable parametric functionals.

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1. Introduction and summary

We shall denote by the triplet $(Y, X\beta, \sigma^2 V)$ the linear model $Y = X\beta + \varepsilon$ where $Y_{n\times 1}$ and $\varepsilon_{n\times 1}$ are random vectors with $E(\varepsilon)=0$ and $D(\varepsilon)=\sigma^2 V$, $X_{n\times m}$ is a non-stochastic matrix (design matrix), $\beta_{m\times 1}$ and σ^2 are unknown parameters. Here V is a known nonnegative definite (n.n.d.) matrix. The following notations will be followed in the paper. For a matrix A, $M(A)$ and $N(A)$ respectively denote the column span and null space of A, A^- denotes a generalised inverse (g-inverse) of A, i.e. any matrix satisfying $AA^-A=A$ and $A^\perp$ denotes a matrix of maximum rank satisfying $A'A^\perp=0$. For an n.n.d. matrix N, $P_{A,N}$ denotes the matrix $A(A'NA)^-A'N$ and P_A denotes $P_{A,I}$. A matrix with $T_1, T_2, \ldots, T_k$ as diagonal blocks and null matrixes as off-diagonal blocks is denoted by $\mathrm{diag}(T_1, T_2, \ldots, T_k)$. The definitions of best linear unbiased estimator (BLUE), linear minimum bias estimator (LIMBE) and best linear minimum bias estimator (BLIMBE) under linear model $(Y, X\beta, \sigma^2 V)$ are well known and we refer to Rao and Mitra (1971, Chapter 7 and 8) for the details.

The problem of robustness of BLUE's with incorrect dispersion matrix received considerable attention in the literature and some significant contributions in this line are Rao (1967, 1968, 1971), Mitra and Rao (1969), Zyskind (1967), Watson (1967) and Mitra and Moore (1973). However, the problem of robustness of BLUE's with incorrect design matrix has not received much attention in the literature except for Mitra and Rao (1969) where they considered the dispersion matrix as $\sigma^2 I$. If we have a linear model $(Y, X\beta, \sigma^2 V)$ with restrictions of the type $R\beta = 0$, then in the restricted model, the expectation of the observations is given by

$$E(Y' : 0')' = (X' : R')'\beta,$$

whereas in the unrestricted model, the expectation can be written as

$$E(Y' : 0')' = (X' : 0')'\beta.$$

Thus we have two linear models which differ in their design matrices and it is of importance to characterise R such that the BLUE of $X\beta$ in the unrestricted model continues to be its BLUE in the restricted model also. Another situation where we come across linear models which differ in their design matrices is when we consider parametric augmentations to a given linear model. In the original model we have $E(Y) = X_1\beta_1$, whereas in the augmented model we have $E(Y) = X_1\beta_1 + X_2\beta_2$. Here it is of interest to characterise X_2 so that the BLUE of $X_1\beta_1$ in the original model continues to be its BLUE in the augmented model also. Recently, one of us, Mathew (1980) solved the following robustness problem of BLUE's. "When is the BLUE of every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2 I)$ its BLIMBE under $(Y, X\beta, \sigma^2 I)$?" What is essentially demanded here is the following: Let $p'\beta$ be estimable under $(Y, X_0\beta, \sigma^2 I)$. If $p'\beta$ is also estimable under $(Y, X\beta, \sigma^2 I)$, its BLUE under the first model should also be its BLUE under the second model. If it is not estimable under the second model, we want it to be the best possible, namely its BLIMBE.

As pointed out by Mitra and Moore (1973), when V is singular, BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$ does not have a unique linear representation except when $\operatorname{rank}(VZ_0) = \operatorname{rank}(Z_0)$ where $Z_0 = X_0^\perp$. In the same spirit as in Mitra and Moore, (1973), we obtain complete solutions to the following problems:

Problem (1). What is the class of all models $(Y, X\beta, \sigma^2 V)$ such that a specific linear representation of BLUE of every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is (a) unbiased estimator (UE), (b) a BLUE, (c) a LIMBE and (d) a BLIMBE of $p'\beta$ under $(Y, X\beta, \sigma^2 V)$?

Problem (2). What is the class of all models $(Y, X\beta, \sigma^2 V)$ such that at least one linear representation of BLUE of every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is (a) an UE, (b) a BLUE, (c) a LIMBE and (d) a BLIMBE of $p'\beta$ under $(Y, X\beta, \sigma^2 V)$?

Problem (3). What is the class of all models $(Y, X\beta, \sigma^2 V)$ such that every linear representation of BLUE of every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is (a) an UE, (b) a BLUE, (c) a LIMBE and (d) a BLIMBE of $p'\beta$ under $(Y, X\beta, \sigma^2 V)$?

It is obvious that if the BLUE of $X_0\beta$ has a unique linear representation under $(Y, X_0\beta, \sigma^2 V)$ then the three problems stated above merge into one. If it is known that $M(X) \subset M(X_0 : V)$, then also the three problems merge into one, since, when this condition holds, the different linear representations of the BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$ coincide with probability one under $(Y, X\beta, \sigma^2 V)$.

