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On a problem connected with quadratic regression*

By R. G. LAHA and E. LUKACS
The Catholic University of America, Washington, D.C.

1. *Introduction.* We consider two random variables X and Y and denote by $E(Y|X)$ the conditional expectation of Y given X. We say that Y has polynomial regression of order k on X if the relation

$$E(Y|X) = \beta_0 + \beta_1 X + \ldots + \beta_k X^k \tag{1·1}$$

holds almost everywhere. We assume that the first moment of Y and the moment of order k of X exist. It follows from (1·1) that

$$E(Y) = \beta_0 + \beta_1 E(X) + \ldots + \beta_k E(X^k). \tag{1·2}$$

If $k = 2$ and $\beta_2 \neq 0$ ($k = 1$ and $\beta_1 \neq 0$) then we speak of quadratic (linear) regression If $k = 0$, that is if $E(Y|X) = E(Y)$ almost everywhere, then we say that Y has constant regression on X. The coefficients $\beta_0, \beta_1, \ldots, \beta_k$ are called the regression coefficients.

Let $X_1, X_2, \ldots, X_n$ be a sample of size n (independently and identically distributed random variables) from a population with distribution function $F(x)$. We write

$$\Lambda = X_1 + X_2 + \ldots + X_n = n\bar{X}$$

for the sum of the observations and $S = S(X_1, X_2, \ldots, X_n)$ for another statistic. In many cases we know that it is possible to find a statistic S which has constant regression on Λ. Conversely, this property determines sometimes the population.

In the present paper we consider a quadratic statistic

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} X_i X_j + \sum_{j=1}^{n} b_j X_j$$

and study all the populations which have the property that Q has quadratic regression on Λ. It will be necessary to distinguish several cases which are defined in terms of relations between the coefficients a_{ij} and b_j of Q and the regression coefficients $\beta_0, \beta_1, \beta_2$. In each of these cases we show that the population is characterized by the property we mentioned.

We note that we consider here only a quadratic statistic which does not reduce to a linear form since it can be shown easily that every linear form $\sum_{j=1}^{n} b_j X_j$ has linear regression on Λ.

In § 2 we derive a fundamental lemma concerning polynomial regression; in § 3 we obtain a differential equation for the characteristic function of the population. Sections 4 and 5 deal with the solutions of this equation and contain the results.

2. *Two lemmas.* We give first a condition on polynomial regression.

Lemma 1. Let X and Y be two random variables and assume that the expectations $E(Y)$ and $E(X^k)$ exist where k is a non-negative integer. The random variable Y has polynomial regression of order k on X if, and only if, the relation

$$E(Ye^{itX}) = \sum_{v=0}^{k} \beta_v E(X^v e^{itX}) \tag{2·1}$$

holds for all real t.

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If one multiplies (1·1) by e^{itX} and takes the expectation then one sees immediately that condition (2·1) is necessary.

To prove the sufficiency of the condition we assume that (2·1) is valid for all real t. Then

$$E\left\{e^{itX}\left[Y-\sum_{v=0}^{k}\beta_v X^v\right]\right\}=0,$$

or

$$\int_{-\infty}^{\infty} e^{itx} E\left(Y-\sum_{v=0}^{k}\beta_v x^v \middle| x\right) dF_1(x)=0.$$

Here $F_1(x)$ is the marginal distribution of X. We introduce here the probability function $P_x(A)$ of the random variable X instead of the distribution function $F_1(x)$. This is a set function defined on all Borel sets of R_1. The preceding equation becomes then

$$\int_{R_1} e^{itx} E\left(Y-\sum_{v=0}^{k}\beta_v x^v \middle| x\right) dP_x=0.$$

Let $\mu(A)=\int_A E\left(Y-\sum_{v=0}^{k}\beta_v x^v \middle| x\right) dP_x$. This is a function of bounded variation which is defined on all Borel sets A of R_1 and we see that

$$\int_{R_1} e^{itx}\, d\mu=0.$$

Since the uniqueness theorem for characteristic functions is valid for the Fourier transforms of functions of bounded variation we conclude that $\mu(A)=\mu(R)=0$ for all Borel sets A. This is only possible if $E\left(Y-\sum_{v=0}^{k}\beta_v x^v \middle| x\right)=0$ almost everywhere, so that the lemma is proven.

We next prove a lemma which is needed in § 5.

