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CHARACTERISATION OF FUETER MAPPINGS AND THEIR JACOBIANS

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We continue the study of a class of real analytic mappings (christened Fueter mappings) from open subsets of $\mathbb{R}^n$ into $\mathbb{R}^n$. This class has connections with spaces of holomorphic functions and functions of quaternionic and octonionic variables. We characterise such mappings by a system of partial differential equations and determine conditions for their K-quasiconformality in the sense of Ahlfors.

We also characterise the Jacobian matrices of these mappings. The Jacobian matrices form a family of subgroups of $GL(n, \mathbb{R})$ (which are mutually isomorphic) parametrised by certain projective spaces.

INTRODUCTION

This paper is a continuation of our study of Fueter's interesting class of mappings. The domains and ranges are open subsets of $\mathbb{R}^n$ ($n \geqslant 2$), and the maps are obtained by certain transformations of complex analytic functions. The mappings are as follows: Let ϕ be a holomorphic mapping whose domain is an open subset of the upper half plane $U = \{z : \text{Im}(z) > 0\}$. The n-dimensional Fueter transform of ϕ, denotedby $F_n(\phi)$, is obtained by substituting in ϕ the expression $(e_1 x_1 + \ldots + e_{n-1} x_{n-1}) / \left(x_1^2 + \ldots + x_{n-1}^2\right)^{1/2}$ for the imaginary unit i. (Here $\mathbb{R}^n$ has coordinate $(x_0, \ldots, x_{n-1})$ with unit vectors $e_0, e_1, \ldots, e_{n-1}$).

One main reason for interest in the 'Fueter maps' stems from the fact that $F_4(\phi)$ and $F_8(\phi)$ are expressible as power series in a quaternionic or octonionic variable (respectively) when ϕ has formally-real expansion around real centres.

Nag *et al.*[8] had proved that Fueter diffeomorphisms (and the corresponding quaternionic and octonionic mapping classes) from pseudogroups. The main interest has been in modelling C^∞ manifolds on these pseudogroups. In the same paper[8] a geometrical interpretation of how Fueter diffeomorphisms arise was discussed via certain "rotations" of complex analytic mappings. Using these principles we had been successful in characterising compact hypercomplex and Fueter manifolds in that paper.

In the present study our main purpose is to make a more algebraic attack on the study of these pseudogroups. First of all we are able to characterise Fueter mappings by a set of algebro-differential conditions.

The problem of whether a given C^{∞} manifold can be assigned hypercomplex/Fueter structure is of course intimately related to whether the structure group of the tangent bundle of the manifold can be reduced to the group of Jacobians of hypercomplex/Fueter diffeomorphisms. In this paper we therefore study the Lie groups of Jacobian matrices and their corresponding Lie algebras. The results of Nag *et al.*[8] might therefore be approchable by pure differential geometric methods using the conclusions of the present study.

Imaeda and Imaeda[6,7], have also pursued analytic functions of hypercomplex variables, extending work of Fueter *et al.*. In section 4 we describe the connection between the functions treated here and Imaedas' functions.

1 The Fueter Transformation

The precise definition of $F_n(\phi)$ is :

Let D be a region in U (the standard upper half plane in $\mathbb{C}$) and $'\mathbb{R}^n = \mathbb{R}^n - \{x_0 - \text{axis}\}$ $(n \geqslant 2)$, we set

$$F_n(D) = \left\{(x_0, x_1, \ldots, x_{n-1}) \in \mathbb{R}^n : \left(x_0, \left(x_1^2 + \ldots + x_{n-1}^2\right)^{1/2}\right) \in D\right\} \subseteq {}'\mathbb{R}^n. \quad \ldots(1)$$

Let $\phi : D \to \mathbb{C}$ be complex analytic with real and imaginary part decomposition $\phi = \xi + i\,\eta$, then

$F_n(\phi)$: $F_n(D) \to \mathbb{R}^n$ is defined by

$$F_n(\phi)(x_0, \ldots, x_{n-1}) = \xi(x_0, y) + \sum_{j=1}^{n-1} \frac{e_j x_j}{y} \eta(x_0\, y) \quad \ldots(2)$$

where $y = \left(x_1^2 + \ldots + x_{n-1}^2\right)^{1/2} > 0$.

If the holomorphic map ϕ has real boundary values where the real axis abuts D then a direct application of the reflection principle guarantees that $F_n(\phi)$ can be defined real analytically on the revolved domain $F_n(D)$ together with the corresponding portions of the x_0 — axis.

We can also define a Fueter transform on analytic maps of several complex variables. [For the sake of simplicity of notation we consider the case of 2-complex variables].

