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OPTIMALITY ASPECTS OF 3-CONCURRENCE MOST BALANCED DESIGNS

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Abstract: Takeuchi (1961, 1963) established E-optimality of Group Divisible Designs (GDDs) with $\lambda_2 = \lambda_1 + 1$. Much later, Cheng (1980) and Jacroux (1980, 1983) demonstrated E-optimality property of the GDDs with $n = 2$, $\lambda_1 = \lambda_2 + 1$ or with $m = 2$, $\lambda_2 = \lambda_1 + 2$. The purpose of this paper is to provide a unified approach for identifying certain classes of designs as E-optimal. In the process, we come up with a complete characterization of all E-optimal designs attaining a specific bound for the smallest non-zero eigenvalue of the underlying C-matrices. This establishes E-optimality of a class of 3-concurrence most balanced designs with suitable intra- and inter-group balancing. We also discuss the MV-optimality aspect of such designs.

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1. Introduction

Jarrett (1983) defined m-concurrence designs and studied the usefulness of 2-concurrence designs in searching for an upper bound for the efficiency factor of block designs. In this paper, we are primarily concerned with the E-optimality criterion and, among other things, we will establish E-optimality of some classes of 3-concurrence designs.

Takeuchi (1961, 1963) established E-optimality of Group Divisible designs (GDDs) with $\lambda_2 = \lambda_1 + 1$. Much later, Cheng (1980) demonstrated E-optimality property of the GDDs with $n = 2$, $m = \frac{1}{2}v$ and $\lambda_1 = \lambda_2 + 1$. Subsequently, Jacroux (1983) deduced that the GDDs with $m = 2$, $n = \frac{1}{2}v$ and $\lambda_2 = \lambda_1 + 2$ are also E-optimal. These are *all* examples of 2-concurrence designs. Some other related papers on this topic are Jacroux (1980, 1982), Constantine (1981) and Sathe and Bapat (1985).

In this paper, we provide a unified approach to the understanding of the above known E-optimality results and, incidentally, we come up with a complete charac-

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terization of E-optimal designs which attain a specific upper bound for the smallest positive eigenvalue of the underlying C-matrices. Such designs include GDDs of the three types mentioned above as members of a class of 3-occurrence designs with *suitable* intra- and inter-group balancing.

Section 2 contains some definitions and the relevant results from Cheng (1980) and Jacroux (1980). Section 3 presents the main result of the paper and some examples. Section 4 contains some concluding remarks.

2. Preliminaries

For given b, v and k, we assume $bk=vr$, r an integer. Denote by $D(b,v,k)$ the class of all connected block designs for comparing v treatments using b blocks each of size k and by $N=((n_{ij}))$ the incidence matrix of order $v\times b$ of a block design in $D(b,v,k)$. Write $NN'=((\lambda_{ii'}))$. A design is said to be *binary* or *generalized binary* if $n_{ij}=[k/v]$ or $[k/v]+1$, $1\le i\le v$, $1\le j\le b$ where $[x]$ = largest integer not exceeding x. A design is said to be *equireplicate* if the replication numbers for the treatments denoted by $r_1,r_2,\dots,r_v$ are all equal to r. Otherwise, it is called *non-equireplicate* in which case $r_{(1)}$ (the smallest replication number) $< r$. In the following, binary is to be understood as binary or generalized binary according as $k<v$ or $k\ge v$. A *3-concurrence* design is an equireplicate binary design for which the $\lambda_{ii'}$'s assume *exactly* three distinct values. Such a design is said to be *most balanced* if the three distinct λ-values are, in fact, three consecutive integers. An *Intra- and Inter-Group Balanced Design* (IIGBBD) is a design in which the treatments are classified into a number of groups, say t groups such that the treatments in the h-th group have each the replication number equal to r_h (say) and for any two treatments i and i', $\lambda_{ii'}$ is determined through the group or groups to which i and i' belong (Rao (1947)). A (binary equireplicate) 3-concurrence most balanced IIGBBD is similarly defined and we aim at establishing E-optimality of such designs in $D(b,v,k)$.

Following Jacroux (1980), we define

$$\begin{aligned} \alpha &= [k/v] = [r/b], \\ r &= l(\alpha+1)+(b-l)\alpha, \quad l=r-b\alpha, \\ \theta &= l(\alpha+1)^2+(b-l)\alpha^2, \\ rk-\theta &= \delta(v-1)+\varepsilon, \quad 0\le\varepsilon<v-1. \end{aligned} \tag{2.1}$$

Also define

$$C=r^\delta-NN\frac{1}{k}, \tag{2.2}$$

$$g_u=rk-\theta+u, \qquad h_u=v(rk-\theta-u)/(v-2), \tag{2.3}$$

$$T_x=kC-x(I_v-J_v/v), \quad x>0, \tag{2.4}$$

where $r^\delta=\operatorname{diag}(r_1,\dots,r_v)$, I_v is the identity matrix of order v and J_v is the matrix of order $v \times v$ formed of all 1's. We denote by $0 < x_1 \le x_2 \le \cdots \le x_{v-1}$ the non-zero eigenvalues of C.