LEMMA 2. *Let a, ρ and λ be three real numbers and suppose that $\rho>0$. The function*

$$f(t)=[\cosh at+i\lambda \sinh at]^{-\rho}$$

is then an infinitely divisible characteristic function.

The statement of the lemma is trivial if $a=0$, it is therefore no restriction to assume that $a \neq 0$.

We consider the function

$$\gamma(z)=\cosh az+i\lambda \sinh az \tag{2·2}$$

of the complex variable $z=t+iy$ (t, y real). We see from (2·2) that $\gamma(z)$ is an entire function of order 1; the zeros of $\gamma(z)$ are the solutions of the system of equations

$$(e^{at}+e^{-at})(\cos ay-\lambda \sin ay)=0, \tag{2·3·1}$$

$$(e^{at}-e^{-at})(\sin ay+\lambda \cos ay)=0. \tag{2·3·2}$$

Equation (2·3·1) has the solutions

$$y_k=\frac{1}{a}\left(\arctan\frac{1}{\lambda}+k\pi\right) \quad (k=0, \pm 1, \pm 2, \ldots), \tag{2·4}$$

while the only real solution of (2·3·2) which is compatible with (2·4) is $t=0$. The zeros of $\gamma(z)$ are therefore given by

$$z_k=iy_k=\frac{i}{a}\left(\arctan\frac{1}{\lambda}+k\pi\right). \quad (k=0, \pm 1, \pm 2, \ldots) \tag{2·5}$$

and are thus purely imaginary. We note that $\gamma(z)$ is real for purely imaginary values of the argument. It follows then from Hadamard's factorization theorem that

$$\gamma(t) = e^{i\delta t} \prod_k \left(1+\frac{it}{y_k}\right) \exp\left(-\frac{it}{y_k}\right), \tag{2·6}$$

where δ is a real constant and where the y_k are given by (2·4).

We conclude from P. Lévy's continuity theorem that $1/\gamma(t)$ is a characteristic function, as a limit of infinitely divisible characteristic functions $1/\gamma(t)$ is necessarily infinitely divisible so that the statement of the lemma is established.

3. *The differential equation for the characteristic function.* Let $X_1, X_2, \ldots, X_n$ be a sample of size n from a population with distribution function $F(x)$ and assume that the second moment of $F(x)$ exists. We denote by

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$$

the characteristic function of $F(x)$. We consider a quadratic statistic

$$Q = \sum_{i,j=1}^{n} a_{ij} X_i X_j + \sum_{j=1}^{n} b_j X_j$$

and suppose that Q has quadratic regression on $\Lambda = \sum_{j=1}^{n} X_j$ so that

$$E(Q|\Lambda) = \beta_0 + \beta_1 \Lambda + \beta_2 \Lambda^2$$

almost everywhere. It follows then from Lemma 1 that

$$E(Q e^{it\Lambda}) = \beta_0 E(e^{it\Lambda}) + \beta_1 E(\Lambda e^{it\Lambda}) + \beta_2 E(\Lambda^2 e^{it\Lambda}) \tag{3·1}$$

holds for all real t.

We denote

$$A_1 = \sum_{i=1}^{n} a_{ii}, \quad A_2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij}, \quad B = \sum_{j=1}^{n} b_j.$$

In a certain neighbourhood of the origin $f(t)$ is different from zero so that we can introduce $\phi(t) = \ln f(t)$. Then

$$\frac{f'(t)}{f(t)} = \phi'(t) \quad \text{and} \quad \frac{f''(t)}{f(t)} = \phi''(t) + [\phi'(t)]^2.$$

We obtain, by means of some elementary computations, the relations

$$E(Q e^{it\Lambda}) = -[f(t)]^n \{A_1 \phi''(t) + (A_1 + A_2)[\phi'(t)]^2 + Bi\phi'(t)\} \tag{3·2·1}$$

and

$$\beta_0 E(e^{it\Lambda}) + \beta_1 E(\Lambda e^{it\Lambda}) + \beta_2 E(\Lambda^2 e^{it\Lambda})$$
$$= -[f(t)]^n \{n\beta_2 \phi''(t) + n^2\beta_2[\phi'(t)]^2 + n\beta_1 i\phi'(t) - \beta_0\}. \tag{3·2·2}$$