Let $\phi \equiv (\phi_1, \phi_2) : D \to \mathbb{C}^2$ be an analytic map.

$$(D \subseteq U^2 = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im}(z_j) > 0;\ j = 1, 2\}.)$$

We define its Fueter transform $F_n^{(2)}(\phi) : F_n^{(2)}(D) \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$F_n^{(2)}(\phi)(x_{0,1} + e_1 x_{1,1} + \ldots + e_{n-1} x_{n-1,1}, x_{0,2} + e_1 x_{1,2} + \ldots + e_{n-1} x_{n-1,2})$$

$$= \left(\xi_1 + \left(\sum_{j=1}^{n-1} e_j x_{j,1}\right)\frac{\eta_1}{y_1}, \xi_2 + \left(\sum_{j=1}^{n-1} e_j x_{j,2}\right)\frac{\eta_2}{y_2}\right) \quad \ldots(3)$$

where $\phi_j = \xi_j + i\eta_j$, $y_j = \left(x^2_{1,j} + \ldots + x^2_{n-1,j}\right)^{1/2}$, $j = 1, 2$.

$$F_n^{(2)}(D) = \{x_{0,1} + e_1 x_{1,1} + \ldots + e_{n-1} x_{n-1,1}, x_{0,2} + e_1 x_{1,2} + \ldots + e_{n-1} x_{n-1,2}) : (x_{0,1} + iy_1, x_{0,2} + iy_2) \in D, y_j \text{ as above}\} \subseteq' \mathbf{R}^n \times {}'\mathbf{R}^n. \quad \ldots(4)$$

Some simple properties are given below.

$$F_n(a\,\phi) = a\,F_n(\phi),\ a \in \mathbf{R} \quad \ldots(5)$$

$$F_n(\phi + \psi) = F_n(\phi) + F_n(\psi) \quad \ldots(6)$$

$$F_n(\phi \circ \psi) = F_n(\phi) \circ F_n(\psi) \quad \ldots(7)$$

$$F_n(j_2 \circ \phi) = j_n \circ F_n(\phi). \quad \ldots(8)$$

(j_n is the conjugation in $\mathbf{R}^n$ i. e., $j_n(x_0, \ldots, x_{n-1}) = (x_0, -x_1, \ldots, -x_{n-1})$)

$$F_n(\phi^{-1}) = F_n(\phi)^{-1} \quad \ldots(9)$$

(whenever ϕ^{-1} is well defined with domain in U).

$$F_n(\phi.\,\psi) = F_n(\phi)).\ F_n(\psi), \text{ for } n = 4 \text{ or } 8. \quad \ldots(10)$$

Remark : A Möbius (conformal) transformation (in dimensions $n > 2$) may not be a Fueter mapping. Indeed even the translation $V \to V + b$, b not purely real, already fails to be a Fueter map.

Definition[1]—Consider a differentiable mapping

$$f : D\,(\subseteq \mathbf{R}^n) \to \mathbf{R}^n.$$

Define,

$$Sf = \tfrac{1}{2}(Df + (Df)^T) - 1/n\,Tr\,(Df)\ I_n$$

where $D\,f$ is the Jacobian matrix of f and $Tr\,(D\,f)$ = Trace of $D\,f$.

$\|S\,f\|$ = (sum of the squares of the entries of $S\,f)^{1/2}$. f is said to be K-quasiconformal if $(1/\sqrt{n})\,\|S\,f\| \leqslant K$.

Proposition 1.1—Suppose $f = F_n(\phi)$ is a Fueter mapping. f is K-quasiconformal if

$$|\,\eta_y - \eta/y\,| \leqslant 2K \text{ over the domain of } \phi = \xi + i\eta. \quad \ldots(11)$$

(The result has already been announced in Nag *et al.*[6]).

PROOF : By direct calculation one obtains

$$\|Sf\|^2 = \left(\eta_y - \frac{\eta}{y}\right)^2 \cdot \frac{(2n^2 - 4n)}{n^2}.$$

$\therefore\ (1/\sqrt{n})\,\|Sf\| = (2n-4)^{1/2}.\, n^{-1}\,|\,\eta_y - \eta/y\,| \leqslant \frac{1}{2}\,|\,\eta_y - \eta/y\,|$.

The result follows.

2. Characterization of Fueter Mappings (Analytically)

Theorem 2.1—Let $f \equiv (f_0, \ldots, f_{n-1}) = F_n(\phi) : F_n(D) \to \mathbf{R}^n$ be a Fueter mapping. Then f satisfies the following relations:

$$\delta_0 f_j = -\,\delta_j f_0 \ (j > 0)\left[\delta_p = \frac{\delta}{\delta x_p},\ p = 0, \ldots, n-1\right] \qquad \ldots(12)$$

$$\delta_k f_j = \delta_j f_k\ (j,\ k > 0) \qquad \ldots(13)$$

$$< \nabla f_0,\ \nabla f_j > = 0\ (j > 0). \qquad \ldots(14)$$

Supertrace of Jac (f) $\left(= \delta_0 f_0 - \sum_{j=1}^{n-1} \delta_j f_j\right)$

$$= (2-n)\left(f_1^2 + \ldots + f_{n-1}^2\right)^{1/2} \Big/ \left(x_1^2 + \ldots + x_{n-1}^2\right)^{1/2}$$

$$at\ (x_0, \ldots, x_{n-1}) \qquad \ldots(15)$$

(in case $n = 2$, (15) becomes $\delta_0 f_0 - \delta_1 f_1 = 0$ which is the second Cauchy-Riemann relation.)

$$(\delta_0 f_0)\, x_k = \sum_{j=1}^{n-1} (\delta_k f_j)\, x_j,\ k > 0 \qquad \ldots(16)$$

(in case $n = 2$, (16) also reduces to the Second Cauchy-Riemann relation.)