We now state a lemma which is needed for the sake of completeness. This is essentially based on the type of reasoning initiated by Takeuchi (1961). In the following modified form, this is to be found in Jacroux (1980). We omit the proof.

Lemma 2.1. (a) *If for some $x > 0$, T_x has* (i) *at least one negative eigenvalue, or* (ii) *at least two eigenvalues as zero, then $kx_1 \le x$.*

(b) *If for some i, $t_{x_0 ii} \le 0$ for a suitable choice of x, say, $x = x_0$, then $kx_1 \le x_0$.*

Next we state the following propositions whose proofs are also to be found in Jacroux (1980) and Cheng (1980) and, hence, are omitted.

Proposition 2.1. *For any non-equireplicate design in $D(b, v, k)$,*

(i) $kx_1 < g_{\delta-1} = rk - \theta + \delta - 1$ *for* $k \ge 3$, $v \ge 4$,

(ii) $kx_1 \le (r-1)(k-1)v/(v-1)$.

It may be noted that the second inequality does *not* require the condition $k \ge 3$ and/or $v \ge 4$. Further, it holds whether or not the design is binary.

Proposition 2.2. *For any equireplicate non-binary design,*

$$kx_1 < g_{\delta-1}.$$

Proposition 2.3. *For a binary equireplicate design in $D(b, v, k)$:*

(i) $kx_1 \le rk - \theta + \lambda_{ii'}$ *for all* $i \ne i'$, *with strict inequality when* $\lambda_{is} \ne \lambda_{i's}$ *for some* $s \ne i, \ne i'$;

(ii) $kx_1 \le v(rk - \theta - \lambda_{ii'})/(v-2)$ *for all* $i \ne i'$, *with strict inequality when* $\frac{1}{2}(\lambda_{is} + \lambda_{i's}) \ne (rk - \theta - \lambda_{ii'})/(v-2)$ *for some* $s \ne i, \ne i'$.

The following is now immediate.

Corollary 2.1. *For a binary equireplicate design in $D(b, v, k)$,*

$$kx_1 \le g_u, \qquad kx_1 \le h_\omega \tag{2.5}$$

where

$$u = \min_{i<i'} (\lambda_{ii'}), \qquad \omega = \max_{i<i'} (\lambda_{ii'}) \tag{2.6}$$

Remark. Since $g_x(h_x)$ is increasing (decreasing) in x, it is clear that for *any* binary equireplicate design (recall that $r(k-1) = \delta(v-1) + \varepsilon$, $0 \le \varepsilon < v-1$),

$$\begin{aligned} kx_1 &\le g_\delta = h_\delta \quad \text{when } \varepsilon = 0, \\ kx_1 &\le \min(g_\delta, h_{\delta+1}) \quad \text{when } \varepsilon > 0. \end{aligned} \tag{2.7}$$

This is because for $\varepsilon = 0$, $u \le \delta \le \omega$ while for $\varepsilon > 0$, $u \le \delta < \delta + 1 \le \omega$. Now it can be easily seen that $g_\delta \gtreqless h_{\delta+1}$ as $\varepsilon \lesseqgtr \frac{1}{2}v$. Accordingly, two cases emerge, viz. $kx_1 \le g_\delta \le h_{\delta+1}$ for $\varepsilon \ge \frac{1}{2}v$ and $kx_1 \le h_{\delta+1} < g_\delta$ for $\varepsilon < \frac{1}{2}v$. Moreover, for $\varepsilon = 0$, it can be checked that $g_\delta = h_\delta$ and kx_1 reaches this bound iff the design is a *Balanced Block Design* (BBD).

The above propositions can now be applied to derive some classes of E-optimal designs. The first result in this direction has been due to Takeuchi (1961, 1963) who studied the case of $kx_1 = g_\delta$ in (2.7). We state below his result without proof.

Theorem 2.1. *For $k \ge 3$ and $v \ge 4$, a design for which $kx_1 = g_\delta$ is necessarily binary and equireplicate, and it is* E*-optimal in $D(b, v, k)$. Further, such a design is necessarily either a* BBD *or a* GDD *with $\lambda_2 = \lambda_1 + 1$.*

In the next section, we give the main result of this paper. This is based on a study of $kx_1 = h_{\delta+1}$ in (2.7). Partial studies of this case have been made earlier by Jacroux (1980), Cheng (1980) and Jacroux (1983).