These relations are valid in the neighbourhood of the origin in which $\phi(t)$ is defined. We see from equations (3·2·1), (3·2·2) and (3·1) that

$$\gamma_1 \phi''(t) + \gamma_2[\phi'(t)]^2 + i\gamma_3 \phi'(t) = \beta_0, \tag{3·3}$$

where

$$\gamma_1 = n\beta_2 - A_1, \quad \gamma_2 = n^2\beta_2 - A_1 - A_2, \quad \gamma_3 = n\beta_1 - B. \tag{3·4}$$

It is convenient to introduce the function

$$\theta(t) = (1/i)\phi'(t). \tag{3·5}$$

Equation (3·3) can then be written as

$$i\gamma_1 \frac{d\theta}{dt} = \gamma_2\theta^2 + \gamma_3\theta + \beta_0, \tag{3·6}$$

where we wrote θ for $\theta(t)$. It follows from (3·5) that

$$\theta(0) = (1/i)\,\phi'(0) = \alpha \tag{3·6·1}$$

and

$$\left.\frac{d\theta}{dt}\right|_{t=0} = \frac{1}{i}\phi''(0) = i\sigma^2. \tag{3·6·2}$$

Here α and σ^2 are the mean and the variance respectively of the distribution $F(x)$. We put $t = 0$ in (3·6) and obtain the relation

$$-\sigma^2\gamma_1 = \gamma_2\alpha^2 + \gamma_3\alpha + \beta_0 \tag{3·7}$$

between the coefficients of Q and the regression coefficients. It would also have been possible to obtain (3·7) from equation (1·2). This is a consistency condition which must be satisfied.

Our next aim is to obtain all the solutions of equation (3·6). For this investigation we must consider several cases.

We need not concern ourselves with the possibility that all coefficients $\gamma_1, \gamma_2, \gamma_3$ vanish. If $\gamma_1 = 0$ while at least one of the coefficients γ_2 and γ_3 is different from zero then we see that $\theta(t)$ and $\phi'(t)$ are constants so that we obtain a degenerate distribution. We can, therefore, assume without loss of generality that $\gamma_1 \neq 0$.

In § 4 we discuss the case $\gamma_2 = 0$. Here we have to study separately the cases where

$$\gamma_1 \neq 0, \quad \gamma_2 = 0, \quad \gamma_3 = 0 \tag{3·8}$$

and

$$\gamma_1 \neq 0, \quad \gamma_2 = 0, \quad \gamma_3 \neq 0. \tag{3·9}$$

In § 5 we deal with the case $\gamma_2 \neq 0$ and must distinguish three possibilities which depend on the sign of the discriminant $\Delta = \gamma_3^2 - 4\gamma_2\beta_0$.

4. *The case* $\gamma_2 = 0$. We first investigate the case (3·8) and prove the following theorem.

THEOREM 1. Let $X_1, X_2, \ldots, X_n$ be a sample of size n taken from a population which has a finite variance σ^2. Consider a quadratic statistic

$$Q = \sum_{i,j=1}^{n} a_{ij}X_iX_j + \sum_{j=1}^{n} b_jX_j$$

such that

$$A = A_1(n-1) - A_2 \neq 0, \tag{4·1}$$

where

$$A_1 = \sum_{i=1}^{n} a_{ii}, \quad A_2 = \sum_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}.$$

Let β_1 and β_2 be two real constants such that

$$\beta_1 = \frac{B}{n} \quad \left(\text{where } B = \sum_{j=1}^{n} b_j\right), \tag{4·2·1}$$

$$\beta_2 = \frac{1}{n^2}(A_1 + A_2). \tag{4·2·2}$$

The relation

$$E(Q|\Lambda) = \beta_0 + \beta_1\Lambda + \beta_2\Lambda^2 \tag{4·3}$$

holds almost everywhere if, and only if, the following two conditions are satisfied

$$\text{(i)}\quad \beta_0 = A\frac{\sigma^2}{n}; \qquad \text{(ii)}\quad \text{the population is normal.}$$

We first note that the relations (4·1), (4·2·1) and (4·2·2) are equivalent to (3·8). We first prove that the conditions (i) and (ii) are necessary and assume therefore that (4·3) is satisfied. The differential equation (3·6) for $\theta(t)$ reduces in view of (3·8) to

$$i\gamma_1 \frac{d\theta}{dt} = \beta_0. \tag{4·4}$$

we put here $t = 0$ and use (3·6·2) to show that $\beta_0 = -\gamma_1\sigma^2$. Condition (i) follows then immediately from (3·4) and (4·2·2). We integrate equation (4·4) for $\theta(t)$ with the initial conditions (3·6·1) and (3·6·2) and obtain

$$\phi(t) = -\tfrac{1}{2}\sigma^2 t^2 + i\alpha t,$$

or

$$f(t) = \exp\left[-\tfrac{1}{2}\sigma^2 t^2 + i\alpha t\right].$$

This is the characteristic function of a normal (possibly degenerate) distribution so that (ii) follows.