Sometimes one might be interested only in inferences regarding a few estimable parametric functionals and not all. The problem of robustness of BLUE's of a subset of estimable parametric functionals is solved in the last section.

The norm defining the bias of a LIMBE is taken to be the Euclidean norm. For matrices X_0 and X, we denote $D = X_0 - X$, $Z_0 = X_0^{\perp}$ and $Z = X^{\perp}$.

2. Solution to Problem (1)

We first prove an algebraic lemma which we need in the sequel.

Lemma 2.1. *Let V be a n.n.d. matrix of order $n \times n$, X_0 be a matrix of order $n \times m$ and G be a specific n.n.d. g-inverse of $V + X_0X_0'$. Then there exists a nonsingular matrix P and order $n \times n$ and an orthogonal matrix Q of order $m \times m$ such that*

$$X_0 = P \operatorname{diag}(I, 0, 0) Q', \tag{2.1}$$

$$V + X_0X_0' = P \operatorname{diag}(\Lambda_1, \Lambda_2, 0) P' \tag{2.2}$$

and

$$G = P'^{-1} \operatorname{diag}(\Lambda_1^{-1}, \Lambda_2^{-1}, S) P^{-1} \tag{2.3}$$

where Λ_1 and Λ_2 are diagonal positive definite matrices and S is n.n.d.

Proof. Since X_0X_0' and $V + X_0X_0'$ are both n.n.d. and since $M(X_0X_0') \subset M(V + X_0X_0')$ there exists a nonsingular matrix T such that

$$X_0X_0' = T \operatorname{diag}(I, 0, 0) T' \quad \text{and} \quad V + X_0X_0' = T \operatorname{diag}(\Lambda_1, \Lambda_2, 0) T'$$

where Λ_1 and Λ_2 are diagonal positive definite (p.d.) matrices (see Rao and Mitra, 1971, p. 121). Then G is an n.n.d. g-inverse of $V + X_0X_0'$ if and only if

$$G = T'^{-1} \begin{pmatrix} \Lambda_1^{-1} & 0 & R_1 \\ 0 & \Lambda_2^{-1} & R_2 \\ R_1' & R_2' & R_3 \end{pmatrix} T^{-1}$$

where R_1, R_2, R_3 are such that $R_3 - R_1'\Lambda_1R_1 - R_2'\Lambda_2R_2$ is n.n.d. Clearly $G = P'^{-1} \operatorname{diag}(\Lambda_1^{-1}, \Lambda_2^{-1}, S) P^{-1}$ where $S = R_3 - R_1'\Lambda_1R_1 - R_2'\Lambda_2R_2$ and

$$P = T \begin{pmatrix} I & 0 & -\Lambda_1R_1 \\ 0 & I & -\Lambda_2R_2 \\ 0 & 0 & I \end{pmatrix}.$$

For this choice of P, there exists an $m\times m$ orthogonal matrix Q satisfying (2.1), (2.2) and (2.3) in view of Problem 1 in Rao and Mitra (1971, p. 17).

Remark 2.1. P in Lemma 2.1 can be chosen such that $\Lambda_1 - I$ is of the form $\operatorname{diag}(\Gamma, 0)$ where Γ is p.d.

Let G be a g-inverse of $V+X_0X_0'$. Rao (1971) and Mitra and Moore (1973) established that BLUE of $X_0\beta$ under the model $(Y, X_0\beta, \sigma^2V)$ has a linear representation $X_0(X_0'GX_0)^- X_0'GY$. Mitra and Moore (1973) also observed that G could be chosen to be n.n.d. without loss of generality. Hence a specific linear representation of BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2V)$ is $P_{X_0,G}Y$ where G is a specific n.n.d. g-inverse of $V+X_0X_0'$.

The following theorem gives the solution to Problem (1) (a) and (b).

Theorem 2.1. *Consider the linear models $(Y, X_0\beta, \sigma^2V)$ and $(Y, X\beta, \sigma^2V)$. Let G be a given n.n.d. g-inverse of $V+X_0X_0'$. Then*

(a) *the linear representation $P_{X_0,G}Y$ of BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2V)$ is unbiased for $X_0\beta$ under $(Y, X_0\beta, \sigma^2V)$ if and only if*

$$X = X_0 + (I - P_{X_0,G})A \tag{2.4}$$

where A is arbitrary, and

(b) *$X_0(X_0'GX_0)^- X_0'GY$ is BLUE of $X_0\beta$ under $(Y, X\beta, \sigma^2V)$ if and only if X is as in* (2.4) *where A satisfies*

$$M(X_0'GV:0)' \subset M(X_0' : A'(I-P_{X_0,G})')'$$

or, equivalently, A satisfies

$$(I-P_{X_0,G})A(X_0^- VGX_0 + (I - X_0^- X_0)B) = 0$$

where B is arbitrary and X_0^- is an arbitrary but fixed g-inverse of X_0.

