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A Joint Universality Theorem for Dirichlet L-Functions

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1. Introduction

In [10] Voronin proved[1] the following universality theorem for the Riemann Zeta function. Let D be any closed disc contained in the strip $\frac{1}{2} < \operatorname{Re}(z) < 1$. Let f be any non-vanishing continuous function on D which is analytic in the interior of D. Let $\varepsilon > 0$. Then there exist real numbers t such that

$$\sup_{z \in D} |\zeta(z+it) - f(z)| < \varepsilon. \tag{1.1}$$

Voronin mentions in [10] that the analogoue of this result for an arbitrary Dirichlet L-function is valid.

In this paper we prove that (1.1) holds for arbitrary $\varepsilon > 0$ provided D is any simply connected compact subset of the strip $\frac{1}{2} < \operatorname{Re}(z) < 1$, and f is any non-vanishing continuous function on D which is analytic in the interior (if any) of D. Further, even under these relaxed assumptions, the set of all real t satisfying (1.1) has positive lower density. The analogue of this result for an arbitrary but fixed Dirichlet L-function is also valid.

These results may easily be deduced from the main theorem (3.1) of this paper. This is a result on simultaneous approximation by vertical translates of all the Dirichlet L-functions with a given modulus.

Section 2 lists the notations used. In Sect. 3 we state the main result and derive some of its consequences. In particular the Riemann hypothesis is shown (Theorem 3.7) to be equivalent to a version of almost periodicity (viz. strong recurrence) of the Zeta function in the strip $\frac{1}{2} < \operatorname{Re}(z) < 1$.

Following Voronin in [10], we base the proof of the main theorem on a suitable Hilbert space result (Proposition 4.3). The requisite lemmas are proved in Sect. 4. The deduction of the main theorem (3.1) from Lemma 4.10, as given in Sect. 5, is based on an idea of Reich in [6]. The results of this paper form a part of the author's Ph.D. thesis [1] where the proofs were based on probabilistic ideas which have been avoided here to the maximum possible extent.

[1] Although the formulation given by Voronin in [10] appears to be more specialised than the one given here, the two statements are easily seen to be equivalent

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2. Notations

In this paper $\mathbb{R}$ and $\mathbb{C}$ will stand for the real line and the complex plane respectively. $\Pi=\{z\in\mathbb{C}: |z|=1\}$ is the unit circle. $\Omega=\{z\in\mathbb{C}: \frac{1}{2}<\mathrm{Re}(z)<1\}$ is the sub-critical strip. For $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$, the modulus $|z|$ of z is defined by $|z|^2=\sum_{j=1}^{n}|z_j|^2$.

For any subset A of a topological space X, we denote by I_A the indicator function of A. That is, I_A: $X\to\{0,1\}$ is defined by $I_A(x)=1$ if $x\in A$, $=0$ otherwise.

For Borel subsets A of the real line, we denote by $\underline{d}(A)$, $\bar{d}(A)$ and $d(A)$ respectively the lower density, the upper density and the density. That is,

$$\underline{d}(A)=\liminf_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}I_A(t)\,dt,$$

$$\bar{d}(A)=\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}I_A(t)\,dt,$$

$$d(A)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}I_A(t)\,dt, \text{ when it exists.}$$

For any planar region U, we denote by $H(U)$ the space of all analytic functions on U, with the topology of uniform convergence on compacta. In particular, we put $H=H(\Omega)$. For any topological space X, and a subspace A of X, we denote by $bd(A)$ the topological boundary of A in X. If l is a positive integer, we denote by X^l the Cartesian product of l copies of X, with product topology.

p will always denote an index running through primes. For positive integers l, p_l will stand for the lth prime in natural order. χ, with or without suffix, will stand for Dirichlet characters. $L(.,\chi)$ and ζ will stand for the Dirichlet L-function with character χ, and the Riemann Zeta function, respectively.

3. The Main Result and Its Consequences

The object of this paper is to prove the following result. The proof appears in Sect. 5 below.

3.1. Theorem (Joint universality of Dirichlet L-function)[2]. *Let $k\geqq 1$, and let $\chi_1,\ldots,\chi_n$ be distinct Dirichlet characters modulo k. For $1\leqq j\leqq n$, let D_j be a simply connected compact subset of Ω, and let f_j be a non-vanishing continuous function on D_j which is analytic in the interior (if any) of D_j. Then the set of all*

[2] Prof. Reich has shown (private communication) that the arguments given in this paper may be made to yield a stronger result. Namely, the Theorem 3.1 remains valid if the D_j's are compact subsets of Ω whose complements relative to the Riemann sphere are connected. Further, this relaxed condition is necessary for the validity of the theorem

$t \in \mathbb{R}$ for which

$$\sup_{1 \leq j \leq n} \sup_{z \in D_j} |L(z+it, \chi_j) - f_j(z)| < \varepsilon \tag{3.1}$$

has positive lower density for every $\varepsilon > 0$.

Theorem 3.1 has the following immediate consequences.

3.2. Corollary. *If $\chi_1, \ldots, \chi_n$ are distinct Dirichlet characters to a fixed modulus then there is no non-trivial algebraic-differential equation relating $L(., \chi_1), \ldots, L(., \chi_n)$.*

3.3. Corollary. *If χ is an arbitrary Dirichlet character, D is a simply connected compact subset of Ω and f is a non-vanishing continuous function on D which is analytic in the interior (if any) of D, then the set of all $t \in \mathbb{R}$ for which*

$$\sup_{z \in D} |L(z+it, \chi) - f(z)| < \varepsilon \tag{3.2}$$

has positive lower density for every $\varepsilon > 0$.

3.4. Corollary. *Let $g \in H$ be given by any one of the following:*

(a) $g(z) = \sum_{\substack{m=1 \\ m \equiv h \bmod k}}^{\infty} m^{-z}$ *for $\operatorname{Re}(z) > 1$, and hence by analytic continuation to Ω. Here $1 \leq h < k$, $k \geq 3$; h and k are relatively prime.*

(b) $g = L^{(m)}(., \chi)$ *where $m \geq 1$ and χ is a Dirichlet character. Then for any simply connected compact subset D of Ω and any continuous function f on D which is analytic in the interior (if any) of D, the set of all $t \in \mathbb{R}$ for which*

$$\sup_{z \in D} |g(z+it) - f(z)| < \delta \tag{3.3}$$

has positive lower density for every $\delta > 0$.

