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THE JACOBIAN MATRIX, GLOBAL UNIVALENCE AND COMPLETELY MIXED GAMES*

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In this paper we prove global univalence results in R^4 when the Jacobian matrices are of the Leontief type. Proofs of these make use of ideas from Gale-Nikaido and some nice game theoretic results on completely mixed games due to Kaplansky.

1. Introduction. Recently global univalence theory has received considerable attention partly because of its usefulness to problems relating to Mathematical Economics, see for example Mas-Colell [8], Nikaido [11, 12] and partly because it is a natural extension to the classical implicit function theorem. Another area where global univalence theory is applied is in nonlinear complementarity theory—see Kojima and Megiddo [7]. For other applications see Parthasarathy [13].

Let F be a differentiable map from $\Omega \subset R^n$ to R^n. We are interested in finding suitable conditions on F and Ω so that F is globally one-one in Ω. It is well known that nonvanishing of the Jacobian alone will not suffice for global univalence to prevail. Generally two approaches are followed to obtain solution to the global univalence problem. For example Hadamard [4], Kestelman [6], McAuley [9] and Plastock [15] have placed topological assumptions on the map F and Ω while Gale-Nikaido-Inada, Garcia-Zangwill [3] and Mas-Colell placed strong conditions on the Jacobian matrices to obtain results on global univalence. For a comprehensive account of their results see Parthasarathy [13].

In this paper we follow the approach taken by Gale-Nikaido and prove univalence results in R^4. It is not clear how to extend Theorem 1 to higher dimensions. Proofs of these results make use of ideas from Gale-Nikaido and some game theoretic results on completely mixed games due to Kaplansky [5]. We also give a counterexample to show that theorem 2 may not be valid in higher dimensions. For notations and terminology we follow [2, 13].

Let Ω be a nonempty set in R^n. Whenever,

$$\Omega = \{x : x \in R^n,\ a_i \leqslant x_i \leqslant b_i \quad \text{for} \quad i = 1, 2, \ldots, n\},$$

Ω is called a rectangular region. Here a_i, b_i are real numbers where we may allow some or all of them to assume $-\infty$ or $+\infty$ and x_i is the ith coordinate of the vector x.

Let $F: \Omega \to R^n$ be a mapping defined by $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ where each $f_i(x)$ is a real valued function on Ω. Let F be differentiable in Ω—in other words

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for x, $a \in \Omega$,

$$f_i(x) = f_i(a) + \sum_{j=1}^{n} f_{ij}(a)(x_j - a_j) + O(\|x - a\|)$$

for $i = 1, 2, \ldots, n$ where $O(\|x - a\|)/\|x - a\| \to 0$ as $x \to a$. The Jacobian matrix J of the mapping F at a point x is given by $J(x) = \|f_{ij}(x)\|$ where $f_{ij}(x) = \partial f_i(x)/\partial x_j$. If the partial derivatives are also continuous then F is called a continuously differentiable map.

Let $A = [a_{ij}]$ be a not necessarily symmetric real $n \times n$ matrix. Call A a *P-matrix* if every principal minor of A is positive. Call A an *N-matrix* if every principal minor of A is negative. A is said to possess *positive dominant diagonal* property if there exists a strictly positive vector $d = (d_1, d_2, \ldots, d_n)$ where each $d_i > 0$ such that $a_{ii}d_i > \sum_{j=1;\, j \neq i}^{n} |a_{ij}| d_j$ for every $i = 1, 2, \ldots, n$. A is said to be of *Leontief type* if the off-diagonal entries of A are nonpositive.

It is well known that a matrix A with positive dominant diagonal property is a P-matrix. If A is of Leontief type and if there exists a positive vector $d = (d_1, d_2, \ldots, d_n)$ with $Ad' > 0$ (here each coordinate of Ad' is positive) then A is a P-matrix. For other interesting properties of P and N matrices see [10, 13]. We are ready to state the following which we need in the sequel.

GLOBAL UNIVALENCE THEOREM (Gale-Nikaido-Inada). *Let F: $\Omega \subset R^n \to R^n$ be a differentiable map where Ω is a rectangular region. Then F is globally univalent in Ω if either one of the following conditions holds good.*

(a) *J (= Jacobian of F) is a-P-matrix for every $x \in \Omega$.*

(b) *J is an N-matrix and F is a C^1 map throughout Ω.*

For a proof see [10, 13].

REMARK 1. We can slightly weaken the hypothesis under (b) where $n \geq 2$. That is global univalence prevails if J is almost an N-matrix (by that we mean that the diagonal entries are nonpositive and other principal minors are negative) and F is a C^1 map throughout Ω.

We will now describe the game theoretic results due to Kaplansky. In order to do that we need some preliminaries.