Proof.

$$\begin{aligned} & E[X_0(X_0'GX_0)^- X_0'GY \mid (Y, X\beta, \sigma^2V)] = X_0\beta \quad \forall\beta \\ \Leftrightarrow\ & X_0(X_0'GX_0)^- X_0'GD = 0 \\ \Leftrightarrow\ & D = (I-P_{X_0,G})A \quad \text{where } A \text{ is arbitrary.} \end{aligned}$$

This completes the proof of (a). Now,

$$\begin{aligned} & P_{X_0,G}Y \text{ is BLUE of } X_0\beta \text{ under } (Y, X\beta, \sigma^2V) \\ \Leftrightarrow\ & P_{X_0,G}VZ = 0 \\ \Leftrightarrow\ & X_0'GVZ = 0 \\ \Leftrightarrow\ & VGX_0 = X_0C + (I-P_{X_0,G})AC \quad \text{for some } C \\ \Leftrightarrow\ & VGX_0 = X_0C \end{aligned} \tag{2.5}$$

and

$$0=(I-P_{X_0,G})AC \quad \text{for some } C \tag{2.6}$$

$$\Leftrightarrow \quad M(X_0'GV:0)'\subset M(X_0':A'(I-P_{X_0,G})')'.$$

Solving for C from (2.5) and substituting in (2.6), we see that (2.5) and (2.6) hold if and only if

$$(I-P_{X_0,G})A(X_0^- VGX_0+(I-X_0^- X_0)B)=0 \quad \text{for some } B.$$

This completes the proof of (b).

Corollary 2.1 (Mitra and Rao, 1969). *If in Theorem 2.1, $V=I$, then BLUE of $X_0\beta$ under $(Y,X_0\beta,\sigma^2 I)$ is its BLUE under $(Y,X\beta,\sigma^2 I)$ if and only if*

$$X=X_0+(I-P_{X_0})A$$

where A satisfies

$$M(X_0')\cap M(A'(I-P_{X_0}))=\{0\}.$$

Proof. If $V=I$, then $G=(I+X_0X_0')^{-1}$ and $P_{X_0,G}=P_{X_0}$. In this case the conditions in Theorem 2.1 reduce to $X=X_0+(I-P_{X_0})A$, where A satisfies

$$M(X_0'G:0)'\subset M(X_0':A'(I-P_{X_0}))' \quad \Leftrightarrow \quad M(X_0')\cap M(A'(I-P_{X_0}))=\{0\}.$$

We state the following lemma.

Lemma 2.2. *Consider the linear model $(Y,X\beta,\sigma^2V)$ and let $\|\zeta\|=(\zeta'\zeta)^{1/2}$. A linear estimator $l'Y$ is LIMBE of a parametric functional $p'\beta$ if and only if $XX'l=Xp$ and $l'Y$ is BLIMBE of $p'\beta$ if and only if $XX'l=Xp$ and $Vl\in M(X)$.*

We prove:

Theorem 2.2. *Consider the linear models $(Y,X_0\beta,\sigma^2V)$ and $(Y,X\beta,\sigma^2V)$ and let G be a given n.n.d. g-inverse of $V+X_0X_0'$. Consider the representations of X_0, $V+X_0X_0'$ and G specified in Lemma 2.1 and Remark 2.1. Then:*

(a) *the representation $p'(X_0'GX_0)^-X_0'GY$ of BLUE of every estimable parametric functional $p'\beta$ under $(Y,X_0\beta,\sigma^2V)$ is its LIMBE under $(Y,X\beta,\sigma^2V)$ if and only if*

$$D=P\begin{pmatrix} B & C_1 & C_2 \\ E_1 & E_2 & E_3 \\ F_1 & F_2 & F_3 \end{pmatrix}Q' \tag{2.7}$$

where B is an arbitrary n.n.d. matrix with eigenvalues in $[0,1]$, C_1 and C_2 are arbitrary matrices satisfying $C_1C_1'+C_2C_2'=B-B^2$ and E_i, F_i $(i=1,2,3)$ are arbitrary matrices satisfying

$$M(E_1:E_2:E_3)'\subset N(B:C_1:C_2) \quad \text{and} \quad M(F_1:F_2:F_3)'\subset N(B:C_1:C_2)$$

and

(b) *the representation $p'(X_0'GX_0)^-X_0'GY$ of BLUE of every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2V)$ is its BLIMBE under $(Y, X\beta, \sigma^2V)$ if and only if D is as given in part* (a) *where $B=((B_{ij}))$ $(i,j=1,2)$ satisfies*

$$(I-B_{11})-B_{12}(I-B_{22})^-B_{12}' \text{ is p.d.},$$

B_{11} being the top left hand corner submatrix of B having the same order as that of Γ in Remark 2.1 *and $M(E_1:E_2:E_3)'$ and $M(F_1:F_2:F_3)'$ are also subspaces of $N(U_1':U_2':U_3')$, U_1, U_2, U_3 being arbitrary matrices satisfying*

$$(I-B)U_1-C_1U_2-C_2U_3=I-\Lambda_1^{-1}.$$

Proof. (a) From Lemma 2.2 it follows that for every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2V)$ its BLUE $p'(X_0'GX_0)^-X_0'GY$ is its LIMBE under $(Y, X\beta, \sigma^2V)$ if and only if

$$\begin{aligned} &XX'GX_0(X_0'GX_0)^-X_0'=XX_0' \\ &\Leftrightarrow\ XD'GX_0=0 \\ &\Leftrightarrow\ X_0'GDX_0'=X_0'GDD'. \end{aligned} \tag{2.8}$$