Proof. First let g be as in (a). Let $\chi_1, \ldots, \chi_n$ be the Dirichlet characters modulo k. For $1 \leq j \leq n$, let $a_j = \frac{1}{n} \overline{\chi_j(h)}$. Then $g = \sum_{j=1}^{n} a_j L(., \chi_j)$.

Since D is simply connected, by Mergelyan's theorem [8, p. 423] there is an $f^* \in H$ such that

$$\sup_{z \in D} |f(z) - f^*(z)| < \frac{\delta}{3}. \tag{3.4}$$

Let S denote the set of all $h \in H$ such that $h \equiv 0$ or $\frac{1}{h} \in H$. Suppose we have $f_1, \ldots, f_n \in S$ such that

$$\sup_{z \in D} \left| \sum_{j=1}^{n} a_j f_j(z) - f^*(z) \right| < \frac{\delta}{3}. \tag{3.5}$$

Clearly there is an $\varepsilon > 0$ such that whenever for some $t \in \mathbb{R}$ (3.1) holds (with $D_1 = \ldots = D_n = D$) it follows that

$$\sup_{z \in D} \left| \sum_{j=1}^{n} a_j L(z+it, \chi_j) - \sum_{j=1}^{n} a_j f_j(z) \right| < \frac{\delta}{3}. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we see that whenever $t\in\mathbb{R}$ satisfies (3.1), it also satisfies (3.3). Hence by Theorem 3.1 it suffices to exhibit $f_1, \dots, f_n \in S$ satisfying (3.5).

Thus it is enough to show that the set $\left\{\sum_{j=1}^{n} a_j f_j : f_j \in S,\ 1 \leqq j \leqq n\right\}$ is dense in H. Since we have $n \geqq 2$ (as $k \geqq 3$) this set contains $S+S=\{f_1+f_2 : f_1, f_2 \in S\}$. Now, for any bounded $h\in H$, there is a constant c such that $h(z) \neq c$ for $z\in\Omega$. Hence both $h-c$ and c are in S. But $h=(h-c)+c$. Thus $S+S$ contains the set of all bounded elements of H. Since the latter set is dense in H, we are done.

In case g is given as in (b), we may similarly deduce the above result from Corollary 3.3 provided we know that $\{f^{(m)} : f\in S\}$ is dense in H for any fixed $m \geqq 1$. In view of the preceding paragraph, this set contains the image, under the continuous surjective operation of m-times differentiation, of the (dense) set of bounded members of H. Hence it is dense.

3.5. Corollary. *Let g be as in 3.4. Then the set of all real parts of the zeros of g is dense in* $[\frac{1}{2}, 1]$.

Proof. Let $\frac{1}{2} \leqq a < b \leqq 1$. We have to show that g has a zero in the strip $a < \mathrm{Re}(z) < b$. Let D be a closed disc contained in this strip. Let's choose $f\in H$ such that f has a zero in D and f is non-vanishing on $bd(D)$. Let's take δ such that $0 < \delta < \inf_{z\in bd(D)} |f(z)|$. By 3.4 there is a $t\in\mathbb{R}$ such that (3.3) holds. Hence by Rouche's theorem [9, p. 116] the function $z \to g(z+it)$ has a zero in D. Therefore g has a zero in $D+it$ and hence in the strip $a < \mathrm{Re}(z) < b$.

3.6. Definition. Let U be a strip $a < \mathrm{Re}(z) < b$. We say that an $f\in H(U)$ is *strongly recurrent* on U in case for every compact subset D of U and every $\varepsilon > 0$, the set of all $t\in\mathbb{R}$ for which $\sup_{z\in D} |f(z+it) - f(z)| < \varepsilon$ has positive upper density.

In an abstract setting, the notion of strong recurrence occurs in [4].

3.7. Theorem. *Let* $\frac{1}{2} \leqq a < b \leqq 1$, *and let* χ *be a Dirichlet character. Then* $L(.,\chi)$ *is zero free in the strip* $a < \mathrm{Re}(z) < b$ *if and only if* $L(.,\chi)$ *is strongly recurrent on that strip. In particular, the Riemann hypothesis holds for* $L(.,\chi)$ *if and only if* $L(.,\chi)$ *is strongly recurrent on* Ω.

Proof. If $L(.,\chi)$ is zero free on $a < \mathrm{Re}(z) < b$, then for any compact set D contained in the strip, and for any $\varepsilon > 0$, we may substitute $L(.,\chi)$ for f in 3.3 above (replacing, if necessary, D by its convex hull) to conclude that $L(.,\chi)$ is strongly recurrent on the strip.

Let's now suppose, if possible, that $L(.,\chi)$ is strongly recurrent on $a < \mathrm{Re}(z) < b$ and that $L(.,\chi)$ has a zero ρ with $a < \mathrm{Re}(\rho) < b$. Let D be a closed disc contained in the strip and containing ρ such that $L(.,\chi)$ has no zero in $bd(D)$. Let's fix ε such that $0 < \varepsilon < \inf_{z\in bd(D)} |L(z,\chi)|$. By the assumed strong recurrence, the set of all real t such that $\sup_{z\in D} |L(z+it,\chi) - L(z,\chi)| < \varepsilon$ has positive upper density. Hence by Rouche's theorem the set of all real t for which $L(.,\chi)$ has a zero in $D+it$ has positive upper density. Therefore, if $c = \inf_{z\in D} \mathrm{Re}(z)$ and

$N_c(T)$ denotes the number of zeros of $L(., \chi)$ in $\mathrm{Re}(z) \geqq c$ and $-T < \mathrm{Im}(z) < T$, then $\limsup_{T\to\infty} \frac{N_c(T)}{T} > 0$. Since $c > \frac{1}{2}$, this contradicts a well-known property of the L-functions. This establishes the converse.

3.8. Examples. (a) If f is given on a strip by an absolutely convergent Dirichlet series then f is strongly recurrent on that strip. If, further, g is strongly recurrent on the same strip then both $f+g$ and $f \cdot g$ are strongly recurrent (see [1, pp. 42–44]).