$\delta_0 f_0$ is a function of x_0 and $\left(x_1^2 + \ldots + x_{n-1}^2\right) = Y$ only, ...(17)

equivalently,

$$\frac{\delta_1(\delta_0 f_0)}{x_1} = \frac{\delta_2(\delta_0 f_0)}{x_2} = \ldots = \frac{\delta_{n-1}(\delta_0 f_0)}{x_{n-1}} \qquad \ldots(17a)$$

(for $x_1, \ldots, x_{n-1} \neq 0$)

$$\frac{\delta_k f_j}{x_k x_j} = \frac{\delta_p f_q}{x_p x_q}$$

for all $p, q, j, k > 0$, $x_j.\ x_k.\ x_p.\ x_q \neq 0$, and $k \neq j$, $p \neq q$. ...(18)

$\dfrac{\delta_k f_j}{x_k x_j}$ is a function of x_0 and $\left(x_1^2 + \ldots + x_{n-1}^2\right) = Y$ only, ...(19)

for $k \neq j$, $j, k > 0$.

Equivalently,

$$\frac{\delta_1\left(\dfrac{\delta_k f_j}{x_k x_j}\right)}{x_1} = \frac{\delta_2\left(\dfrac{\delta_k f_j}{x_k x_j}\right)}{x_2} = \ldots = \frac{\delta_{n-1}\left(\dfrac{\delta_k f_j}{x_k x_j}\right)}{x_{n-1}} \qquad \ldots(19a)$$

for $x_1, x_2, \ldots, x_{n-1} \neq 0$

$$f_p\,(x_0, \ldots, x_{n-1}) = \left(\delta_0 f_0 - \left(x_1^2 + \ldots + x_{n-1}^2\right)\frac{\delta_k f_j}{x_k\, x_j}\right) x_p \qquad \ldots(20)$$

for $k, j > 0\; k \neq j,\, x_k \cdot x_j \neq 0,\, p > 0.$

$$y\,\delta_0^2\, f_0 - y^3\,\delta_0^2\left(\frac{\delta_k f_j}{x_k\, x_j}\right) + 2y\,\frac{\delta_k(\delta_0 f_0)}{x_k} + \frac{y^3}{x_p}\delta_p\left(\frac{\delta_q(\delta_0 f_0)}{x_q}.\,y\right)$$
$$- 6y\,\frac{\delta_k f_j}{x_k\, x_j} - \frac{6y^3}{x_r}\,\delta_r\left(\frac{\delta_k f_j}{x_k\, x_j}\right) - \frac{y^4}{x_s}\,\delta_s\left[\frac{y}{x_r}\,\delta_r\left(\frac{\delta_k f_j}{x_k\, x_j}\right)\right]$$
$$= 0 \qquad \ldots(21)$$

$$k \neq j,\, k, j, r, p, q, s > 0,\;\; x_j.\, x_p.\, x_q.\, x_r,\, x_s \neq 0,$$

$$y = \left(x_1^2 + \ldots + x^{n-1}\right)^{1/2}.$$

[(21) is unambiguous because of (17a) and (19a)].

$$y^2\,\frac{\delta_p(\delta_0 f_0)}{x_p} - 3y^2\,\frac{\delta_k f_j}{x_k\, x_j} - \frac{y^4}{x_q}\delta_q\left(\frac{\delta_k f_j}{x_k\, x_j}\right) = 0 \qquad \ldots(22)$$

$$x_j.\, x_k.\, x_p.\, x_q \neq 0,\;\; k \neq j,\;\; j, k, p, q > 0.$$

PROOF : The Cauchy-Riemann relation between ξ and η and Laplace's equation $\Delta\eta = 0$ (which is (21)) implies all the above formulae.

Theorem 2.2—$f = (f_0, \ldots, f_{n-1}) : \widetilde{D}\,(\subseteq' \mathbf{R}^n \to \mathbf{R}^n$ is a Fueter map if and only if it satisfies formulas (12) and (17) – (22).

PROOF : Let $y = \left(x_1^2 + \ldots + x_{n-1}^2\right)^{1/2}$

Put $\eta\,(x_0, y) = y\left(\delta_0 f_0 - y^2\,\frac{\delta_k f_j}{x_k\, x_j}\right)$

(it is unambiguous because of (17), (18) and (19)) then by (20)

$$f_k = \frac{x_k}{y}\,\eta,\, k > 0.$$

The equation (12) says

$$\delta_k\, f_0 = -\,\delta_0\, f_k = -\,\frac{x_k}{y}\eta_{x_0},\, k > 0.$$

Therefore,

$$\frac{\delta_1 f_0}{x_1} = \ldots = \frac{\delta_{n-1} f_0}{x_{n-1}}$$

equivalently f_0 depends on x_0 and y only (which means that on x_0 and y constant loci f_0 takes constant values).

One may therefore unambiguously define

$$\xi\,(x_0, y) = f_0\,\,(x_0 + e_1 x_1 + \ldots + e_{n-1} x_{n-1}).$$

Then by eqn. (12) $\xi_y = -\eta_{x_0}$

and by eqn. (22) $\xi_{x_0} = \delta_0 f_0 = \eta_y$.

Equation (21) says η is harmonic in the relavent domain of the $x_0 - y$ plane.

Now one verifies that $f = F_n\ (\xi + i\eta)$ on the relavent domain.