Write $P^{-1}DQ=T=((T_{ij}))$, $i,j=1,2,3$, where the partitioning is clear from the context. Then (2.8) holds if and only if

$$T_{11}=\sum_{i=1}^{3}T_{1i}T_{1i}',\quad 0=\sum_{i=1}^{3}T_{1i}T_{2i}',\quad 0=\sum_{i=1}^{3}T_{1i}T_{3i}',$$

which are equivalent to

$$T_{11}=B,\quad T_{12}=C_1,\quad T_{13}=C_2,\qquad T_{2i}=E_i,\quad T_{3i}=F_i,\qquad i=1,2,3,$$

where these quantities are as specified in the statement of part (a) of the theorem. This completes the proof of part (a).

(b) For every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2V)$, its BLUE $p'(X_0'GX_0)^-X_0'GY$ is its BLIMBE under $(Y, X\beta, \sigma^2V)$ if and only if $X_0'GDX_0'=X_0'GDD'$ and $Z'VGX_0=0$, or, equivalently $M(VGX_0)\subset M(X_0-D)$ where D is given by part (a). Using the representations of X_0, $V+X_0X_0'$ and G specified in Lemma 2.1 and Remark 2.1, we see that the above condition holds if and only if

$$(I-B)U_1-C_1U_2-C_2U_3=I-\Lambda_1^{-1}, \tag{2.9}$$

$$E_1U_1+E_2U_2+E_3U_3=0, \tag{2.10}$$

$$F_1U_1+F_2U_2+F_3U_3=0, \tag{2.11}$$

for some U_1, U_2 and U_3.

When (2.9) is consistent, it follows from (2.10) and (2.11) that E_i and F_i, $i=1,2,3$, should be as stated in the theorem. Since $C_1C_1'+C_2C_2'=B-B^2$ (from part

(a) of the theorem), $M(C_1 : C_2) \subset M(I-B)$ and hence the equation (2.9) is consistent if and only if

$$M(I-\Lambda_1^{-1}) \subset M(I-B). \tag{2.12}$$

Let $\Lambda_1 = \text{diag}(\Lambda_{11}, \Lambda_{22})$, where Λ_{11} and B_{11} are of the same order equal to that of Γ in Remark 2.1. Then

$$I-\Lambda_1^{-1} = (\Lambda_1 - I)\Lambda_1^{-1} = \text{diag}(\Gamma\Lambda_{11}^{-1}, 0),$$

in view of Remark 2.1. Hence (2.12) holds if and only if

$$M(\Lambda_{11}^{-1}\Gamma : 0)' \subset M(I-B). \tag{2.13}$$

Since

$$M(I-B) = M\begin{pmatrix} (I-B_{11}) - B_{12}(I-B_{22})^- B_{12}' & -B_{12} \\ 0 & I-B_{22} \end{pmatrix},$$

(2.13) holds if and only if $(I-B_{11}) - B_{12}(I-B_{22})^- B_{12}'$ is p.d. The proof of Theorem 2.2 is now complete.

3. Solution to Problem (2)

The class of all solutions to Problem (2) can be obtained as union of solutions to Problem (1), the union being taken over all n.n.d. g-inverses of $V+X_0X_0'$. However, given X, one does not know from the above whether there is at least one linear representation of BLUE of every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2 V)$ with the desired optimality condition. We give below several methods for finding this and also obtaining one such linear representation, whenever it exists.

At least one linear representation of the BLUE of every estimable $p'\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is unbiased for $p'\beta$ (or is a LIMBE of $p'\beta$) under $(Y, X\beta, \sigma^2 V)$ if and only if there exists G, a g-inverse of $V+X_0X_0'$ satisfying $X_0'GD=0$ (respectively $X_0'GDX'=0$). This is equivalent to demanding that the following system of equations (3.1) (respectively (3.2)) should have a common solution in G, which can be verified using Theorem 2.2 of Mitra (1973).

$$X_0X_0'GDD'=0, \qquad (V+X_0X_0')G(V+X_0X_0') = V+X_0X_0'; \tag{3.1}$$

$$X_0X_0'GDX'XD'=0, \qquad (V+X_0X_0')G(V+X_0X_0') = V+X_0X_0'. \tag{3.2}$$

In order to verify if X satisfies the requirement in Problem (2) (b) (or (d)), in addition to examining the consistency of (3.1) (respectively (3.2) it is enough to check the condition $X_0'GVZ=0$ for any G, a g-inverse of $V+X_0X_0'$.