(b) If f is strongly recurrent on a strip U and for some $m \geqq 1$, $P: \mathbb{C}^m \to \mathbb{C}$ is analytic then $P(f, f^{(1)}, \ldots, f^{(m-1)})$ is strongly recurrent on U (this is a particular case of Proposition 1.3.2 in [1, p. 25]).

(c) If g is as in 3.4 then g is strongly recurrent on Ω. This follows from 3.4.

(d) If g is given by $g(z) = \frac{d}{dz}((1-2^{1-z})\zeta(z))$, $z \in \mathbb{C}$, then it can be shown that g is strongly recurrent on the strip $\frac{1}{2} < \mathrm{Re}(z) < 1$ as well as on the half plane $\mathrm{Re}(z) > 1$; but there is no $\delta > 0$ for which g is strongly recurrent on the strip $1 - \delta < \mathrm{Re}(z) < 1 + \delta$. Therefore, even under reasonable growth assumptions, a function given by a convergent Dirichlet series need not have "the half-plane of strong recurrence". (Thus the recurrence conjecture in [1, pp. 56–57] is false.)

4. Preliminary Lemmas

4.1. Lemma. *Let $x_1, \ldots, x_n$ be linearly dependent vectors in a complex vector space. Let $a_1, \ldots, a_n$ be complex numbers with $|a_j| \leqq 1$ $(1 \leqq j \leqq n)$. Then there exist complex numbers $b_1, \ldots, b_n$ with $|b_j| \leqq 1$ $(1 \leqq j \leqq n)$ and at least one $|b_j| = 1$, such that $\sum_{j=1}^n a_j x_j = \sum_{j=1}^n b_j x_j$.*

Proof. By assumption there exist complex numbers $c_1, \ldots, c_n$, not all of them zero, such that $\sum_{j=1}^n c_j x_j = 0$.

Let $K = \{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}: |\alpha_j| \leqq 1$ for $1 \leqq j \leqq n\}$, and let $I = \{t \in \mathbb{R}: a + tc \in K\}$.

Here $a = (a_1, \ldots, a_n)$ and $c = (c_1, \ldots, c_n)$.

Since $a \in K$, $0 \in I$, so that I is nonempty. Since K is convex, so is I; therefore I is an interval. Since K is compact and $c \neq 0$, I is bounded. Let t_0 be one of the end points of I, and let $b = (b_1, \ldots, b_n) = a + t_0 c$. Clearly $b \in bd(K)$. That is, $|b_j| \leqq 1$ $(1 \leqq j \leqq n)$ and $|b_j| = 1$ for at least one j. Also,

$$\sum_{j=1}^n b_j x_j = \sum_{j=1}^n a_j x_j + t_0 \sum_{j=1}^n c_j x_j = \sum_{j=1}^n a_j x_j.$$

So we are done.

4.2. Lemma. *Let $x_1, \ldots, x_n$ be points in a complex Hilbert space and let $a_1, \ldots, a_n$ be complex numbers with $|a_j| \leqq 1$ $(1 \leqq j \leqq n)$. Then there exist complex numbers $b_1, \ldots, b_n$ with $|b_j| = 1$ $(1 \leqq j \leqq n)$ such that*

$$\left\| \sum_{j=1}^{n} a_j x_j - \sum_{j=1}^{n} b_j x_j \right\|^2 \leqq 4 \sum_{j=1}^{n} \|x_j\|^2.$$

Proof. We prove it by induction on n. It is trivial for $n=1$. So suppose it is true for n, and we prove it for $n+1$.

Let $x_1, \ldots, x_{n+1}$ be points in our Hilbert space, and let $a_1, \ldots, a_{n+1}$ be complex numbers with $|a_j| \leqq 1$ $(1 \leqq j \leqq n+1)$. Let y_{n+1} be the orthogonal projection of x_{n+1} into the span of $x_1, \ldots, x_n$. Thus $x_1, \ldots, x_n, y_{n+1}$ are linearly dependent vectors. So by Lemma 4.1 there exist $c_1, \ldots, c_{n+1}$ with $|c_j| \leqq 1$ $(1 \leqq j \leqq n+1)$ and $|c_{j_0}| = 1$ for some j_0 and

$$\sum_{j=1}^{n} a_j x_j + a_{n+1} y_{n+1} = \sum_{j=1}^{n} c_j x_j + c_{n+1} y_{n+1}.$$

Case I. $j_0 = n+1$. That is, $|c_{n+1}| = 1$. By induction hypothesis there exist $b_1, \ldots, b_n$ with $|b_j| = 1$ $(1 \leqq j \leqq n)$ such that

$$\left\| \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} b_j x_j \right\|^2 \leqq 4 \sum_{j=1}^{n} \|x_j\|^2.$$

Let us put $b_{n+1} = c_{n+1}$. We have:

$$\sum_{j=1}^{n+1} a_j x_j - \sum_{j=1}^{n+1} b_j x_j = \left(\sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} b_j x_j \right) + (a_{n+1} - c_{n+1}) z_{n+1}$$

where $z_{n+1} = x_{n+1} - y_{n+1}$, so that z_{n+1} is orthogonal to $x_1, \ldots, x_n$. Hence

$$\begin{aligned} &\left\| \sum_{j=1}^{n+1} a_j x_j - \sum_{j=1}^{n+1} b_j x_j \right\|^2 \\ &\quad = \left\| \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} b_j x_j \right\|^2 + |a_{n+1} - c_{n+1}|^2 \, \|z_{n+1}\|^2 \\ &\quad \leqq 4 \cdot \sum_{j=1}^{n} \|x_j\|^2 + 4 \cdot \|z_{n+1}\|^2. \end{aligned}$$

But $\|z_{n+1}\|^2 = \|x_{n+1}\|^2 - \|y_{n+1}\|^2 \leqq \|x_{n+1}\|^2$. So in this case we are done.