3. Jacobians of Fueter Mappings

Definition—For any $\tilde{k} = (k_1, \ldots k_{n-1}) \in S^{n-2}$, we consider a subgroup of $GL\,(n, \mathbb{R})$:

$$J_n(\tilde{k}) = \left\{\lambda(a, b, c, \tilde{k}) = \begin{bmatrix} a & -bk_1 & -bk_2 & \ldots bk_{n-1} \\ bk_1 & a-\left(1-k_1^2\right)c & ck_1k_2 & \ldots ck_1k_{n-1} \\ bk_2 & ck_1k_2 & a-\left(1-k_2^2\right)c & \ldots ck_2k_{n-1} \\ \vdots & & & \\ bk_{n-1} & ck_1k_{n-1} & ck_2k_{n-1} & \ldots a-\left(1-k_{n-1}^2\right)c \end{bmatrix}\right.$$

$$\left.(a, b) \neq (0, 0) \text{ and } a \neq c\right\} \subseteq GL\,(n, \mathbb{R})$$

Remark : $\lambda(a, b, c, \tilde{k}) = M(\tilde{k})^T\, \lambda(a, b, c, \tilde{1})\, M(\tilde{k})$

where the 'base-point' in S^{n-2} is

$$\tilde{1} = (1, 0, \ldots, 0) \in S^{n-2}$$

and

$$M(\tilde{k}) = \begin{bmatrix} 1 & 0 & 0 & \ldots 0 \\ 0 & k_1 & k_2 & \ldots k_{n-1} \\ 0 & & & \\ \vdots & & N & \\ 0 & & & \end{bmatrix}$$

N is any $(n-2) \times (n-1)$ matrix such that

$M(\tilde{k})$ is orthogonal i. e. $M(\tilde{k})^T = M(\tilde{k})^{-1}$.

Lemma 3.1—$\det(\lambda(a, b, c, \tilde{k})) = (a^2 + b^2)(a - c)^{n-2}$.

Proof : From the remark $\det(\lambda(a, b, c, \tilde{k})) = \det(\lambda(a, b, c, \tilde{1}))$

$= (a^2 + b^2)(a - c)^{n-2}$.

[Also by direct calculation it can be proved that

$$\det(\lambda(a, b, c, \tilde{k})) = (a^2 + b^2)(a - c)^{n-2}].$$

Theorem 3.2— (i) $\widetilde{J}_n(k)$ are commutative subgroups of $GL(n, R)$, (ii) Any two such subgroups are isomorphic to each other, (iii) $J_n(\widetilde{k}) = J_n(-\widetilde{k})$, (iv) for $\widetilde{k}^{(1)} \neq \pm \widetilde{k}^{(2)}$, $J_n(\widetilde{k}^{(1)}) \cap J_n(\widetilde{k}^{(2)}) = \{aI_n/a \neq 0,\ I_n \text{ is identity matrix}\}$.

PROOF : (i) follows from the fact that

$$\lambda(a_1, b_1 c_1, \widetilde{k}).\ \lambda(a_2, b_2, c_2, \widetilde{k}) = \lambda(a_2, b_2, c_2, \widetilde{k}).\ \lambda(a_1, b_1, c_1, \widetilde{k})$$

$$= \lambda(a_1a_2 - b_1b_2, a_1b_2 + b_1a_2, a_1c_2 + c_1a_2 - c_1c_2 - b_1b_2, \widetilde{k})$$

and

$$a_1 \neq c_1\ a_2 \neq c_2 \text{ implies } a_1 a_2 - b_1 b_2 \neq a_1 c_2 + c_1 a_2 - c_1c_2 - b_1b_2.$$

And

$$\lambda(a, b, c, \widetilde{k})^{-1} = \lambda\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}, \frac{-(ac + b^2)}{(a^2 + b^2)(a - c)}, \widetilde{k}\right)$$

and

$$a \neq c \text{ implies } \frac{a}{a^2 + b^2} \neq \frac{-(ac + c^2)}{(a^2 + b^2)(a - c)}.$$

(ii) $\lambda(a, b, c, \widetilde{k}^{(1)}) \mapsto \lambda(a, b, c, \widetilde{k}^{(2)})$ gives an isomorphism.

(iii) follows from the definition of $J_n(\widetilde{k})$

(iv) $aI_n = \lambda(a, 0, 0, \widetilde{k}^{(1)}) = \lambda(a, 0, 0, \widetilde{k}^{(2)})$ belongs to $J_n(\widetilde{k}^{(1)}) \cap J_n(\widetilde{k}^{(2)})$.

Conversly, let $\lambda(a_1, b_1, c_1, \widetilde{k}^{(1)}) = \lambda(a_2, b_2, c_2, \widetilde{k}^{(2)}) \in J_n(\widetilde{k}^{(1)}) \cap J_n(\widetilde{k}^{(2)})$.

If $b_1 \neq 0$ ($\therefore b_2 \neq 0$) compairing the terms of $\lambda(a_1, b_1, c_1, \widetilde{k}^{(1)})$ and $\lambda(a_2, b_2, c_2, \widetilde{k}^{(2)})$ we get, either both $k_i^{(1)}$ and $k_i^{(2)}$ are zero or both

non zero and for non zero $k_i^{(1)}$ and $k_i^{(2)}$

$$\frac{\left(k_i^{(1)}\right)^2}{\left(k_i^{(2)}\right)^2} = \frac{\sum\limits_{k_i^{(1)} \neq 0} \left(k_i^{(1)}\right)^2}{\sum\limits_{k_i^{(2)} \neq 0} \left(k_i^{(2)}\right)^2} = 1$$

which implies $\widetilde{k}^{(1)} = \pm \widetilde{k}^{(2)}$.