The proceding discussion suggests a method of verifying whether the requirements stated in Problem (2) are satisfied for two given linear models. In what follows, we shall present several equivalent solutions to problem (2).

Theorem 3.1. (a) *At least one linear representation of the BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2V)$ is unbiased for $X_0\beta$ under $(Y, X\beta, \sigma^2V)$ if and only if any one of the following equivalent conditions holds:*

(i) $$D = PTQ' \tag{3.3}$$

where P and Q are as given in Lemma 2.1 and $T=((T_{ij}))$ $(i,j=1,2,3)$ satisfies

$$M(T_{11}:T_{12}:T_{13})' \subset M(T_{31}:T_{32}:T_{33})', \tag{3.4}$$

(ii) $M(X_0:0:0)' \subset M(X_0:VZ_0:D)'$,

(iii) $M(X_0) \cap M(VZ_0:D) = \{0\}$,

(iv) $M(VZ_0:D)' = M(Z_0'VZ_0:Z_0'D)'$.

(b) *At least one linear representation of the BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2V)$ is its BLUE under $(Y, X\beta, \sigma^2V)$ if and only if any one of the following equivalent conditions holds:*

(i) *D is given by (3.3) and (3.4) with the further condition*

$$M\begin{pmatrix} T_{21}(\Lambda_1 - I) \\ T_{31}(\Lambda_1 - I) \end{pmatrix} \subset M\begin{pmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{pmatrix} \tag{3.5}$$

where Λ_1 is as given in (2.2),

(ii) $M(X_0:0:0:0)' \subset M(X_0:VZ_0:VZ:D)'$,

(iii) $M(X_0) \cap M(VZ_0:VZ:D) = \{0\}$,

(iv) $M(VZ_0:VZ:D)' = M(Z_0'VZ_0:Z_0'VZ:Z_0'D)'$.

Proof. It is fairly easy to establish the equivalence of (ii), (iii) and (iv) in Theorem 3.1 (a) or (b). In part (a) we shall prove (i) and establish its equivalence with (iii).

We want D to satisfy the condition $X_0'GD=0$ for some n.n.d. g-inverse G of $V+X_0X_0'$. Since $V+X_0X_0'$ is given by (2.2), an n.n.d. g-inverse is

$$G = P'^{-1}\begin{bmatrix} \Lambda_1^{-1} & 0 & R_1 \\ 0 & \Lambda_2^{-1} & R_2 \\ R_1' & R_2' & S \end{bmatrix} P^{-1}$$

where $S - R_1'\Lambda_1R_1 - R_2'\Lambda_2R_2$ is n.n.d.

Let $P^{-1}DQ = T = ((T_{ij}))$, $i,j = 1,2,3$. Then $X_0'GD=0$ for some n.n.d. g-inverse G of $V+X_0X_0'$ if and only if

$$\Lambda_1^{-1}T_{11} + R_1T_{31} = 0, \quad \Lambda_1^{-1}T_{12} + R_1T_{32} = 0, \quad \Lambda_1^{-1}T_{13} + R_1T_{33} = 0,$$

for some R_1, or equivalently

$$M(T_{11}:T_{12}:T_{13})' \subset M(T_{31}:T_{32}:T_{33})',$$

which establishes condition (i) in part (a). To prove the equialence of (i) and (iii) in (a), observe that when X_0 and $V+X_0X_0'$ are given by (2.1) and (2.2), one choice of Z_0 is $Z_0 = P'^{-1}\operatorname{diag}(0, I, I)$ and then $VZ_0 = P\operatorname{diag}(0, \Lambda_2, 0)$. Let D be given by (3.3) and let

$$a = (a_1' : a_2' : a_3')', \quad b = (b_1' : b_2' : b_3')', \quad c = (c_1' : c_2' : c_3')'$$

be any three vectors.

Then $M(X_0) \cap M(VZ_0 : D) = \{0\}$ if and only if

$$X_0 a + VZ_0 b + Dc = 0 \quad \Rightarrow \quad X_0 a = 0$$

or equivalently

$$\operatorname{diag}(I, 0, 0)a + \operatorname{diag}(0, \Lambda_2, 0)b + Tc = 0 \quad \Rightarrow \quad a_1 = 0,$$

that is (3.4) holds. This completes the proof of part (a) of Theorem (3.1). In (b), we shall prove (i) and (ii). To prove (i), observe that at least one linear representation of the BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if D is given by (3.5) and (3.6) with the condition $X_0' GVZ = 0$ or, equivalently, $M(VGX_0) \subset M(X_0 - D)$.

$$(\Lambda_1 - I)\Lambda_1^{-1} = (I - T_{11})K_1 - T_{12}K_2 - T_{13}K_3 \tag{3.6}$$

if and only if

$$0 = T_{21}K_1 + T_{22}K_2 + T_{23}K_3, \tag{3.7}$$

$$0 = T_{31}K_1 + T_{32}K_2 + T_{33}K_3. \tag{3.8}$$

Using (3.4) and (3.8) in (3.6) we get $K_1 = (\Lambda_1 - I)\Lambda_1^{-1}$. Hence (3.6), (3.7) and (3.8) are equivalent to (3.5). To prove (ii), we notice that $L'Y$ is BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$ and $(Y, X\beta, \sigma^2 V)$ if and only if

$$X_0' L = X_0', \quad Z_0' VL = 0, \quad D'L = 0, \quad Z'VL = 0$$

or, equivalently,

$$(X_0 : VZ_0 : VZ : D)'L = (X_0 : 0 : 0 : 0)',$$

that is (ii) holds. The proof of Theorem 3.1 is now complete.