Case II. $1 \leqq j_0 \leqq n$. So without loss of generality we may take $j_0 = 1$, so that $|c_1| = 1$. By induction hypothesis there exist $b_2, \ldots, b_{n+1}$ with $|b_j| = 1$ $(2 \leqq j \leqq n+1)$ such that

$$\left\| \sum_{j=2}^{n} c_j x_j + c_{n+1} y_{n+1} - \sum_{j=2}^{n} b_j x_j - b_{n+1} y_{n+1} \right\|^2 \leqq 4 \sum_{j=2}^{n} \|x_j\|^2 + 4\|y_{n+1}\|^2.$$

Let us put $b_1 = c_1$. Then we have:

$$\sum_{j=1}^{n+1} a_j x_j - \sum_{j=1}^{n+1} b_j x_j = \left(\sum_{j=2}^{n} c_j x_j + c_{n+1} y_{n+1} - \sum_{j=2}^{n} b_j x_j - b_{n+1} y_{n+1} \right) + (a_{n+1} - b_{n+1}) z_{n+1}.$$

Therefore,

$$\begin{aligned} \left\| \sum_{j=1}^{n+1} a_j x_j - \sum_{j=1}^{n+1} b_j x_j \right\|^2 &= \left\| \sum_{j=2}^{n} c_j x_j + c_{n+1} y_{n+1} - \sum_{j=2}^{n} b_j x_j - b_{n+1} y_{n+1} \right\|^2 \\ &\quad + |a_{n+1} - b_{n+1}|^2 \|z_{n+1}\|^2 \\ &\leqq 4 \cdot \sum_{j=2}^{n} \|x_j\|^2 + 4 \cdot \|y_{n+1}\|^2 + 4 \cdot \|z_{n+1}\|^2 \\ &= 4 \cdot \sum_{j=2}^{n+1} \|x_j\|^2 \leqq 4 \sum_{j=1}^{n+1} \|x_j\|^2. \end{aligned}$$

This settles the second case too.

4.3. Proposition. *Let $\{x_n : n \geqq 1\}$ be a sequence in a complex Hilbert space X satisfying:*

(i) $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, *and*

(ii) $\sum_{n=1}^{\infty} |(x_n, x)| = \infty$ *for* $x \in X$, $x \neq 0$.

Then the set $\left\{ \sum_{n=1}^{m} a_n x_n : m \geqq 1, a_n \in \Pi \text{ for } 1 \leqq n \leqq m \right\}$ *is dense in* X.

Proof. We have to show that for an arbitrary $x_0 \in X$ and $\varepsilon > 0$ there exists a sequence $\{a_n : 1 \leqq n \leqq m\}$ in Π such that

$$\left\| x_0 - \sum_{n=1}^{m} a_n x_n \right\| < \varepsilon. \tag{4.1}$$

Let us fix an integer M so large that

$$\sum_{n=M+1}^{\infty} \|x_n\|^2 < \frac{\varepsilon^2}{16}. \tag{4.2}$$

Let $K = \left\{ \sum_{n=M+1}^{m} b_n x_n : |b_n| \leqq 1 \text{ for } M < n \leqq m \text{ and } m > M \right\}$. Clearly K is convex. First we show that K is dense in X. If not, then the closure of K is a proper closed and convex subset of X, and hence by the separation theorem 3.4(b) of

[7, p. 58], there exists $x \in X$ $x \neq 0$ such that $\{\mathrm{Re}(x, y): y \in K\}$ is bounded above. That is, there exists a finite constant c such that $\mathrm{Re}\left(\sum_{n=M+1}^{m} b_n(x_n, x)\right) \leq c$ whenever $m > M$ and $|b_n| \leq 1$ for $M < n \leq m$. In particular, choosing $b_n \in \Pi$ such that $b_n(x_n, x) = |(x_n, x)|$ for $n > M$, we find that $\sum_{n=M+1}^{m} |(x_n, x)| \leq c$ for $m > M$. Hence $\sum_{n=1}^{\infty} |(x_n, x)| < \infty$. Since $x \neq 0$, this contradicts the assumption on $\{x_n\}$. Thus K is dense. Hence there exists a sequence $\{b_n: M < n \leq m\}$ with $|b_n| \leq 1$ such that

$$\left\| \sum_{n=M+1}^{m} b_n x_n + \sum_{n=1}^{M} x_n - x_0 \right\| < \frac{\varepsilon}{2}. \tag{4.3}$$

By Lemma 4.2, there exists a sequence $\{a_n: M < n \leq m\}$ in Π such that

$$\left\| \sum_{n=M+1}^{m} a_n x_n - \sum_{n=M+1}^{m} b_n x_n \right\|^2 \leq 4 \sum_{n=M+1}^{m} \|x_n\|^2.$$

Hence from (4.2), we obtain

$$\left\| \sum_{n=M+1}^{m} a_n x_n - \sum_{n=M+1}^{m} b_n x_n \right\| \leq \frac{\varepsilon}{2}. \tag{4.4}$$

Putting $a_n = 1$ for $1 \leq n \leq M$, and combining (4.3) and (4.4), we obtain (4.1).

4.4. Proposition. *Let $\{f_m: m \geq 1\}$ be a sequence in H^n ($f_m = (f_m^1, \ldots, f_m^n)$, $m \geq 1$) which satisfies:*

(i) *whenever $\mu_1, \ldots, \mu_n$ are complex Borel measures with compact supports contained in Ω such that*

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int f_m^j \, d\mu_j \right| < \infty, \text{ we have } \int z^r \, d\mu_j(z) = 0 \text{ for } 1 \leq j \leq n, \qquad r = 0, 1, 2, \ldots$$

and (ii) $\sum_{m=1}^{\infty} \operatorname{Sup}_{z \in K} |f_m(z)|^2 < \infty$ *for each compact $K \subseteq \Omega$. Then the set of all sums $\sum_{m=1}^{M} a_m f_m$ with $M \geq 1$ and $a_m \in \Pi$ is dense in H^n.*

Proof. Let $g = (g^1, \ldots, g^n) \in H^n$ be arbitrary, let $\varepsilon > 0$ and K be a compact subset of Ω. We have to exhibit a finite sequence $\{a_m: 1 \leq m \leq M\}$ in Π such that $\operatorname{Sup}_{z \in K} \left| g(z) - \sum_{m=1}^{M} a_m f_m(z) \right| < \varepsilon$.