To prove the sufficiency of condition (4·3) we assume that the population is normal so that

$$\phi(t) = \log f(t) = -\tfrac{1}{2}\sigma^2 t^2 + i\alpha t.$$

It follows then from (3·2·1) and (3·2·2) that

$$E(Q\,e^{it\Lambda}) = -[f(t)]^n\{\sigma^4(A_1+A_2)t^2 - \sigma^2[2\alpha(A_1+A_2)+B]\,it - A_1\sigma^2 - B\alpha - \alpha^2(A_1+A_2)\} \tag{4·5·1}$$

and

$$\beta_0 E(e^{it\Lambda}) + \beta_1 E(\Lambda\,e^{it\Lambda}) + \beta_2 E(\Lambda^2 e^{it\Lambda})$$
$$= -[f(t)]^n\{\sigma^4 n^2\beta_2 t^2 - \sigma^2[2\alpha n^2\beta_2 + \beta_1 n]\,it - \beta_0 - \alpha\beta_1 n - n\beta_2\sigma^2 - n^2\alpha^2\beta_2\}. \tag{4·5·2}$$

In view of (4·2·1), (4·2·2) and (i) we see from (4·5·1) and (4·5·2) that

$$E(Q\,e^{it\Lambda}) = \beta_0 E(e^{it\Lambda}) + \beta_1 E(\Lambda\,e^{it\Lambda}) + \beta_2 E(\Lambda^2 e^{it\Lambda})$$

for all real t. We conclude then from Lemma 1 that (4·3) holds almost everywhere. We must examine the case where (3·9) holds and prove the following theorem.

THEOREM 2. Let $X_1, X_2, \ldots, X_n$ be a sample of size n taken from a population which has finite variance σ^2. Consider a quadratic statistic

$$Q = \sum_{i,j=1}^{n} a_{ij}X_iX_j + \sum_{j=1}^{n} b_jX_j$$

such that condition (4·1) of Theorem 1 holds. Let β_1 and β_2 be two real constants such that

$$\beta_1 \neq \frac{B}{n} \quad \left(\text{where } B = \sum_{j=1}^{n} b_j\right), \tag{4·6·1}$$

$$\beta_2 = \frac{1}{n^2}(A_1+A_2). \tag{4·6·2}$$

The relation

$$E(Q|\Lambda) = \beta_0 + \beta_1\Lambda + \beta_2\Lambda^2 \tag{4·3}$$

holds almost everywhere if, and only if, the following three conditions are satisfied:

(i) the population has the Poisson type characteristic function

$$f(t) = \exp\left[\lambda(e^{i\rho t} - 1) + i\mu t\right],$$

where $\lambda > 0$, $\rho \neq 0$ and μ are three real constants,

$$\text{(ii)}\quad \beta_0 = -\frac{A\rho\mu}{n}, \qquad \text{(iii)}\quad \beta_1 = \frac{1}{n^2}(A\rho + nB).$$

Here the relations (4·1), (4·6·1) and (4·6·2) are equivalent to (3·9). To prove the necessity of conditions (i), (ii) and (iii) we assume that (4·3) is satisfied. The differential equation (3·6) for $\theta(t)$ becomes then

$$i\gamma_1 \frac{d\theta}{dt} = \gamma_3\theta + \beta_0. \tag{4·7}$$

We integrate equation (4·7) with the initial conditions (3·6·1) and (3·6·2) and obtain

$$\phi(t) = \frac{\sigma^2\gamma_1^2}{\gamma_3^2}\{\exp(-i\gamma_3 t/\gamma_1) - 1\} - \frac{\beta_0}{\gamma_3} it.$$

We write

$$\lambda = \frac{\sigma^2\gamma_1^2}{\gamma_3^2}, \tag{4·8·1}$$

$$\rho = -\frac{\gamma_3}{\gamma_1}, \tag{4·8·2}$$

$$\mu = -\frac{\beta_0}{\gamma_3} \tag{4·8·3}$$

and obtain the Poisson-type characteristic function (i). We see then easily from (3·4), (4.6.2), (4.8.2) and (4.8.3) that (ii) and (iii) are satisfied.