A two-person zero-sum matrix game can be described as follows: Player 1 selects an integer i ($i = 1, 2, \ldots, m$), and player 2 selects an integer j ($j = 1, 2, \ldots, n$) simultaneously. Then player 1 pays player 2 an amount a_{ij} (which may be positive, zero or negative).

A strategy for player 1 is a probability vector $(p_1, p_2, \ldots, p_m)$. The idea is that he will choose integer i with probability p_i. From von Neumann's fundamental results we know that there exist strategies $(p_1, p_2, \ldots, p_m)$ and $(q_1, q_2, \ldots, q_n)$ and a real number v such that

$$\sum_i p_i a_{ij} \leq v \quad \text{for } j = 1, 2, \ldots, n \quad \text{and}$$

$$\sum_j q_j a_{ij} \geq v \quad \text{for } i = 1, 2, \ldots, m.$$

This v is called the *value* of the game and the strategies are called *optimal* strategies for the two players. In the game we described, player I is the minimizer (that is he wants to give player II as little as possible) and player II is the maximizer.

A strategy is pure if it has the form $(0, 0, 1, 0, \ldots, 0)$, otherwise it is mixed. In case each $p_i > 0$ we call the strategy $p = (p_1, p_2, \ldots, p_m)$ *completely mixed*. If the only optimal or good strategies are completely mixed, we shall call the *game completely mixed*. We are ready to state Kaplansky's results.

(i) If player 1 has a completely mixed optimal strategy $p = (p_1, p_2, \ldots, p_m)$ then any optimal strategy $q = (q_1, \ldots, q_n)$ for player 2 satisfies $\Sigma_j a_{ij} q_j = v$, for all i.

(ii) If $m = n$ and the game is not completely mixed then both players 1 and 2 have optimal strategies that are not completely mixed.

(iii) A game of value zero is completely mixed if and only if (a) its matrix is square ($m = n$) and has rank $n - 1$ and (b) all cofactors are different from zero and have the same sign.

(iv) The value v of a completely mixed game is given by $v = |A|/\Sigma\Sigma A_{ij}$ where A_{ij} are the cofactors of a_{ij} and $|A|$ = determinant of A. For a proof of these results see Kaplansky [5]. We freely make use of these results throughout the paper and we refer them as simply Kaplansky. In fact Raghavan made use of these results to give several equivalent characterizations of nonsingular M-matrices—interested readers should see [16].

2. Main results and their proofs. In this section we state and prove two univalence results for rectangular regions in R^4 when the Jacobian matrices are of Leontief type. We also give a counterexample in R^6 to show that Theorem 2 may not be valid in higher dimensions. We make use of Kaplansky's results on completely mixed games to prove the following two lemmas, that are needed in the sequel.

LEMMA 1. *Let*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

be a 4×4 *nonsingular matrix with the sign structure*

$$\begin{bmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}.$$

[*For example* $a_{11} < 0$, $a_{21} > 0$ *etc.*] *Then exactly one of the following statements holds good always*:

(i) *the value of* A *and the value of* A' *are positive*,

(ii) *the value of* A *and the value of* A' *are negative*.

If (ii) *holds good, then* $-A$ *is a* P-*matrix*.

PROOF. Suppose the value of A is less than or equal to zero. Then from the sign structure of A, we can conclude that every optimal for the minimizer has to be completely mixed. Hence from Kaplansky we may infer that the game is completely mixed. Since $|A| \neq 0$, value can never be zero. Thus either (i) must hold or (ii) must hold always.

If (ii) holds good, then $-A$ has positive dominant diagonal property [10] and consequently $-A$ is a P-matrix.

Lemma 1 holds good for any n where A is a nonsingular matrix of order n whose diagonal entries are negative and off-diagonal entries are positive. If $|A| < 0$ in Lemma 1, then clearly (ii) cannot hold good since $-A$ is a P-matrix of order 4 and consequently (i) must hold good always. In other words when n is even and when $|A| < 0$ then the value of A and the value of A' are positive with the above sign structure.

What happens to Lemma 1 if we allow some of the entries in A to be zero? We have the following lemma.

LEMMA 2. *Let A be a 4×4 nonsingular matrix with the following (weak) sign structure*

$$A = \begin{bmatrix} \leqslant 0 & \geqslant 0 & \geqslant 0 & \geqslant 0 \\ \geqslant 0 & \leqslant 0 & \geqslant 0 & \geqslant 0 \\ \geqslant 0 & \geqslant 0 & \leqslant 0 & \geqslant 0 \\ \geqslant 0 & \geqslant 0 & \geqslant 0 & \leqslant 0 \end{bmatrix}.$$

Further suppose $|A| > 0$. Then exactly one of the following statements holds good always:
(i) *the value of A and the value of A' are positive,*
(ii) *the value of A and the value of A' are negative.*
If (ii) *holds good then $-A$ is a P-matrix.*