Let γ be an analytic simple closed curve lying inside Ω and properly enclosing K. Let U denote the region enclosed by γ. Consider the Hardy space $H_2(U)$ (see [3, pp. 168–175] for definition and other properties used below). Let X be the Cartesean product of n copies of $H_2(U)$. $H_2(U)$ is a Hilbert space with an inner product $(.,.)$. Let's define $\langle .,. \rangle$ from $X \times X$ to $\mathbb{C}$ by $\langle g_1, g_2 \rangle = \sum_{j=1}^{n} (g_1^j, g_2^j)$. Then $\langle .,. \rangle$ is an inner product on X which makes X into a complex Hilbert space. Let $\|.\|$ be the induced norm. f_m, g may be regarded as points in X. Clearly there is $\delta>0$ such that whenever $\left\|\sum_{m=1}^{M} a_m f_m - g\right\| < \delta$, it follows that $\sup_{z\in K}\left|\sum_{m=1}^{M} a_m f_m(z) - g(z)\right| < \varepsilon$. Thus it suffices to show that the set

$$\left\{\sum_{m=1}^{M} a_m f_m : M\geqq 1, a_m\in\Pi \text{ for } 1\leqq m\leqq M\right\}$$

is dense in X.

Now the assumption (ii) above implies that $\sum_{m=1}^{\infty}\|f_m\|^2<\infty$. Hence by Proposition 4.3, it suffices to show that whenever $h\in X$ satisfies

$$\sum_{m=1}^{\infty}|\langle f_m, h\rangle|<\infty \tag{4.5}$$

it follows that $h=0$.

So let us assume (4.5). There exist complex Borel measures $\mu_1, \ldots, \mu_n$, with compact supports contained in γ, such that whenever $g\in H$ has continuous extension to the closure of U, it follows that $\langle g, h\rangle = \sum_{j=1}^{n}\int g^j d\mu_j$. (Indeed, the density of μ_j relative to the normalised arc-length measure of γ equals the complex conjugate of the boundary value of h_j on γ).

Hence (4.5) may be rewritten as $\sum_{m=1}^{\infty}\left|\sum_{j=1}^{n}\int f_m^j d\mu_j\right|<\infty$. Therefore, by assumption (i), $\int z^r d\mu_j=0$ for $1\leqq j\leqq n$, $r=0,1,2,\ldots$. Thus h^j is orthogonal to all the polynomials. But since γ is analytic, the polynomials are dense in $H_2(U)$. Hence $h^j=0$ for $1\leqq j\leqq n$. That is, $h=0$. So we are done.

4.5. Lemma. *Let μ be a complex Borel measure with compact support contained in $\mathrm{Re}(z)>a$. Let f be given by $f(z)=\int e^{sz}d\mu(s)$, $z\in\mathbb{C}$. Let's assume that $f\not\equiv 0$. Then* $\limsup_{\substack{x\to\infty\\ x\in\mathbb{R}}}\frac{\log|f(x)|}{x}>a$.

Proof. Clearly f is an entire function of exponential type. A simple computation shows that its Borel transform F is given by $F(z)=\int\frac{d\mu(s)}{z-s}$ for z outside the support of μ. Therefore the conjugate indicator diagram of f is contained in the convex hull of the support of μ, and hence it is contained in $\mathrm{Re}(z)>a$. Since by assumption $f\not\equiv 0$, an appeal to Theorem 5.3.7 of [2, p. 74] completes the proof.

4.6. Lemma (Bernstein). *Let f be an entire function of exponential type, let $\{\lambda_n : n\geqq 1\}$ be a sequence of complex numbers. Let α, β, δ be positive reals such that*

(i) $\limsup_{\substack{y\to\infty\\ y\in\mathbb{R}}}\frac{\log|f(\pm iy)|}{y}\leqq\alpha$,

(ii) $|\lambda_m-\lambda_n|\geqq\delta\cdot|m-n|, \quad m,n\geqq 1$,

(iii) $\lim_{n\to\infty}\frac{\lambda_n}{n}=\beta$

and

(iv) $\alpha\beta<\pi$.

$$\textit{Then } \limsup_{n\to\infty} \frac{\log|f(\lambda_n)|}{|\lambda_n|} = \limsup_{\substack{x\to\infty\\ x\in\mathbb{R}}} \frac{\log|f(x)|}{x}.$$

Proof. First suppose $\beta=1$, so that $0<\alpha<\pi$. Also, if h is the indicator function of f then by hypothesis (i), $h\left(\frac{\pi}{2}\right)\leqq\alpha$, $h\left(-\frac{\pi}{2}\right)\leqq\alpha$. So by Theorem 5.1.2 of [2, p. 66], $h(\theta)\leqq h(0)\operatorname{Cos}\theta+\alpha|\operatorname{Sin}\theta|$ for $|\theta|\leqq\frac{\pi}{2}$. Also, as $\beta=1$, $\lim_{n\to\infty}\frac{\lambda_n}{n}=1$. Hence by Bernstein's theorem, the stated conclusion follows in this case (see [2, p. 185] and also the remark in 4.7 below).

One obtains the general result by applying this special case to the system $f^*, \lambda_n^*, \alpha^*, \delta^*$ where $f^*(z)=f(\beta z)$, $\lambda_n^*=\frac{\lambda_n}{\beta}$, $\alpha^*=\alpha\beta$, $\delta^*=\frac{\delta}{\beta}$.

4.7. Remarks. (a) The statement of Bernstein's theorem as given in [2, p. 185] is false. A counterexample is given by $f(z)=\operatorname{Sin}(\pi z)$, $\alpha=\frac{\pi}{3}$, $a=2$, $b=0$, $\lambda_n=n$, $\delta=1$. An examination of the proof shows that the correct statement is obtained by replacing the hypothesis "$h(\theta)\leqq a\operatorname{Cos}\theta+b|\operatorname{Sin}\theta|$" by "$h(\theta)\leqq h(0)\operatorname{Cos}\theta+b|\operatorname{Sin}\theta|$". It is this rectified version that we have used in 4.6 above.

(b) Arguments similar to those in 4.1 to 4.3 above have been used by Drobot in Trans. Amer. Math. Soc. **142**, 239–248 (1969), and by Fonf in Math. Notes **11**, 129–132 (1972), in order to prove results regarding rearrangement of series in Hilbert spaces.