3. Main result

We state and prove the following Theorem which gives a complete characterization of T_x when kx assumes the value $h_{\delta+1}$. This in its turn will lead to a complete characterization of the underlying E-optimal designs in $D(b, v, k)$.

Theorem 3.1. *For given b, v and k with $k \ge 3$, $v \ge 4$ and $bk = vr$, r an integer, suppose $0 < \varepsilon < \frac{1}{2}v$. Then a design for which $kx_1 = h_{\delta+1}$ is necessarily binary and equireplicate, and it is* E*-optimal in $D(b, v, k)$. Further, for such a design, the resulting T_x matrix with $x = h_{\delta+1}$ necessarily assumes the form of a block diagonal matrix with the component matrices given by*

$$\begin{pmatrix} J_{p_i} & -J_{p_i} \\ -J_{p_i} & J_{p_i} \end{pmatrix}, \quad i = 1, 2, \ldots, t \ (say)$$

where the p_i's are positive integers satisfying $\sum_1^t p_i = \frac{1}{2}v$.

Proof. First observe that $\varepsilon > 0$ implies $g_{\delta-1} < h_{\delta+1}$. Hence the first part of the theorem follows. The second part on characterization of the form of T_x needs close arguments which we develop below.

Since kx_1 attains the bound $h_{\delta+1}$, it is clear that $\omega = \delta + 1$. Suppose then that $\lambda_{12} = \delta + 1$. Then for $x = v(rk - \theta - \delta - 1)/(v-2)$, using $\lambda_{ii} = \theta$ (Jacroux, 1980) we get

$$t_{x11} = t_{x22} = -t_{x12} = \frac{v - 1 - \varepsilon}{v - 2} \doteq t' \text{ (say)}, \quad 0 < t' \le 1.$$

Referring to Proposition 2.3(ii), since T_x is n.n.d., we must have for every $s \ne 1, \ne 2$,

$$\lambda_{1s}+\lambda_{2s}=2(rk-\theta-\lambda_{12})/(v-2)=2\delta+2(\varepsilon-1)/(v-2).$$

As $0<\varepsilon<\frac{1}{2}v$, this gives $2\delta\leq\lambda_{1s}+\lambda_{2s}<2\delta+1$ so that essentially $\lambda_{1s}+\lambda_{2s}=2\delta$ for every $s\neq 1, \neq 2$. This forces $\varepsilon=1$ and, hence, $t'=1$. Further, $\lambda_{1s}=\lambda_{2s}=\delta$ or $\lambda_{1s}=\delta\pm 1$, $\lambda_{2s}=\delta\mp 1$ are the only possibilities (as $\omega=\delta+1$). Note incidentally that $\lambda_{ii'}=\delta, \delta\pm 1$ is equivalent to $t_{xii'}=0, \mp 1$ where $x=v(rk-\theta-\delta-1)/(v-2)$. As to the elements of T_x, we then have the following observations:

(i) $t_{xii}=1$, $1\leq i\leq v$; $t_{xii'}=0, \pm 1$, $1\leq i\neq i'\leq v$;

(ii) T_x is n.n.d., $\sum_{i'=1}^{v} t_{xii'}=0$ for every i, $1\leq i\leq v$;

(iii) $t_{x12}=-1$ (assumed);

(iv) $t_{xii'}=-1 \Rightarrow t_{xis}+t_{xi's}=0$ i.e., $t_{xis}=t_{xi's}=0$ or $t_{xis}=\pm 1$, $t_{xi's}=\mp 1$ for every s, $s\neq i, \neq i'$.

Without any loss, set now $t_{x1s}=1$, $t_{x1s'}=-1$ for some s, s', $s\neq s'\neq 2$. Then we immediately get the following structural form of the 4×4 submatrix of T_x corresponding to the rows and columns numbered $(1,2,s,s')$:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

This is equivalent (up to a permutation) to

$$\begin{pmatrix} J_2 & -J_2 \\ -J_2 & J_2 \end{pmatrix}.$$

Moreover, if now $t_{x1s''}=0$ for some $s''\neq s\neq s'\neq 2$, then we immediately deduce that $t_{x2s''}=t_{xss''}=t_{xs's''}=0$. Thus starting with the first entry t_{x11} of T_x, we end up with a block diagonal matrix of the form

$$\begin{pmatrix} J & -J \\ -J & J \end{pmatrix}.$$

Certainly this can be carried further starting with a diagonal entry *not* covered by the above submatrix and using the previous argument. This settles the claim.

Remark. The extreme cases are

(a) $$t=1,\quad p_1=\tfrac{1}{2}v,\quad T_x=\begin{pmatrix} J & -J \\ -J & J \end{pmatrix}$$

and

(b) $$t=\tfrac{1}{2}v,\quad p_1=\cdots=p_t=1,\quad T_x=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\otimes I_{v/2},$$

where $\otimes$ = Kronecker product.