Now let $V + X_0 X_0' = \sum_{i=1}^{n} E_i$ $(k \le n)$ and $DD' = \sum_{i=1}^{n} \lambda_i E_i$ be a spectral representation of DD' relative to $V + X_0 X_0'$ as defined by Mitra and Moore (1973). Let G_0 be a p.d. matrix such that G_0 is a p.d. g-inverse of $V + X_0 X_0'$ and $E_i G_0 E_j = \delta_{ij} E_i$. We now prove:

Theorem 3.2. (a) *At least one linear representation of the BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is unbiased for $X_0\beta$ under $(Y, X\beta, \sigma^2 V)$ if and only if*

$$X = X_0 + (I - P_{X_0, G_0})A \tag{3.9}$$

where A is arbitrary.

(b) *At least one linear representation of the BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$ is* *its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if X is given by (3.9) where A satisfies*

$$M(X_0'G_0V:0)' \subset M(X_0':A'(I-P_{X_0,G_0})')'$$

or equivalently

$$(I-P_{X_0,G_0})A[X_0^- VG_0X_0+(I-X_0^- X_0)B]=0,$$

X_0^- being an arbitrary but fixed g-inverse of X_0 and B arbitrary.

Proof. In view of Theorem 2.1, sufficiency is obvious. To prove the necessity of (3.9), assume that there exists a g-inverse G of $V+X_0X_0'$ such that $X_0(X_0'GX_0)^- X_0'GY$ is unbiased for $X_0\beta$ under $(Y, X\beta, \sigma^2 V)$. Then it is necessary and sufficient that

$$X_0'GD=0,$$

or equivalently

$$\sum_{i=1}^{n} \lambda_i X_0'GE_i=0, \tag{3.10}$$

that is

$$\lambda_i X_0'GE_i=0 \quad (\text{using } E_iG_0E_j=\delta_{ij}E_i) \quad \text{for } i=1,2,\ldots,n. \tag{3.11}$$

Using (3.10), (3.11), and the fact that $M(E_i) \subset M(V+X_0X_0')$ for $i=1,2,\ldots,k$ and $X_0'G_0E_i=0$ for $i>k$, we get

$$\sum_{i=1}^{n} \lambda_i X_0'GE_i=0 \quad \Rightarrow \quad \sum_{i=1}^{n} \lambda_i X_0'G_0E_i=0$$
$$\Leftrightarrow \quad X_0'G_0D=0$$
$$\Leftrightarrow \quad \text{(3.9) holds.}$$

This proves the necessity of (a). The necessity of (b) is proved similarly.

Remark 3.1. Theorems 3.1 and 3.2 not only provide a method of verifying whether there exists a linear representation of the BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$ which is an unbiased estimator (or a BLUE) of $X_0\beta$ under $(Y, X\beta, \sigma^2 V)$, but also suggests a procedure for constructing such a linear representation, whenever one exists.

Remark 3.2. Theorem 3.2 and the conditions (ii), (iii) and (iv) in Theorem 3.1 (a) and (b) are analogous to Theorem 3.1 and Theorem 3.2 respectively in Mitra and Moore (1973).

Remark 3.3. From (3.1) and (3.2) it is clear that the solution to Problem (2) (c) (or (d)) can be obtained by replacing D with DX' in the solution to problem (2) (a) (respectively (b)) given in part (a) (respectively (b)) of Theorem 3.1 and Theorem 3.2.

4. Solution to Problem (3)

We prove:

Theorem 4.1. (a) *The BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$, irrespective of its linear representation is unbiased for $X_0\beta$ under $(Y, X\beta, \sigma^2 V)$ if and only if*

$$X = X_0 + VZ_0A \tag{4.1}$$

where A is arbitrary.

(b) *The BLUE of $X_0\beta$ under $(Y, X_0\beta, \sigma^2 V)$, irrespective of its linear representation is its BLUE under $(Y, X\beta, \sigma^2 V)$ if and only if X is given by* (4.1) *where A satisfies*

$$M(X_0'GV:0)' \subset M(X_0':A'Z_0'V)'$$

or equivalently A satisfies

$$VZ_0A[X_0^- VGX_0 + (I - X_0^- X_0)B] = 0,$$

X_0 being an arbitrary but fixed g-inverse of X_0, B is arbitrary and G is any g-inverse of $V + X_0X_0'$.