Similarly if we assume $c_1 \neq 0$ ($\therefore c_2 \neq 0$) then for $k_i^{(1)} = 0$ if and only if $c_1 k_1^{(1)} k_i^{(1)}$

$= \ldots = c_1 k^{(1)}_{n-1}\ k^{(1)}_i = 0$; if and only if $c_2 k^{(2)}_1\ k^{(2)}_i = \ldots = c_2 k^{(2)}_{n-1}\ k^{(2)}_i = 0$; if and only if $k^{(2)}_i = 0$ and hence by the same argument $\widetilde{k^{(1)}} = \pm \widetilde{k^{(2)}}$. Thus for $\widetilde{k^{(1)}} \neq \pm k^{(2)}$ $b_1 = 0 = c_1$ $(\therefore b_2 = 0 = c_2)$.

Remark : Note that in virtue of Theorem 3.2 (iii) and 3.2 (iv) the distinct sub groups are parametrised by $\mathbb{P}^{n-2}$ (R) (real projective space).

Theorem 3.3—For any $\lambda (a, b, c, \widetilde{k}) \in J_n(\widetilde{k}) = J_n(-\widetilde{k})$, $(\widetilde{k} = (k_1, \ldots, k_{n-1}) \in S^{n-2}$ and $n \geqslant 2)$ and for any point p in {the 2-plane generated by the vectors (1,0,0, ..., 0) and $(0, k_1, \ldots, k_{n-1})$}\real line, (note, if $p = (x_0, x_1, \ldots, x_{n-1})$ then $x_1 : k_1 = x_2 : k_2 \equiv \ldots \equiv x_{n-1} . k_{n-1})$, there exists a Fueter mapping $f \equiv (f_0, \ldots, f_{n-1})$ [i. e. $\exists$ holo ϕ with $f = F_n(\phi)$] such that

$$d_p f \equiv [\text{Jac}(f)]_p = \lambda (a, b, c, \widetilde{k}).$$

PROOF : $\lambda (a, b, c, \widetilde{k}) \in J_n(\widetilde{k})$.

Let

$$p = (p_0, p_1 k_1, \ldots, p_1 k_{n-1})$$

we may assume $p_1 > 0$.

[Since if $p_1 < 0$ we can replace p_1 by $-p_1$, $\widetilde{k}$ by $-\widetilde{k}$ and b by $-b$].

Consider the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \lambda^* (a, b)$.

There exists a complex analytic function $\phi = \xi + i\eta$ defined in a neighbourhood D of (p_0, p_1) in $\cup$ to $\mathbb{C}$ with $\eta(p_0, p_1) = (a - c) p_1$ such that

$$[\text{Jac}(\phi)]_{(p_0, p_1)} = \lambda^* (a, b).$$

Consider the Fueter function $f = F_n(\phi)$

$$\text{i. e. } f(x_0, \ldots, x_{n-1}) = \xi(x_0, y) + \frac{e_1 x_1 + \ldots + e_{n-1} x_{n-1}}{y} \eta(x_0, y)$$

where

$$y = \left[x_1^2 + \ldots + x_{n-1}^2\right]^{1/2}.$$

Then,

$$[\text{Jac}(f)]_p = \begin{bmatrix} \partial_0 f_0 & \ldots & \partial_{n-1} f_0 \\ \partial_0 f_1 & \ldots & \partial_{n-1} f_1 \\ \cdots & \cdots & \cdots \\ \partial_0 f_{n-1} & \ldots & \partial_{n-1} f_{n-1} \end{bmatrix}_p$$

$$= \begin{bmatrix} \eta_y & -\eta_{x_0}\frac{x_1}{y} & -\eta_{x_0}\frac{x_2}{y} & \dots & -\eta_{x_0}\frac{x_{n-1}}{y} \\ \eta_{x_0}\frac{x_1}{y} & \frac{x_1^2}{y^2}\left(\eta_y-\frac{\eta}{y}\right)+\frac{\eta}{y} & \frac{x_1x_2}{y^2}\left(\eta_y-\frac{\eta}{y}\right) & \dots & \frac{x_1x_{n-1}}{y^2}\left(\eta_y-\frac{\eta}{y}\right) \\ \eta_{x_0}\frac{x_2}{y} & \frac{x_1x_2}{y}\left(\eta_y-\frac{\eta}{y}\right) & \frac{x_2^2}{y^2}\left(\eta_y-\frac{\eta}{y}\right)+\frac{\eta}{y} & \dots & \frac{x_2x_{n-1}}{y^2}\left(\eta_y-\frac{\eta}{y}\right) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \eta_{x_0}\frac{x_{n-1}}{y} & \frac{x_1x_{n-1}}{y^2}\left(\eta_y-\frac{\eta}{y}\right) & \frac{x_2x_{n-1}}{y^2}\left(\eta_y-\frac{\eta}{y}\right) & \dots & \frac{x_{n-1}^2}{y^2}\left(\eta_y-\frac{\eta}{y}\right)+\frac{\eta}{y} \end{bmatrix},$$

$$= \lambda(a, b, c, \tilde{k}).$$

Definition—From (7) and (9) it follows that the Fueter diffeomorphisms form pseudogroups. We call a n-dimensional C^∞ manifold a n-dimensional Fueter manifold if it is modelled on one such pseudogroup.