4.8. Lemma. *Let f be an entire function of exponential type such that* $\limsup_{\substack{x\to\infty\\ x\in\mathbb{R}}}\frac{\log|f(x)|}{x}>-1$. *Let h and k be mutually prime positive integers. Then* $\sum_{\substack{p\equiv h\\ \bmod k}}|f(\log p)|=\infty$.

Proof. Let $\alpha>0$ be such that $\limsup_{\substack{y\to\infty\\ y\in\mathbb{R}}}\frac{\log|f(\pm iy)|}{y}\leqq\alpha$. Let's fix β such that $0<\beta<\frac{\pi}{\alpha}$.

Suppose, if possible,

$$\sum_{\substack{p\equiv h\\ \bmod k}}|f(\log p)|<\infty. \tag{4.6}$$

Let A be the set of all positive integers n such that there exists x with $(n-\frac{1}{4})\beta<x<(n+\frac{1}{4})\beta$ and $|f(x)|\leqq e^{-x}$. Then

$$\sum_{\substack{p\equiv h\\ \bmod k}}|f(\log p)|\geqq\sum_{n\notin A}\Sigma_n^*|f(\log p)|\geqq\sum_{n\notin A}\Sigma_n^*\frac{1}{p},$$

where Σ_n^* denotes a sum over all primes p such that $p \equiv h \bmod k$ and $(n-\frac{1}{4})\beta < \log p < (n+\frac{1}{4})\beta$.

Let $\Pi_{h,k}(x)$ denote the number of primes $p \leqq x$ such that $p \equiv h (\bmod k)$. Well known estimates of Hadamard and de la Vallee Poussin, together with the asymptotic expansion of the logarithmic integral, yields the result:

$$\Pi_{h,k}(x) = \frac{1}{\phi(k)}(x(\log x)^{-1} + x(\log x)^{-2} + O(x(\log x)^{-3})),$$

where $\phi(k)$, as usual, denotes the number of Dirichlet characters modulo k. Hence, summation by parts gives the estimate

$$\sum_{\substack{p \leqq x \\ p \equiv h \\ \bmod k}} \frac{1}{p} = \frac{1}{\phi(k)} \log\log x + C_k + O(\log x)^{-2}),$$

where C_k is a constant depending only on k. Therefore, putting $x_1 = \exp(n-\frac{1}{4})\beta$ and $x_2 = \exp(n+\frac{1}{4})\beta$ we have

$$\begin{aligned}
\Sigma_n^* \frac{1}{p} &= \sum_{\substack{p \leqq x_2 \\ p \equiv h \\ \bmod k}} \frac{1}{p} - \sum_{\substack{p \leqq x_1 \\ p \equiv h \\ \bmod k}} \frac{1}{p} + O(x_2^{-1}) \\
&= \frac{1}{\phi(k)}(\log\log x_2 - \log\log x_1) + O(x_2^{-1}) + O(\log x_1)^{-2}) \\
&= \frac{1}{\phi(k)} \log \frac{n+\frac{1}{4}}{n-\frac{1}{4}} + O\left(\frac{1}{n^2}\right) \\
&= \frac{1}{2\phi(k)} \frac{1}{n} + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n \notin A} \left(\frac{1}{2\phi(k)} \frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) &= \sum_{n \notin A} \Sigma_n^* \frac{1}{p} \\
&\leqq \sum_{n \notin A} \Sigma_n^* |f(\log p)| \quad \text{(by definition of } A) \\
&\leqq \sum_{\substack{p \equiv h \\ \bmod k}} |f(\log p)| \\
&< \infty \quad \text{(by assumption (4.6))}.
\end{aligned}$$

Hence $\sum_{n \notin A} \frac{1}{n} < \infty$. A fortiori, the natural density of A equals one. That is, if we write $A = \{a_n : n \geqq 1\}$ with $a_1 < a_2 < \dots$ then $\lim_{n\to\infty} \frac{a_n}{n} = 1$.

Now, by definition of A, there exists a sequence $\{\lambda_n\}$ such that

$$(a_n - \tfrac{1}{4})\beta \leqq \lambda_n \leqq (a_n + \tfrac{1}{4})\beta \quad \text{and} \quad |f(\lambda_n)| \leqq e^{-\lambda_n}.$$

Therefore

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \beta \quad \text{and} \quad \limsup_{n\to\infty} \frac{\log|f(\lambda_n)|}{\lambda_n} \leqq -1.$$

Hence by Lemma 4.6,

$$\limsup_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{\log|f(x)|}{x} \leqq -1.$$

This contradicts the hypothesis on f, so that (4.6) must be false.

4.9. Lemma. *Let $k \geqq 1$, and let $\chi_1, \dots, \chi_n$ be distinct Dirichlet characters modulo k. For primes p, let $f_p \in H^n$ be defined by $f_p^j(z) = \chi_j(p) p^{-z}$, $z \in \Omega$, $1 \leqq j \leqq n$. Let $x_0 > 0$. Then the set of all sums of the form $\sum_{x_0 < p \leqq x} a_p f_p$, with $a_p \in \Pi$ and $x > x_0$, is dense in H^n.*

Proof. In view of Proposition 4.4, it suffices to show that whenever $\mu_1, \dots, \mu_n$ are complex Borel measures, with compact supports contained in Ω, satisfying:

$$\sum_p \left| \sum_{j=1}^{n} \int f_p^j \, d\mu_j \right| < \infty \tag{4.7}$$

it follows that

$$\int z^r \, d\mu_j(z) = 0 \quad \text{for } 1 \leqq j \leqq n,\ r = 0, 1, 2, \dots \tag{4.8}$$

For $1 \leqq h \leqq k$ with $(h, k) = 1$, let the complex Borel measure ν_h be defined by $\nu_h = \sum_{j=1}^{n} \chi_j(h) \mu_j$. Then (4.7) may be rewritten as:

$$\sum_{\substack{p \equiv h \\ \bmod k}} \left| \int p^{-z} \, d\nu_h(z) \right| < \infty \quad \text{for } 1 \leqq h \leqq k,\ (h, k) = 1.$$