In case (a), the corresponding design is immediately identified as a GDD with $m=2$, $n=\frac{1}{2}v$, $\lambda_2=\lambda_1+2$ and in case (b), we identify the resulting design as a GDD with $n=2$, $m=\frac{1}{2}v$, $\lambda_1=\lambda_2+1$. Earlier, Cheng (1980) and Jacroux (1983) derived these results using quite different arguments. This study seems to unify all the previously known results and, further, it reflects various other structures on the nature of such E-optimal designs. This we elaborate further below.

Clearly, except for the particular cases (a) and (b) presented above, in all other cases, the off-diagonal elements of the T_x-matrix will involve elements 0, ± 1 which, in their turn, will determine the $\lambda_{ii'}$'s as assuming values $\delta, \delta\pm 1$. Thus the resulting design is a most balanced 3-concurrence design with the following group structure of the v treatments.

The treatments fall into t groups with $2p_s$ treatments in the sth group so that $\sum_1^t p_s=\frac{1}{2}v$. Divide the treatments of the sth group into two sets G_s and $\bar{G}_s$ each having p_s treatments. Then $G=\bigcup(G_s\cup\bar{G}_s)$ is the set of all v treatments. As regards the $\lambda_{ii'}$'s we have that

$$\begin{aligned}\lambda_{ii'} &= \delta-1 \quad \text{for both } i,i'\in G_s \text{ or } \bar{G}_s,\ i\neq i',\\ &= \delta+1 \quad \text{for } i\in G_s,\ i'\in \bar{G}_s \text{ or the reverse},\\ &= \delta \qquad\ \ \text{for } i\in G_s\cup\bar{G}_s;\ i'\in G_{s'}\cup\bar{G}_{s'},\ s\neq s'.\end{aligned}$$

Such designs form very special subclasses of what are generally termed *Intra- and Inter-Group Balanced Block Designs* (IIGBBDs). (See Rao (1947).) In the literature, combinatorial and constructional aspects of such designs with unequal replications have been studied quite extensively. See, for example, Adhikary (1965). Below we give an example of an E-optimal 3-concurrence IIGBBD with $\lambda_{ii'}=0$, 1 or 2.

Example. $b=v=12$, $r=k=4$ and

$$\begin{aligned} G_1 &= (1,2), & \bar{G}_1 &= (3,4),\\ G_3 &= (5), & \bar{G}_3 &= (6),\\ G_4 &= (7), & \bar{G}_4 &= (8),\\ G_5 &= (9), & \bar{G}_5 &= (10),\\ \dot{G}_6 &= (11), & \bar{G}_6 &= (12).\end{aligned}$$

See Table 1.

It may be noted that a GDD with $m=6$, $n=2$, $\lambda_1=2$, $\lambda_2=1$ also exists in this design set-up.

Table 1

Blocks	Treatments	Blocks	Treatments	Blocks	Treatments
1	1 4 5 7	5	2 4 9 12	9	5 6 9 11
2	1 4 6 8	6	2 4 10 11	10	5 6 10 12
3	1 3 9 10	7	2 3 5 8	11	7 8 9 10
4	1 3 11 12	8	2 3 6 7	12	7 8 11 12

Concluding remarks

It can be seen that the non-zero eigenvalues of T_x in its most general representation above are $2p_1, 2p_2, \ldots, 2p_t$ each with multiplicity 1. Hence the non-zero eigenvalues of C for such a design are given by $(x+2p_i)/k$, $i=1,2,\ldots,t$, each with multiplicity one and x/k with multiplicity $v-1-t$ where $x=v(rk-\theta-\delta-1)/(v-1)$. From this one can construct C^+, the Moore–Penrose inverse of C and verify that the maximum variance for a paired treatment contrast is $2k\sigma^2/x$ if *some* $p_i \geq 2$ and, otherwise, it is $k\sigma^2\{x^{-1}+(x+2)^{-1}\}$.

The above analysis leads us to the following conclusions as regards A-, D- and MV-optimality.

(1) If a GDD with $n=2$, $\lambda_1=\lambda_2+1$ exists, it is A-, D- and MV-optimal within the class of IIGBBDs. Further, by a result of Jacroux (1983), it is MV-optimal in the entire class $D(b,v,k)$.

(2) If the above GDD does *not* exist, *all* the others in this class are *equivalent* with regard to the MV-optimality criterion. Hence, by a result of Jacroux (1983) (which asserts that the GDD with $m=2$, $\lambda_2=\lambda_1+2$ is MV-optimal in the entire class), these are all MV-optimal in the entire class. As regards A- and D-optimality, however, the GDD with $m=2$, $\lambda_2=\lambda_1+2$ is *least* preferred within this class. At any rate, GDDs of this type seem to be rather rare for $k>2$.

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