Thus, a Fueter manifold is a manifold with transition functions from the class of Fueter diffeomorphisms.

Theorem 3.4— (i) $\text{Det}\,[\text{Jac} f_n(\phi)]_{(x_0,x_1,\dots,x_{n-1})} = \left(\frac{\eta}{y}\right)^{n-2} [\det(\text{Jac}(\phi))]_{(x_0,y)}$

where

$$\phi = \xi + i\eta\, y = \left[x_1^2 + \dots + x_{n-1}^2\right]^{1/2} \neq 0.$$

(ii) Fueter manifold of even dimension (with coordinate charts $\phi : U \to \psi(U) \subseteq R^n$) are orientable. Also those odd dimensional manifolds (we also called them Fueter manifolds) modelled on the pseudogroup of Fueter maps obtained from holomorphic maps of the type $\phi : D\,(\subseteq U) \to \varphi(D) \subseteq U$ are orientable (for any dimension $n \geqslant 2$).

PROOF : (i) $\text{Det}\,[\text{Jac}\,F_n(\phi)]_{(x_0,x_1,\dots,x_{n-1})})$

$$= \det\left(\begin{bmatrix} \eta_y & -\eta_{x_0}\frac{x_1}{y} & \dots\eta_{x_0}\frac{x_{n-1}}{y} \\ \eta_{x_0}\frac{x_1}{y} & \frac{x_1^2}{y^2}\left(\eta_y-\frac{\eta}{y}\right)+\frac{\eta}{y} & \dots \frac{x_1x_{n-1}}{y^2}\left(\eta_y-\frac{\eta}{y}\right) \\ \dots & \dots & \dots \\ \eta_{x_0}\frac{x_{n-1}}{y} & \frac{x_1x_{n-1}}{y^2}\left(\eta_y-\frac{\eta}{y}\right) & \dots\frac{x_{n-1}^2}{y^2}\left(\eta_y-\frac{\eta}{y}\right)+\frac{\eta}{y} \end{bmatrix}_{(x_0,y)}\right)$$

$\left(\text{where } y = \left[x_1^2 + \dots + x_{n-1}^2\right]^{1/2}\right)$

$$= \det(\lambda(a, b, c, \widetilde{k}))\Big[\text{where } k_i = \frac{x_i}{y} \quad i = 1, \dots, n-1,$$

$$\tilde{k} = (k_1, \ldots, k_{n-1}),\ b = \eta_{x_0}\ (x_0, y))]$$

$$= (a^2 + b^2)\,(a - c)^{n-2}\left[\ a = \eta_y\,(x_0, y),\ c = \eta_y\,(x_0, y) - \frac{\eta(x_0, y)}{y}\right]$$

$$= \left(\frac{n}{y}\right)^{n-2} \det\,([\mathrm{Jac}\,(\phi)]_{(x_0, y)}).$$

(ii) From (i) if n is even and $\eta \neq 0$ then the determinant of the Jacobian of the Fueter transformation is positive. And therefore such manifolds are orientable.

We observe that $X \equiv (x^1, x^2, x^3)$: $\lambda\,(a, b, c, \tilde{k}) \mapsto (a, b, c)$ defines a global C^∞ structure on $J_n(\tilde{k})$. The multiplication in $J_n\,(\tilde{k})$ is equivalent to the mapping! $R^6 \equiv R^3 \times R^3 \to R^3$ given by $((a_1, b_1, c_1), (a_2, b_2, c_2)) \mapsto (a_1a_2 - b_1b_2,\ a_1b_2 + b_1a_2,\ a_1c_2 + c_1a_2 - c_1c_2 - b_1b_2)$ which is C^∞, and $\lambda\,(a, b, c, \tilde{k}) \mapsto \lambda\,(a, b, c, \tilde{k})^{-1}$ is equivalent to the map:

$$'R^3 \to 'R^3 \text{ as } (a, b, c) \mapsto \left(\frac{a}{a^2 + b^2},\ \frac{-b}{a^2 + b^2},\ \frac{-(ac + b^2)}{(a^2+b^2)\,(a - c)}\right)$$

which is also C_∞. This shows directly that $J_n\ (\tilde{k})$ is a 3-dimenional Lie subgroup of $GL\,(n, R)$.

Remark : The isomorphisms defined in Th. 3.2 are actually Lie group isomorphisms.

Remark : We can calculate explicitly the Lie algebra $L\,(J_n\,(\tilde{k}))$ (the Lie algebra of all left invariant vector fields) $T_e\ (J_n\ (\tilde{k}))$ (the tangent space of $J_n\ (\tilde{k})$ at the identity $(e = \lambda\,(1, 0, 0, \bar{k})) = < e_1, e_2, e_3 : e_i = \left(\frac{\partial}{\partial x^i}\right)_e >$

Let $\mathcal{L}\ (J_n\,(\tilde{k}))$ be all the left invariant vector fields. Then

$$\mathcal{L}\,(J_n\,(\tilde{k})) = < X_1, X_2, X_3 \mid {}_iX : \lambda \to L_\lambda * \left(\frac{\partial}{\partial x^i}\right)_e ;\ i = 1, 2, 3 >.$$

Since $J_n\,(k)$ is commutative than of course all Lie backets vanish. Also by some simple calculations

$$X_1 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}$$

[these are a basis for the left-invariant vector fields]

$$X_2 = x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} - x^3 \frac{\partial}{\partial x^3}$$

$$X_3 = (x^1 - x^3)\ \frac{\partial}{\partial x^3}.$$

And,

$$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0.$$

For higher dimensional Fueter transforms $\left(F_n^{(l)}\right)$ the Jacobians form families of non-commutative Lie subgroups (of dimension $2l^2 + l$) in $GL\,(\eta l, R)$. The Lie algebra of any one of these subgroups is non trivial and has been explicitly computable, (details are available with the author).