Or, defining g_h by $g_h(z) = \int e^{-sz} \, d\nu_h(s)$, $z \in \mathbb{C}$, we have

$$\sum_{p \equiv h \bmod k} |g_h(\log p)| < \infty. \tag{4.9}$$

If for some h, $g_h \not\equiv 0$ then, by Lemma 4.5,

$$\limsup_{\substack{x\to\infty \\ x\in\mathbb{R}}} \frac{\log|g_h(x)|}{x} > -1. \tag{4.10}$$

But (4.9) and (4.10) together contradicts Lemma 4.8. Hence $g_h \equiv 0$. Differentiating the integral representation of g_h r times, and evaluating at $z = 0$, we obtain $\int z^r \, d\nu_h(z) = 0$. That is, $\sum_{j=1}^{n} \chi_j(h) \int z^r \, d\mu_j(z) = 0$ for $1 \leqq h \leqq k$.

The orthogonality relation of the Dirichlet characters imply that they are linearly independent over $\mathbb{C}$. Hence we have (4.8).

4.10. Lemma. *Let $k \geqq 1$, and let $\chi_1, \dots, \chi_n$ be distinct Dirichlet characters modulo k. For primes p, and $\alpha \in \Pi$, let's define $g_p(\cdot, \alpha) \in H^n$ by $g_p^j(z, \alpha) = (1 - \alpha \chi_j(p) p^{-z})^{-1}$, $z \in \Omega$, $1 \leqq j \leqq n$. Then the closure of the set of all products $\prod_{p \leqq x} g_p(\cdot, \alpha_p)$, with $x \geqq 2$ and $\alpha_p \in \Pi$, is the set S^n where $S = \left\{ f \in H : f \equiv 0 \text{ or } \frac{1}{f} \in H \right\}$.*

Proof. For primes p and $\alpha\in\Pi$ let $h_p(.,\alpha)\in H^n$ be defined by $h_p^j(.,\alpha)=\log g_p^j(.,\alpha)$, $1\leqq j\leqq n$. Clearly it suffices to show that the set of all sums $\sum_{p\leqq x} h_p(.,\alpha_p)$ with $x\geqq 2$ and $\alpha_p\in\Pi$ is dense in H^n. Let $h\in H^n$, K a compact subset of Ω and $\varepsilon>0$. We have to exhibit $x\geqq 2$ and a sequence $\{\alpha_p:p\leqq x\}$ in Π such that

$$\operatorname{Sup}_{z\in K}|h(z)-\sum_{p\leqq x} h_p(z,\alpha_p)|<\varepsilon. \tag{4.11}$$

Clearly $\sum_p \operatorname{Sup}_{z\in K}|\alpha_p f_p(z)-h_p(z,\alpha_p)|<\infty$ uniformly for all sequences $\{\alpha_p\}$ in Π, where f_p is as in Lemma 4.9.

Hence we may choose x_0 so large that

$$\sum_{p>x_0}\operatorname{Sup}_{z\in K}|\alpha_p f_p(z)-h_p(z)|<\frac{\varepsilon}{2} \tag{4.12}$$

for all sequences $\{\alpha_p\}$ in Π.

By Lemma 4.9 there is $x>x_0$ and a sequence $\{\alpha_p: x_0<p\leqq x\}$ such that

$$\operatorname{Sup}_{z\in K}|h(z)-\sum_{p\leqq x_0} h_p(z,1)-\sum_{x_0<p\leqq x}\alpha_p f_p(z)|<\frac{\varepsilon}{2}. \tag{4.13}$$

Let us put $\alpha_p=1$ for $p\leqq x_0$. Then (4.12) and (4.13) together imply (4.11).

5. Proof of the Main Theorem

We need one final lemma from which Theorem 3.1 will be readily deduced.

5.1. Lemma. *Let $k\geqq 1$, and let $\chi_1,\ldots,\chi_n$ be distinct Dirichlet characters modulo k. Let $f=(f^1,\ldots,f^n)\in S^n$, where S is as in Lemma 4.10. Let D be a compact subset of Ω and $\varepsilon>0$. Then the set of all real t such that*

$$\operatorname{Sup}_{1\leqq j\leqq n}\operatorname{Sup}_{z\in D}|L(z+it,\chi_j)-f^j(z)|<\varepsilon$$

has positive lower density.

Proof. Let $A\in H^n$ be defined by $A=(A^1,\ldots,A^n)$, $A^j=L(.,\chi_j)$. Clearly it suffices to show that the set of all real t such that

$$\operatorname{Sup}_{z\in D}|A(z+it)-f(z)|<\varepsilon \tag{5.1}$$

has positive lower density for every $\varepsilon>0$.

For positive integers l, let $A_l\in H^n$ be defined by

$$A_l^j(z)=\prod_{i=1}^{l}(1-\chi_j(p_i)p_i^{-z})^{-1},\qquad z\in\Omega,\ 1\leqq j\leqq n.$$

Also, for positive integers l, and $\alpha=(\alpha_1,\dots,\alpha_l)\in\Pi^l$, let $A_l(.,\alpha)\in H^n$ be defined by

$$A_l^j(z,\alpha)=\prod_{i=1}^{l}(1-\chi_j(p_i)\alpha_i p_i^{-z})^{-1},\qquad z\in\Omega,\ 1\leqq j\leqq n.$$

Let E be a compact subset of Ω such that D is contained in the interior of E. Let $\delta>0$ be such that whenever $g\in H^n$ satisfies $\int_E |g(z)|^2\,dz<\delta$, it follows that $\operatorname{Sup}_{z\in D}|g(z)|<\frac{\varepsilon}{2}$. Let l be a large positive integer (to be specified later). By Lemma 4.10, there exists an $\alpha=(\alpha_1,\dots,\alpha_\lambda)$ in Π^l such that

$$\operatorname{Sup}_{z\in E}|A_l(z,\alpha)-f(z)|<\frac{\varepsilon}{2}. \tag{5.2}$$

Since the real-valued map on $E\times\Pi^l$ sending (z,α) to $|A_l(z,\alpha)-f(z)|$ is uniformly continuous, there exists a nonempty open set $U\subseteq\Pi^l$ such that (5.2) holds for all $\alpha\in U$. We choose U to be a μ-continuity set (i.e., $\mu(bdU)=0$) where μ is the Haar probability measure on Π^l.