Proof. (a) D is required to satisfy the condition

$$X_0'GD = 0 \tag{4.2}$$

for every g-inverse G of $V + X_0X_0'$, which happens if and only if $D = (V + X_0X_0')K$, for some K. From (4.2), we get $K = Z_0A$ for some A and thus $X = X_0 + VZ_0A$, which completes the proof of (a). Proof of (b) is similar to that of Theorem 2.1(b).

Theorem 4.2. *Consider the representations of X_0 and $V + X_0X_0'$ given in* (2.1), (2.2) *and Remark* 2.1. *Then* (a) *the BLUE of every estimable parametric functional under $(Y, X_0\beta, \sigma^2 V)$ irrespective of the linear representation is its LIMBE under $(Y, X\beta, \sigma^2 V)$ if and only if*

$$D = P\begin{pmatrix} B & C_1 & C_2 \\ E_1 & E_2 & E_3 \\ 0 & 0 & 0 \end{pmatrix} Q'$$

where B is an arbitrary n.n.d. matrix with eigenvalues in $[0, 1]$, C_1, C_2 and E_i $(i = 1, 2, 3)$ are arbitrary matrices satisfying

$$C_1C_1' + C_2C_2' = B - B^2 \quad \text{and} \quad M(E_1 : E_2 : E_3)' \subset N(B : C_1 : C_2).$$

(b) *the BLUE of every estimable parametric functional under $(Y, X_0\beta, \sigma^2 V)$, irrespective of the linear representation is its BLIMBE under $(Y, X\beta, \sigma^2 V)$ if and only if D is as given in part* (a), *where $B = ((B_{ij}))$ $(i, j = 1, 2)$ satisfies*

$$I - B_{11} - B_{12}(I - B_{22})^- B_{12}' \text{ is p.d.},$$

B_{11} being the top left hand corner submatrix of B having the same order as that of Γ in remark 2.1 and $M(E_1:E_2:E_3)'$ is also a subspace of $N(U_1':U_2':U_3')$, U_1, U_2 and U_3 being arbitrary matrices satisfying

$$(I-B)U_1-C_1U_2-C_2U_3=I-\Lambda_1^{-1}.$$

Proof. (a) We want D to satisfy the condition (2.8) for every n.n.d. g-inverse G of $V+X_0X_0'$. Using (2.1), (2.2) and the partitioned forms of $T=P^{-1}DQ$ and G as given in the proof of Theorem 3.1(a) (i), we see that (2.8) holds for every n.n.d. g-inverse G of $V+X_0X_0'$ if and only if the equations

$$\Lambda_1^{-1}T_{11}+R_1T_{31}=\Lambda_1^{-1}\sum_{i=1}^{3}T_{1i}T_{1i}'+R_1\sum_{i=1}^{3}T_{3i}T_{1i}',$$

$$0=\Lambda_1^{-1}\sum_{i=1}^{3}T_{1i}T_{2i}'+R_1\sum_{i=1}^{3}T_{3i}T_{2i}',$$

$$0=\Lambda_1^{-1}\sum_{i=1}^{3}T_{1i}T_{3i}'+R_1\sum_{i=1}^{3}T_{3i}T_{3i}'$$

hold for every R_1. The last equation holds for every R_1 if and only if $T_{3i}=0$, $i=1,2,3$. The first two equations then reduce to

$$T_{11}=\sum_{i=1}^{3}T_{1i}T_{1i}' \quad\text{and}\quad 0=\sum_{i=1}^{3}T_{1i}T_{2i}'$$

respectively. The proof of Theorem 4.2 can now be completed along the same lines as that of Theorem 2.2.

Remark 4.1. Application of the results so far obtained to the restricted linear model and some other cases, are discussed in Mathew (1981).

5. Optimality of BLUE's of a subclass of parametric functions

In practice it happens that one may not be interested in inferences on all estimable parametric functionals, but may be interested in only a subset of them, say, for example, certain contrasts in the models of design of experiments. In this section, we shall study the robustness regarding the BLUE's of a subclass of estimable parametric functionals.

Let A be a specified $k\times m$ matrix such that $M(A')\subset M(X_0')$. We shall obtain conditions on X such that a specified linear representation/ some linear representation/ every linear representation of the BLUE of $A\beta$ under $(Y, X_0\beta, \sigma^2V)$ is its BLUE under $(Y, X\beta, \sigma^2V)$ also.

Consider the BLUE $A(X_0'GX_0)^-X_0'GY$ of $A\beta$ under $(Y, X_0\beta, \sigma^2V)$, where G is a specified g-inverse of $V+X_0X_0'$. Let $B=A(X_0'GX_0)^-X_0'G$. BY is also BLUE of $A\beta$

under $(Y, X\beta, \sigma^2 V)$ if and only if $BD=0$ and $BVZ=0$, which are equivalent to the conditions

$$D=(I-B^- B)T \tag{5.1}$$

and

$$BV=W[X_0'+T'(I-B^- B)' \tag{5.2}$$

for some T and W. It is easily verified that for any fixed W, (5.2) is consistent in T if and only if $BVB'=WA'$. Once we obtain matrices W satisfying this condition T can be solved for from (5.2). The following lemma gives a characterization of matrices W satisfying $BVB'=WA'$.