For 2-dimensional Fueter transformation $F_n^{(2)}$ the Jacobians are of the form

$$\lambda(a, b, c, d, e, f, g, h, p, q, \tilde{k}, \tilde{m})$$

$$= \begin{bmatrix} a & c & k_1e & -m_1g & -k_2e & m^2g & \ldots -k_{n-1}e & -m_{n-1}g \\ b & d & -k_1f & -m_1h & -k_2f & -m_2h & \ldots -k_{n-1}f & -m_{n-1}h \\ lk_1 & gk_1 & a-\left(1-k_1^2\right)p & k_1m_1c & k_1k_2p & k_1m_2c & \ldots k_1k_{n-1}p & k_1k_{n-1}c \\ m_1f & m_1h & k_1m_1p & d-\left(1-m_1^2\right)q & m_1k_2b & m_1m_2q & \ldots m_1k_{n-1}b & m_1m_{n-1}q \\ k_2e & k_2g & k_1k_2b & m_1k_2c & a-\left(1-k_2^2\right)c & k_1m_2c & \ldots k_2k_{n-1}p & ck_2m_{n-1} \\ m_2f & k_2h & k_1m_2b & m_1m_2q & k_2m_2b & d-\left(1-m_2^2\right)q & \ldots m_2k_{n-1}b & m_2m_{n-1}q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{n-1}e & k_{n-1}g & k_1k_{n-1}p & m_1k_{n-1}c & \ldots & \ldots & \ldots a-\left(1-k_{n-1}^2\right)p & ck_{n-1}m_{n-1} \\ m_{n-1}f & m_{n-1}h & k_2m_{n-1}b & m_2m_{n-1}q & \ldots & \ldots & \ldots k_{n-1}m_{n-1}b & d-\left(1-m_{n-1}^2\right)q \end{bmatrix} \quad \ldots(23)$$

(where $\tilde{k}, \tilde{m} \in s^{n-2}$)

with $\det(\lambda(a, b, c, d, e, f, g, h, p, q, \tilde{k}, \tilde{m}))$.

$$= (ad - bc)^2 + (eh - gf)^2 \; (a - p)^{n-2} (b - q)^{n-2} \qquad \ldots(24)$$

And for $(\tilde{k}, \tilde{m}) \in S^{n-2} \times S^{n-2}$

$$J_n \; (\tilde{k}, \tilde{m}) = \{\lambda (a, b, c, d, e, f, g, h, p, q, \tilde{k}, \vec{m}) : a \neq p, b \neq q$$
$$(ad, eh) \neq (bc, gf)\} \subseteq GL\,(2n, \mathrm{R})$$

is a non-commutative Lie group with multiplication:

$$\lambda (a_1, b_1, c_1, d_1, e_1, f_1, g_1, h_1, p_1, q_1, \tilde{k}, \tilde{m}).\; \lambda (a_2, b_2, c_2, d_2, e_2, f_2, g_2, h_2 \tilde{k}, \tilde{m})$$

$$= \lambda \left(\begin{bmatrix} a_1 a_2 \\ +c_1 b_2 \\ -e_1 e_2 \\ -g_1 f_2 \end{bmatrix}, \begin{bmatrix} b_1 a_2 \\ +d_1 b_2 \\ -f_1 e_2 \\ -h_1 f_2 \end{bmatrix}, \begin{bmatrix} a_1 c_2 \\ +c_1 d_2 \\ -e_1 g_2 \\ -g_1 h_2 \end{bmatrix}, \begin{bmatrix} b_1 c_2 \\ +d_1 d_2 \\ -f_1 g_2 \\ -h_1 h_2 \end{bmatrix}, \right.$$

$$, \begin{bmatrix} a_1 e_2 \\ +c_1 f_2 \\ +c_1 a_2 \\ +g_1 b_2 \end{bmatrix}, \begin{bmatrix} b_1 e_2 \\ +d_1 f_2 \\ +f_1 a_2 \\ +h_1 b_2 \end{bmatrix}, \begin{bmatrix} a_1 g_2 \\ +c_1 h_2 \\ +e_1 c_2 \\ +g_1 d_2 \end{bmatrix}, \begin{bmatrix} b_1 g_2 \\ +d_1 h_2 \\ +f_1 c_2 \\ +h_1 d_2 \end{bmatrix}$$

$$\left. \begin{bmatrix} c_1 b_2 \\ -e_1 e_2 \\ -g_1 f_2 \\ +a_1 p_2 \\ +p_1 a_2 \\ -p_1 p_2 \end{bmatrix}, \begin{bmatrix} b_1 c_2 \\ -f_1 g_2 \\ -h_1 h_2 \\ +d_1 q_2 \\ +q_1 d_2 \\ -q_1 q_2 \end{bmatrix}, \tilde{k}\; \tilde{m} \right) \qquad \ldots(25)$$