To prove the sufficiency of (4·3) we assume that the characteristic function of the population is given by (i) and that β_0 and β_1 are defined by (ii) and (iii). We see from (3·2·1) and (3·2·2) that

$$E(Q\,e^{it\Lambda}) = [f(t)]^n\{\rho^2\lambda^2(A_1+A_2)\,e^{2it\rho} + \rho\lambda[\rho A_1 + 2\mu(A_1+A_2) + B]\,e^{it\rho} + \mu^2(A_1+A_2) + B\mu\}, \tag{4·9·1}$$

$$\beta_0 E(e^{it\Lambda}) + \beta_1 E(\Lambda\, e^{it\Lambda}) + \beta_2 E(\Lambda^2 e^{it\Lambda})$$
$$= [f(t)]^n\{\rho^2\lambda^2 n^2\beta_2 e^{2it\rho} + \rho\lambda[n\beta_2 + 2\mu n^2\beta_2 + \beta_1 n]\,e^{it\rho} + \beta_0 + \beta_1 n\mu + \mu^2 n^2\beta_2\}. \tag{4·9·2}$$

We use the relations (4·6·2), (ii) and (iii) and Lemma 1 to show that (4·3) holds almost everywhere.

5. *The case* $\gamma_2 \neq 0$. Our next theorem deals with the case where $\gamma_2 \neq 0$ while

$$\Delta = \gamma_3^2 - 4\gamma_2\beta_0 > 0.$$

THEOREM 3. Let $X_1, X_2, \ldots, X_n$ be a sample of size n taken from a population with finite variance σ^2. Consider a quadratic statistic

$$Q = \sum_{i,j=1}^{n} a_{ij} X_i X_j + \sum_{j=1}^{n} b_j X_j$$

such that condition (4·1) holds. Let β_0, β_1 and β_2 be three real constants such that

$$\beta_2 \neq \frac{A_1 + A_2}{n^2}, \tag{5·1·1}$$

$$(n\beta_1 - B)^2 > 4\beta_0(n^2\beta_2 - A_1 - A_2), \tag{5·1·2}$$

where

$$B = \sum_{j=1}^{n} b_j.$$

The relation

$$E(Q|\Lambda) = \beta_0+\beta_1\Lambda+\beta_2\Lambda^2 \tag{4·3}$$

holds almost everywhere if, and only if, the following four conditions are satisfied:

(i) The population has the characteristic function

$$f(t) = [p\,e^{it\mu_1}+q\,e^{it\mu_2}]^\rho \quad (p+q = 1),$$

$$\text{(ii)}\quad \beta_0 = -\frac{\rho^2 A\mu_1\mu_2}{n\rho-1}, \qquad \text{(iii)}\quad \beta_1 = \frac{B}{n}+\frac{\rho A(\mu_1+\mu_2)}{n(n\rho-1)}, \qquad \text{(iv)}\quad \beta_2 = \frac{A_1(\rho-1)+A_2\rho}{n(n\rho-1)}$$

Thus, the population is either a binomial population (if ρ is a positive integer, then necessarily $p > 0$, $q > 0$) or a negative binomial population (if $\rho < 0$ then $pq < 0$).

We prove first that condition (4·3) is necessary. It follows from (3·4), (5·1·1) and (5·1·2) that $\gamma_2 \neq 0$ and $\Delta = \gamma_3^2-4\gamma_2\beta_0 > 0$. Since Δ is the discriminant of the quadratic form $\gamma_2\theta^2+\gamma_3\theta+\beta_0$ one can write the differential equation (3·6) for $\theta(t)$ in the form

$$i\gamma_1\frac{d\theta}{dt} = \gamma_2(\theta-\eta_1)(\theta-\eta_2), \tag{5·2}$$

where η_1 and η_2 are both real and $\eta_1 \neq \eta_2$. We integrate first this equation, then the equation which results from (3·5) and use the initial conditions (3·6·1) and (3·6·2). In this way we obtain

$$\phi(t) = \frac{\gamma_1}{\gamma_2}\ln\left\{\frac{\alpha-\eta_2}{\eta_1-\eta_2}\exp\left(\frac{\eta_1\gamma_2}{\gamma_1}it\right)-\frac{\alpha-\eta_1}{\eta_1-\eta_2}\exp\left(\frac{\eta_2\gamma_2}{\gamma_1}it\right)\right\}.$$