Lemma 5.1. *$BVB'=WA'$ if and only if $W=(L_1:L_2)(S_1':S_2')'$ where $(L_1:L_2)$ is an orthogonal matrix such that the columns of L_1 form a basis for $M(BV)$, S_1 is any solution (of full row rank) of the equation $L_1'BVB'=S_1A'$ and S_2 is any solution of $S_2A'=0$.*

Proof. If L_1 and L_2 are as specified in the lemma, then we can write $W=(L_1:L_2)(S_1':S_2')'$ for some S_1 and S_2. Then

$$BVB'=WA' \quad \Leftrightarrow \quad BVB'=(L_1:L_2)(S_1':S_2')'A'$$

or equivalently

$$L_1'BVB'=S_1A' \quad \text{and} \quad 0=S_2A'.$$

Since

$$\begin{aligned}\text{rank}(L_1'BVB') &= \text{rank}(BV) \text{ (which cannot exceed rank}(A))\\ &= \text{the number of columns of } L_1\\ &= \text{the number of rows of } S_1,\end{aligned}$$

it is easily seen that S_1 satisfying $L_1'BVB'=S_1A'$ is of full row rank. This completes the proof of Lemma 5.1.

If $V=I$, then $B=A(X_0'X_0)^- X_0'$ and the equations (5.1) and (5.2) can be written as

$$D=(I-P_{B'})T \tag{5.3}$$

and

$$B=W[X_0'P_{B'}+(X_0'+T')(I-P_{B'})] \tag{5.4}$$

for some T and W. (5.4) holds if and only if

$$B=WX_0'P_{B'} \quad \text{and} \quad 0=W(X_0'+T')(I-P_{B'}),$$

for some W, which is equivalent to the condition

$$M(B:0)' \subset M(X_0'P_{B'}:(X_0'+T)(I-P_{B'}))', \tag{5.5}$$

that is

$$M(X_0'P_{B'}) \cap M((X_0'+T')(I-P_{B'})) = \{0\}. \tag{5.6}$$

Hence D is given by (5.3), where T satisfies (5.5) or (5.6). Matrices T satisfying (5.5) or (5.6) are also solution of

$$(I-P_{B'})(X_0+T)[(P_{B'}X_0)^- B' + (I-(P_{B'}X_0)^-(P_{B'}X_0))Q] = 0$$

where Q is arbitrary. Observe that the solution obtained here is analogous to that given in Corollary 2.1.

Let G_0 be the p.d. g-inverse of $V+X_0X_0'$ considered in Theorem 3.2. Arguments similar to those in the proof of Theorem 3.2 will lead us to the conclusion that at least one linear representation of the BLUE of $A\beta$ under $(Y, X_0\beta, \sigma^2V)$ is an unbiased estimator (or a BLUE) of $A\beta$ under $(Y, X\beta, \sigma^2V)$ if and only if $A(X_0'G_0X_0)^- X_0'G_0Y$ is an unbiased estimator (respectively BLUE) of $A\beta$ under $(Y, X\beta, \sigma^2V)$. Equivalent conditions similar to those given in theorem 3.1 can be derived in a straightforward manner.

Now let G be any g-inverse of $V+X_0X_0'$ and let $C = A(X_0'GX_0)^- X_0'$. Every linear representation of the BLUE of $A\beta$ under $(Y, X_0\beta, \sigma^2V)$ is its BLUE under $(Y, X\beta, \sigma^2V)$ if and only if

$$D = (V+X_0X_0')(I-C^-C)T \tag{5.7}$$

and

$$CGV = W[X_0' + T'(I-C^-C)'(V+X_0X_0')] \tag{5.8}$$

for some W and T. Let K be a matrix of maximum rank such that $(I-CC^-)'(V+X_0X_0')K = 0$. Observe that K satisfies $M((V+X_0X_0')K) = M(C')$. Then for every fixed W, (5.8) is consistent in T if and only if $CGVK = WX_0'K$. It is fairly easy to observe that

$$M(K'VGC') \subset M(K'X_0) \quad \text{and} \quad \text{rank}(CGVK) = \text{rank}(CGV).$$

Using these facts and arguments similar to those given in the proof of Lemma 5.1, one can easily arrive at the fact that

$$CGVK = WX_0'K \iff W = (L_1 : L_2)(S_1' : S_2')',$$

where $(L_1 : L_2)$ is an orthogonal matrix such that $M(L_1) = M(CGV)$, S_1 is any solution (of full row rank) of $L_1'CGVK = S_1X_0'K$ and S_2 is any solution of $S_2X_0'K = 0$. After obtaining W, T can be solved for from (5.8). We thus have a characterisation of the class of design matrices X such that every linear representation of the BLUE of $A\beta$ under $(Y, X_0\beta, \sigma^2V)$ remains its BLUE under $(Y, X\beta, \sigma^2V)$ also.

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