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Univariate Stochastic Programming with Random Decision Variable

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The problem of minimising $E(X)$ subject to the constraints $X \geqslant 0$, $P(X \geqslant b) \geqslant a (0 < a < 1)$ has been considered, where b is a non-negative random variable with continuous probability distribution. A necessary and sufficient condition for randomised decisions to be superior to the non-randomised one has been derived.

INTRODUCTION

CONSIDER the problem of minimising $E(X)$ subject to the constraints $X \geqslant 0$ and $P(X \geqslant b) \geqslant a$, where $0 < a < 1$ and b is a random variable distributed independently of X with known cumulative distribution function $F(b)$. The following assumptions are made about $F(b)$ throughout the subsequent development:

A1. $F(b) = 0$ for $b < 0$, i.e. b is a non-negative random variable.

A2. $F(b)$ is everywhere continuous and is strictly increasing at every b such that $0 < F(b) < 1$.

By the above assumptions, for each $a(0 < a < 1)$ there exists unique $z(>0)$ such that

$$F(z) = a. \tag{1}$$

A non-randomised solution to the present optimisation problem is, obviously, to take $X = z$.

Vajda[1] and later Mukherjee[2] showed, through examples, that if randomised decisions are admitted (by treating the decision variable as random), the minimum $E(X)$ can be made still smaller. However, both of these authors restricted themselves to some particular form of the distribution function of b. In the present work, keeping $F(b)$ quite general, a necessary and sufficient condition for randomised decisions to be superior to the non-randomised one has been derived.

SUPERIORITY OF RANDOMISATION

For a given form of $F(b)$ and a given a, randomisation is said to be *superior* if there exists a non-degenerate probability distribution of X (say characterized by a distribution function $G(x)$) such that

$$P_G(X \geqslant 0) = 1, \; P_G(X \geqslant b) \geqslant a, \tag{2}$$

$$E_G(X) < z, \tag{3}$$

where z is as in (1). To determine the conditions under which randomisation is superior, define

$$H(x) = \{x - z\}/\{F(x) - a\}, \; x \geqslant 0, \; x \neq z. \tag{4}$$

Theorem 1

For a given form of $F(b)$ and a given $a(0 < a < 1)$ randomisation is superior if and only if

$$\sup_{0 \leqslant x < z} H(x) > \inf_{x > z} H(x) \tag{5}$$

Proof

If: Let (5) hold. Then there exist $x_1 (0 \leq x_1 < z)$ and $x_2 (> z)$ such that

$$H(x_1) > H(x_2). \tag{6}$$

Since $x_1 < z < x_2$, by A2, $F(x_1) < a < F(x_2)$. Consider the randomised decision rule

$$P(X = x_1) = \{F(x_2) - a\}/\{F(x_2) - F(x_1)\},\ P(X = x_2) = \{a - F(x_1)\}/\{F(x_2) - F(x_1)\}.$$

Obviously $P(X \geq 0) = 1$ and

$$P(X \geq b) = F(x_1)\{F(x_2) - a\}/\{F(x_2) - F(x_1)\} + F(x_2)\{a - F(x_1)\}/\{F(x_2) - F(x_1)\} = a.$$

Further,

$$E(X) = x_1\{F(x_2) - a\}/\{F(x_2) - F(x_1)\} + x_2\{a - F(x_1)\}/\{F(x_2) - F(x_1)\} < z,$$

after some simplification using (4) and (6). Hence randomisation is superior.

Only if: Suppose (5) does not hold. Writing

$$h_1 = \operatorname*{Sup}_{0 \leq x < z} H(x),\ h_2 = \operatorname*{Inf}_{x > z} H(x),$$

then

$$0 \leq h_1 \leq h_2. \tag{7}$$

Since for $x < z$, $F(x) < a$ and for $x > z$, $F(x) > a$, we have $H(x)\{F(x) - a\} \geq h_1\{F(x) - a\}$ for $0 \leq x < z$ and $H(x)\{F(x) - a\} \geq h_2\{F(x) - a\}$ for $x > z$, which together with (7) yield

$$H(x)\{F(x) - a\} \geq h_1\{F(x) - a\} \text{ for } x \neq z,\ x \geq 0. \tag{8}$$

By (4) and (8)

$$x \geq z + h_1\{F(x) - a\} \text{ for } x \neq z,\ x \geq 0. \tag{9}$$

Trivially (9) holds for $x = z$ as well. Hence (9) holds for all $x \geq 0$.