PROOF. We will first show that the value cannot be zero. Suppose the value is zero. Clearly no optimal strategy is completely mixed. For if $x = (x_1, x_2, x_3, x_4)$ with $x_i > 0$ (for all i) is optimal for one player then from Kaplansky, there exists a probability vector y with $Ay = 0$ but this is impossible as A is a nonsingular matrix. Suppose $x = (0, x_2, x_3, x_4)$ with $x_i > 0$ $(i = 2, 3, 4)$ is an optimal strategy for the minimizer = (who chooses the rows) then $xA \leqslant 0$. This means

$$A = \begin{bmatrix} < 0 & \geqslant 0 & \geqslant 0 & \geqslant 0 \\ 0 & \leqslant 0 & \geqslant 0 & \geqslant 0 \\ 0 & \geqslant 0 & \leqslant 0 & \geqslant 0 \\ 0 & \geqslant 0 & \geqslant 0 & \leqslant 0 \end{bmatrix}.$$

Since $|A| > 0$, it is clear the 3×3 principal minor (leaving out the first row and first column) is negative. Now one can easily verify that $-A$ is a P-matrix. This means the value of A is negative contradicting the value of $A = 0$.

Suppose $x = (0, 0, x_3, x_4)$ with $x_3 > 0$, $x_4 > 0$ is an optimal strategy. Then $xA \leqslant 0$. This means

$$A = \begin{bmatrix} \leqslant 0 & \geqslant 0 & \geqslant 0 & \geqslant 0 \\ \geqslant 0 & \leqslant 0 & \geqslant 0 & \geqslant 0 \\ 0 & 0 & \leqslant 0 & \geqslant 0 \\ 0 & 0 & \geqslant 0 & \leqslant 0 \end{bmatrix}.$$

Since $|A| > 0$, the two principal minors of order 2×2 (the leading one and the other got by omitting the first two rows and two columns) should keep the same sign. If both are negative then this will imply that the value of A to be positive which contradicts our hypothesis, namely, the value of $A = 0$. If both are positive then $-A$ is a P-matrix which will again contradict the fact that the value of $A = 0$.

If $x = (0, 0, 0, 1)$ is an optimal strategy for the minimizer then once again one can prove that $-A$ is a P-matrix which will lead to a contradiction. Thus we have shown that the value of A cannot be equal to zero.

If the value of $A < 0$ then clearly (from the sign structure) game is completely mixed. Hence from Kaplansky the value of $A' < 0$. (Here A' = transpose of A.)

This terminates the proof of Lemma 2.

Lemma 2 may not hold good if $|A| < 0$ as the following simple example shows. Let

$$A = \begin{bmatrix} -4 & 7 & 8 & 9 \\ 7 & 0 & 7 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}.$$

Clearly A satisfies the sign structure imposed in Lemma 2. Observe that $|A| < 0$, the value of $A = 0$ and the value of $A' > 0$. [While computing the value of A, we assume as usual minimizer chooses the rows.]

Let A satisfy the conditions of Lemma 2 and let $B = -A$. Then the conclusion of Lemma 2 is valid also for the matrix B.

Let D be a diagonal matrix of order 4 with exactly two entries $+1$ and the other two entries -1. Let A satisfy the conditions of Lemma 2. Let $C = DAD$. Then conclusion of Lemma 2 is valid also for the matrix C. (Note that the sign structure of C is not the same as that of A.) This fact is crucial to the proof of theorems given below.

THEOREM 1. *Let F: $\Omega \subset R^4 \to R^4$ be a C^1 differentiable function where Ω is an arbitrary rectangular region. Suppose the Jacobian J is nonsingular and has the sign structure*

$$J = \begin{bmatrix} + & - & - & - \\ - & + & - & - \\ - & - & + & - \\ - & - & - & + \end{bmatrix}$$

throughout Ω. Then F is globally one-one throughout Ω.

PROOF. Since F is a C^1 differentiable function it is clear from Lemma 1 that value of J and value of J' are positive or negative throughout Ω. If value of J is positive throughout Ω then it is a P-matrix for all $x \in \Omega$ and univalence prevails from Gale-Nikaido's theorem.

Suppose value of J (as well as value of J') is negative throughout Ω. In this case consider the map $G = -F$. Note that the sign structure of J_G will be

$$\begin{bmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}.$$

To complete the proof we have to prove the following: If $a \in \Omega$ and if $G(x) \leqslant G(a)$ and $x \geqslant a$ then $x = a$. We imitate the proof as given in Theorem 3 in [2].