And the Lie Algebra of any of the Lie groups is

$$L\,(J_n\,(\tilde{k}, \tilde{m})) = \langle X_1, \ldots, X_{10}, [\;,\;] \rangle \text{ where}$$

$$X_1 = x^1 \partial_1 + x^2 \partial_2 + x^5 \partial_5 + x^6 \partial_6 + x^9 \partial_9,$$

$$X_2 = x^3 \partial_1 + x^4 \partial_2 + x^7 \partial_5 + x^8 \partial_6 + x^3 \partial_9,$$

$$X_3 = x^1 \partial_3 + x^2 \partial_4 + x^5 \partial_7 + x^6 \partial_8 + x^9 \partial_{10},$$

$$X_4 = x^3 \partial_3 + x^4 \partial_4 + x^7 \partial_7 + x^8 \partial_8 + x^{10} \partial_{10},$$

$$X_5 = - x^5 \partial_1 - x^6 \partial_2 + x^1 \partial_5 + x^2 \partial_6 - x^5 \partial_9,$$

$$X_6 = - x^7 \partial_1 - x^8 \partial_2 + x^3 \partial_5 + x^4 \partial_6 - x^7 \partial_9,$$

$$X_7 = - x^5 \partial_3 - x^6 \partial_4 + x^1 \partial_7 + x^2 \partial_8 - x^6 \partial_{10},$$

$$X_8 = -x^4\partial_3 - x^8\partial_4 + x^8\partial_7 + x^4\partial_8 - x^8\partial_{10},$$
$$X_9 = (x^1 - x^9)\,\partial_9,\ X_{10} = (x^4 - x^{10})\partial_{10} \qquad \text{...(26)}$$

and

$$[X_1, X_2] = [X_3, X_4] = [X_7, X_8] = [X_6, X_7] = X_3,$$
$$[X_3, X_1] = [X_8, X_7] = [X_4, X_6] = [X_2, X_8] = X_6,$$
$$[X_1, X_7] = [X_6, X_3] = [X_3, X_6] = [X_7, x_4] = X_7,$$
$$[X_4, X_2] = [X_6, X_8] = X_2, [X_2, X_7] = [X_6, X_3] = X_9 - X_8,$$
$$[X_6, X_7] = X_1, [X_2, X_8] = X_4 + X_{10} - X_1 - X_9,$$
$$[X_i, X_j] = 0 \qquad \text{...(27)}$$

where $[X_i, X_j]$ or $[X_j, X_i]$ are not amongst the above.

4. Connection between Fueter and Regular Maps

Ima eda and Imaeda[6] had defined some generalizations of analytic functions of an octonionic variable which are similar to those for quaternionic variables defined and discussed by Fueter[5]. Imaeda and Imaeda[7] also generalized these concepts over v ariables from more general algebras. The Fueter maps which are discussed in Section 1 an d 2 and the regular maps defined by Imaeda and Imaeda actually from disjoint c lasses (except for the constant maps, being members of both). However, there is some connection between these two classes which we are going to discuss.

Definition —(due to Imaeda and Imaeda, of regular functions). A map $f \equiv (f_0, \ldots, f_{n-1}) \equiv e_0 f_0 + \ldots + e_{n-1} f_{n-1}$: $\tilde{D} (\subseteq R^n) \to R^n$ is called left regular (resp. right regular) if $Df = 0$ (resp. $fD = 0$) where $D = \sum_{j=0}^{n-1} e_j\,\partial_j$ and $e_j e_k + e_k e_j = -2\delta_{j,k}$.

Proposition 4.1—A map f both Fueter and regular in the sense of Imaeda and Imaeda if and only if it is a real constant.

Proof : Let $f : \tilde{D}$ ($\subseteq' R^n$) $\to R^n$ be both left regular and Fueter $\tilde{D}$ i. e., $Df(x_0,\ldots,x_{n-1}) = 0$ for $(x_0, \ \ ,x_{n-1}) \in \tilde{D}$ and there exists holomorphic map $\phi = \xi + i\eta$ on the relevant domain such that $f = F_n(\phi)$ on $\tilde{D}$. Equivalently

$$\left(\partial_0 + \sum_{j=1}^{n-1} e_j\,\partial_j\right)\left(\xi(x_0, y) + \sum_{j=1}^{n-1} \frac{e_j\, x_j}{y}\,\eta\, x_0, y)\right) = 0$$

(where $y = \left(x_1^2 + \ldots + x_{n-1}^2\right)^{1/2}$) which gives by using Cauchy-Riemann relation between ξ and η, $\dfrac{(n-2)\,\eta\,(x_0, y)}{y} = 0.$

Therefore for $n > 2$, $\eta = 0$, which implies ϕ and therefore f are constant maps which constant real values. Similarly for right regularity.

For even dimensions one can generate regular maps in the sense of Imaeda from Fueter maps. For interest's sake we state this connection below (see Imaeda Imaeda[6]).

Proposition 4.2—If $f : \widetilde{D}\,(\subseteq R^{2r}) \to R^{2r}$ is a Fueter map then $\Box^{r-1} f$ is both left and right regular. $\left(\text{where } \Box = D\,\bar{D} = \sum_{j=0}^{2r-1} \partial_j^2\right)$.

Remark 1: Note that for the trivial 2-dimensional case Fucier maps and Imaeda's regular maps all coincide with usual holomorphic maps.

Remark 2 : As a general reference for previous work in this and allied areas we refer to Brackx *et al.*[2].

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