Now let X be a non-negative random (degenerate or non-degenerate) variable with distribution function $G(x)$. Then, under the constraints (2), by (9) we have

$$\begin{aligned} E_G(X) &= \int_0^\infty x\,dG(x) \geq \int_0^\infty [z + h_1\{F(x) - a\}]\,dG(x) \\ &= z - h_1 a + h_1 \int_0^\infty \int_0^x dF(b)\,dG(x) = z - h_1 a + h_1 P_G(X \geq b) \\ &\geq z - h_1 a + h_1 a = z, \end{aligned}$$

so that randomisation cannot be superior.

Q.E.D.

OPTIMAL RANDOMISED DECISION

The following two rules, which can be proved easily, lead to optimal randomised decisions in two different situations, which, however, do not exhaust all possibilities:

Rule 1

Let $b^*(> 0)$ be such that

$$\operatorname*{Inf}_{b > 0} \{b/F(b)\} = b^*/F(b^*).$$

Suppose $F(b^*) \geqslant a$. Then an optimal solution is provided by the following probability distribution of X

$$P(X = 0) = 1 - \{a/F(b^*)\},\ P(X = b^*) = a/F(b^*).$$

If the effective range of b is finite (i.e. if there exists a finite positive b_0 such that $F(b) < 1$ for $b < b_0$ and $F(b_0) = 1$), then we have yet another rule:

Rule 2

Let $b^{**}(0 \leqslant b^{**} < b_0)$ be such that

$$\sup_{0 \leqslant b < b_0} [\{b_0 - b\}/\{1 - F(b)\}] = \{b_0 - b^{**}\}/\{1 - F(b^{**})\}.$$

Suppose $F(b^{**}) \leqslant a$. Then an optimal solution is provided by the following probability distribution of X:

$$P(X = b_0) = 1 - \{1 - a\}/\{1 - F(b^{**})\},\ P(X = b^{**}) = \{1 - a\}/\{1 - F(b^{**})\}.$$

Example

Let

$$\begin{aligned} F(b) &= 0,\ b < 0 \\ &= 3b^2 - 2b^3,\ 0 \leqslant b \leqslant 1 \\ &= 1,\ b > 1. \end{aligned}$$

For any $a(0 < a < 1)$, let z be as in (1) and $H(x)$ be as in (4). Applying the techniques of differential calculus, it may be checked that (5) holds if and only if $z < 3/4$ (i.e. if and only if $a < 27/32$). Hence by theorem 1, randomisation is superior if and only if $a < 27/32$. As is easy to check, here b^* as in rule 1, equals 3/4, i.e. $F(b^*) = 27/32$. Hence by rule 1, for $a < 27/32$, the optimal (randomised) solution is

$$P(X = 0) = 1 - (32/27)a,\ P(X = 3/4) = (32/27)a.$$

CONCLUDING REMARKS

In the present work the problem of (linear) stochastic programming involving one random decision variable has been considered. Though the rules presented in the preceding section have wide coverage, they are not universally applicable (for all choices of $F(b)$ and a). The author is working on an algorithm yielding optimum decision rules for all choices of $F(b)$ and a. It is also anticipated that the results described here can be extended to the univariate (non-linear) and multivariate (linear and non-linear) cases, though any such attempt is bound to be much more complicated. It is intended to present these developments in subsequent communications.

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REFERENCES

[1] S. Vajda (1972) *Probabilistic Programming*. Academic Press, New York.

[2] S. P. Mukherjee (1980) Mixed strategies in chance-constrained programming. *J. Opl Res. Soc.* 31, 1045–1047.