Since $\log p_1,\dots,\log p_l$ are linearly independent over rationals, it follows by Weyl's criterion (see [5]) that the net $\{(p_1^{-it},\dots,p_l^{-it}): t\in\mathbb{R}\}$ is uniformly distributed over Π^l. Hence if V denotes the set of all real t such that $\chi_t=(p_1^{-it},\dots,p_l^{-it})$ is in U, then V has positive density $d(V)=\mu(U)>0$.

For $t\in V$, $\alpha_t\in U$, and hence $\operatorname{Sup}_{z\in E}|A_l(z,\alpha_t)-f(z)|<\frac{\varepsilon}{2}$. That is, we have,

$$\operatorname{Sup}_{z\in E}|A_l(z+it)-f(z)|<\frac{\varepsilon}{2}\quad\text{for } t\in V. \tag{5.3}$$

Let W be the set of all $t\in V$ such that

$$\int_E |A(z+it)-A_l(z+it)|^2\,dz<\delta.$$

By choice of δ, we have

$$\operatorname{Sup}_{z\in D}|A(z+it)-A_l(z+it)|<\frac{\varepsilon}{2}\quad\text{for } t\in W. \tag{5.4}$$

(5.3) and (5.4) together imply that (5.1) holds for $t\in W$. Hence it suffices to show that $\underline{d}(W)>0$. Suppose not.

Then, clearly,

$$I=\liminf_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} I_v(t)(1-I_w(t))\int_E |A(z+it)-A_l(z+it)|^2\,dz\,dt\geqq\delta d(V). \tag{5.5}$$

That is, $I\geqq\delta\,\mu(U)$.
On the other hand,

$$I\leqq c_0\operatorname{Sup}_{z\in E}\limsup_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} I_v(t)|A(z+it)-A_l(z+it)|^2\,dt,$$

where c_0 is the Lebesgue measure of E, so that $0<c_0<\infty$. (5.3) shows that for $t\in V$, $\operatorname{Sup}_{z\in E}|A_l(z+it)|^2\leqq c_1<\infty$, where $c_1=\left(\frac{\varepsilon}{2}+\operatorname{Sup}_{z\in E}|f(z)|\right)^2$.

Hence

$$I\leqq c_0 c_1 \operatorname{Sup}_{z\in E} \limsup_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} I_v(t)\left|1-\frac{A(z+it)}{A_l(z+it)}\right|^2 dt,$$

where 1 is the constant function $(1,1,\ldots,1)\in H^n$.

Let l be so large that all the prime divisors of k occur among $p_1,\ldots,p_l$. Then, since U is a μ-continuity set and since $\{\alpha_t: t\in\mathbb{R}\}$ is uniformly distributed on Π^l, it can be shown by imitating the proof of Theorem 9.51 in [9, pp. 304–306], that $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} I_v(t)$.

$$\left|1-\frac{A(z+it)}{A_l(z+it)}\right|^2 dt=\mu(U)\, n\, \Sigma^*|m^{-z}|^2,$$

the convergence being uniform for $z\in E$. Here Σ^* denotes a sum over all $m>1$ which are relatively prime to $p_1,\ldots,p_l$. If we put $x_0=\min\{\operatorname{Re}(z): z\in E\}$, so that $x_0>\frac{1}{2}$, then we obtain

$$I\leqq n\, c_0\, c_1\, \mu(U) \sum_{m=l+1}^{\infty} n^{-2x_0}. \tag{5.6}$$

Combining (5.5) and (5.6), and noting that $0<\mu(U)\leqq 1$, we obtain:

$$\sum_{m=l+1}^{\infty} n^{-2x_0}\geqq c>0 \tag{5.7}$$

where $c=\dfrac{\delta}{n c_0 c_1}$ does not depend on l.

But if l is chosen sufficiently large, then (5.7) is falsified, so that for such an l we must have $\underline{d}(W)>0$.

5.2. Proof of Theorem 3.1. For $1\leqq j\leqq n$, since f_j is continuous on the simply connected compact set D_j and analytic in the interior of D_j, there is a sequence $\{P_{m,j}\}$ of polynomials such that $P_{m,j}(z)\to f_j(z)$ uniformly for $z\in D_j$ (Mergelyan's theorem). Hence, if we put $g_j=P_{m,j}$ for a sufficiently large m, then $g_j(z)\neq 0$ for $z\in D_j$ and

$$\operatorname{Sup}_{z\in D_j}|f_j(z)-g_j(z)|<\frac{\varepsilon}{4},\qquad 1\leqq j\leqq n. \tag{5.8}$$

Since g_j is a polynomial, it has only finitely many zeros. So we may choose a simply connected region E_j containing D_j such that $g_j(z)\neq 0$ for $z\in E_j$. Hence there is a continuous version $\log g_j$ of the logarithm of g_j on E_j. Clearly $\log g_j$ is analytic in the interior of D_j. There is a sequence $Q_{m,j}$ of polynomials such that $Q_{m,j}\to\log g_j$ uniformly on D_j. Hence, if we put $h_j=\exp(Q_{m,j})$ for a sufficiently large m, then $h_j\in S$ and

$$\operatorname{Sup}_{z\in D_j}|g_j(z)-h_j(z)|<\frac{\varepsilon}{4},\qquad 1\leqq j\leqq n. \tag{5.9}$$

Combining (5.8) and (5.9) we obtain

$$\sup_{1 \leq j \leq n} \sup_{z \in D_j} |f_j(z) - h_j(z)| < \frac{\varepsilon}{2} \tag{5.10}$$

where $h = (h_1, \ldots, h_n) \in S^n$.

Also, by Lemma 5.1, the set of all real t for which

$$\sup_{1 \leq j \leq n} \sup_{z \in D} |L(z + it, \chi_j) - h_j(z)| < \frac{\varepsilon}{2} \tag{5.11}$$

has positive lower density for any compact subset D of Ω which contains all the D_j's. But in view of (5.10), any real t which satisfies (5.11) also satisfies (3.1). This completes the proof.

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