If we write $\quad f(t) = \exp[\phi(t)] \quad$ and $\quad p = \dfrac{\alpha-\eta_2}{\eta_1-\eta_2}, \quad q = -\dfrac{\alpha-\eta_1}{\eta_1-\eta_2},$

$$\mu_1 = \frac{\eta_1\gamma_2}{\gamma_1}, \quad \mu_2 = \frac{\eta_2\gamma_2}{\gamma_1}, \quad \rho = \frac{\gamma_1}{\gamma_2}$$

we see that $f(t)$ has the form specified in (i).

We put $t = 0$ in (5·2) and see, using (3·6·1) and (3·6·2) that $-\sigma^2\rho = (\alpha-\eta_1)(\alpha-\eta_2)$. From this it follows easily that $p > 0$, $q > 0$ in case $\rho > 0$ while $pq < 0$ if $\rho < 0$. We note that

$$\eta_1+\eta_2 = -\frac{\gamma_3}{\gamma_2} \quad\text{and}\quad \eta_1\eta_2 = \frac{\beta_0}{\gamma_2}.$$

Using these relations and the formulae defining μ_1, μ_2 and ρ as well as (3·4) we can derive the expressions (ii), (iii) and (iv) for β_0, β_1 and β_2.

To prove that our conditions are sufficient we assume that the population has the characteristic function specified by (i) and that β_0, β_1 and β_2 are given by (ii), (iii) and (iv), respectively. We proceed then as in the proof of Theorem 2, and can show that (4·3) holds almost everywhere.

We consider next the case where $\Delta = \gamma_3^2-4\gamma_2\beta_0 = 0$ and prove the following theorem.

THEOREM 4. Let $X_1, X_2, \ldots, X_n$ be a sample of size n taken from a population with finite variance σ^2. Consider a quadratic statistic

$$Q = \sum_{i,j=1}^{n} a_{ij}X_iX_j+\sum_{j=1}^{n} b_jX_j$$

such that condition (4·1) holds. Let β_0, β_1 and β_2 be three real constants such that

$$\beta_2 \neq \frac{A_1+A_2}{n^2}, \tag{5·3·1}$$

$$(n\beta_1 - B)^2 = 4\beta_0(n^2\beta_2 - A_1 - A_2), \tag{5.3.2}$$

where
$$B = \sum_{j=1}^{n} b_j.$$

The relation
$$E(Q|\Lambda) = \beta_0 + \beta_1\Lambda + \beta_2\Lambda^2 \tag{4·3}$$

holds almost everywhere if, and only if, the following four conditions are satisfied:

(i) The population is a Gamma population with characteristic function

$$f(t) = e^{it\mu}\left(1-\frac{it}{\alpha}\right)^{-\rho} \quad (\rho > 0,\ \alpha \neq 0),$$

$$\text{(ii)}\quad \beta_0 = \frac{\mu^2 A}{n\rho+1}, \qquad \text{(iii)}\quad \beta_1 = \frac{B}{n} - \frac{2\mu A}{n(n\rho+1)}, \qquad \text{(iv)}\quad \beta_2 = \frac{A_1(\rho+1)+A_2\rho}{n(n\rho+1)}.$$

We see from (3·4), (5·3·1) and (5·3·2) that $\gamma_2 \neq 0$ and that $\Delta = \gamma_3^2 - 4\gamma_2\beta_0 = 0$. Since Δ is the discriminant of the quadratic from $\gamma_2\theta^2 + \gamma_3\theta + \beta_0$ we can write the differential equation (3·6) for $\theta(t)$ in the form

$$i\gamma_1\frac{d\theta}{dt} = \gamma_2(\theta-\eta)^2, \tag{5·4}$$

where η is real. To prove that condition (4·3) is necessary we solve this equation and determine then $f(t)$ and obtain

$$f(t) = e^{it\eta}\left[1 + \frac{(\alpha-\eta)\gamma_2}{\gamma_1}it\right]^{\gamma_1/\gamma_2}$$

We put $t = 0$ in (5·4) and see that

$$\frac{\gamma_1}{\gamma^2} = -\frac{(\alpha-\eta)^2}{\sigma^2} < 0.$$

If we write
$$\rho = -\frac{\gamma_1}{\gamma_2}, \quad a = -\frac{\gamma_1}{\gamma_2(\alpha-\eta)}, \quad \mu = \eta,$$

then we see that $f(t)$ has the form specified in (i). We note that

$$\frac{\beta_0}{\gamma_2} = \eta^2 \quad \text{and} \quad \frac{\gamma_3}{\gamma_2} = -2\eta.$$

From these relations and from (3·4) we get easily the expressions (ii), (iii) and (iv) for β_0, β_1 and β_2.

The proof of the sufficiency of our conditions is given in the same way as in the preceding theorems and is therefore omitted.

Finally, we consider the case where $\Delta = \gamma_3^2 - 4\gamma_2\beta_0 < 0$ and prove the following theorem.

THEOREM 5. Let $X_1, X_2, \ldots, X_n$ be a sample of size n taken from a population with variance σ^2. Consider a quadratic statistic

$$Q = \sum_{i,j=1}^{n} a_{ij}X_iX_j + \sum_{j=1}^{n} b_jX_j$$

such that condition (4·1) holds. Let β_0, β_1 and β_2 be three constants such that

$$\beta_2 \neq \frac{A_1+A_2}{n^2}, \tag{5·5·1}$$

$$(n\beta_1 - B)^2 < 4\beta_0(n^2\beta_2 - A_1 - A_2), \tag{5·5·2}$$

where $$B = \sum_{j=1}^{n} b_j.$$

The relation $$E(Q|\Lambda) = \beta_0 + \beta_1\Lambda + \beta_2\Lambda^2 \tag{4·3}$$

holds almost everywhere if, and only if, the following four conditions are satisfied:

(i) the characteristic function of the population is given by

$$f(t) = e^{it\mu}[\cosh at + i\lambda \sinh at]^{-\rho},$$

where a, λ, μ, ρ are real constants and $\rho > 0$, $a \neq 0$,

(ii) $\beta_0 = \dfrac{A(\mu^2+\rho^2a^2)}{n\rho+1}$, (iii) $\beta_1 = \dfrac{B}{n} - \dfrac{2A\mu}{n(n\rho+1)}$, (iv) $\beta_2 = \dfrac{A_1(1+\rho)+\rho A_2}{n(n\rho+1)}$.

We see from (3·4), (5·5·1) and (5·5·2) that $\gamma_2 \neq 0$ and $\Delta < 0$. We can then write the differential equation (3·6) for $\theta(t)$ in the form

$$i\gamma_1 \frac{d\theta}{dt} = \gamma_2(\theta-\eta)(\theta-\bar{\eta}), \tag{5·6}$$

where $\eta = \mu + i\nu$, $\bar{\eta} = \mu - i\nu$ (μ, ν real). We determine $f(t)$ from this equation and obtain

$$f(t) = e^{i\mu t}\left[\cosh\frac{\nu\gamma_2}{\gamma_1}t + i\lambda \sinh\frac{\nu\gamma_2}{\gamma_1}t\right]^{\gamma_1/\gamma_2}$$

We put $t = 0$ in (5·6) and see that

$$\frac{\gamma_1}{\gamma_2} = -\frac{(\alpha-\mu)^2+\nu^2}{\sigma^2} < 0.$$

We write $$\rho = -\frac{\gamma_1}{\gamma_2} > 0, \quad a = \frac{\nu\gamma_2}{\gamma_1}$$

and see that $f(t)$ has the form specified in (i). We note that

$$\eta + \bar{\eta} = -\frac{\gamma_3}{\gamma_2} \quad \text{and} \quad \eta\bar{\eta} = \frac{\beta_0}{\gamma_2}.$$

From these relations and from (3·4) we get easily (ii), (iii), and (iv).

The proof of the sufficiency of our conditions is given in the same way as in Theorems 1, 2 and 3.

In conclusion we note that Tweedie (1946) considered earlier the regression of the sample variance on the sample mean. He obtained particular cases of some of the theorems of the present paper. The authors are indebted to Mr Tweedie for calling their attention to his paper.

Reference

Tweedie, M. C. K. (1946). The regression of the sample variance on the sample mean. *J. Lond. Math. Soc.* 21, 22–8.