Let $X = \{x\colon x \in \Omega,\ G(x) \leqslant G(a) \text{ and } x \geqslant a\}$ and $Y = X \setminus \{a\}$. Let $\bar{x}$ be a minimal element of Y. Two possibilities occur) $\bar{x} > a$, (ii) $\bar{x}_i = a_i$, $\bar{x}_j \geqslant a_j$, $j \neq i$. Case (i) can be disposed of as in Gale-Nikaido [2]. If $\bar{x}_i = a_i$, and $\bar{x}_j \geqslant a_j$, $j \neq i$, then clearly from the sign structure of the Jacobian of G we have $g_i(\bar{x}) > g_i(a)$ contradicting $G(\bar{x}) \leqslant G(a)$ where $G = (g_1, g_2, g_3, g_4)$.

Now univalence of G is immediate. This can be seen as follows. Suppose $G(b) = G(a)$, $b \neq a$. If $b > a$, the above observation implies $b = a$ leading to a contradiction. If $b_1 < a_1$, $b_i > a_i$ then $g_1(b) > g_1(a)$ (from the sign structure) leading to a contradiction. If $b_1 < a_1$, $b_2 < a_2$ and $b_i > a_i$, $i = 3, 4$, then define $H = D \circ G \circ D$ where $Dx = (-x_1, -x_2, x_3, x_4)$. (Here $x = (x_1, x_2, x_3, x_4)$). Now one can verify that the sign structure of the Jacobian of the map H will be

$$\begin{bmatrix} - & + & - & - \\ + & - & - & - \\ - & - & - & + \\ - & - & + & - \end{bmatrix}.$$

Let $H = (h_1, h_2, h_3, h_4)$ and $\overline{H} = (h_2, h_1, h_4, h_3)$. Then the sign structure of the Jacobian of $\overline{H}$ will be of the Leontief type. Note that $\overline{H}(b^*) = \overline{H}(a^*)$ where $D(a) = a^*$ and $D(b) = b^*$ and $b^* > a^*$ Now we can conclude from what we have shown already that $a^* = b^*$ or $a = b$ leading to a contradiction. This terminates the proof of Theorem 1.

What happens to Theorem 1 if we allow the partial derivatives to vanish at some points? We are able to settle the question when $|J| > 0$ throughout Ω. The problem remains open when $|J| < 0$ for every $x \in \Omega$. We are ready to state the following.

THEOREM 2. *Let F: $\Omega \subset R^4 \to R^4$ be a C^1 differentiable function where Ω is an arbitrary rectangular region. Suppose the Jacobian J is nonsingular with $|J| > 0$ throughout Ω and has the following (weak) sign structure*

$$J = \begin{bmatrix} \geqslant 0 & \leqslant 0 & \leqslant 0 & \leqslant 0 \\ \leqslant 0 & \geqslant 0 & \leqslant 0 & \leqslant 0 \\ \leqslant 0 & \leqslant 0 & \geqslant 0 & \leqslant 0 \\ \leqslant 0 & \leqslant 0 & \leqslant 0 & \geqslant 0 \end{bmatrix}$$

throughout Ω. Then F is globally one-one throughout Ω.

PROOF. We will only indicate the relevant modifications required as the proof is similar to Theorem 1. Since F is C^1 differentiable, value of J (as well as value of J') is positive or negative throughout Ω (see Lemma 2). If value of J is positive then J is a P-matrix and we are done. Suppose value of J (as well as value of J') is negative throughout Ω. Let $G = -F$. Then the Jacobian of the map G satisfies the conditions of Lemma 2 throughout Ω and further we assume value of J_G is positive in Ω. To complete the proof it is enough if we prove the following. If $a \in \Omega$, $G(x) \leqslant G(a)$ and $x \geqslant a$ then $x = a$. [To prove this we may and do assume that Ω is a compact rectangle and this entails no loss of generality.] As before (see Theorem 1) let X be the solution set, $Y = X \setminus \{a\}$ and $\bar{x}$ be a minimal element of Y. Since value of J_G is positive, $\bar{x} > a$ is impossible. Suppose $\bar{x}_1 = a_1$ and $\bar{x}_i > a_i$, $i = 2, 3, 4$. As the value of J_G is positive at $\bar{x}$, the first row of J_G must contain at least one positive entry. This means $g_1(x) > g_1(a)$ where $G = (g_1, g_2, g_3, g_4)$ but this leads to a contradiction to the assumption that $G(\bar{x}) \leqslant G(a)$.

Suppose $\bar{x}_1 = a_1$, $\bar{x}_2 = a_2$ and $\bar{x}_i > a_i$, $i = 3, 4$. If the first or the second row of J_G at $\bar{x}$ contains one positive entry we can conclude either $g_1(\bar{x}) > g_1(a)$ or $g_2(\bar{x}) > g_2(a)$ which will lead to a contradiction. Suppose J_G at $\bar{x}$ takes the following form

$$\begin{bmatrix} \leqslant 0 & \geqslant 0 & 0 & 0 \\ \geqslant & \leqslant 0 & 0 & 0 \\ \geqslant 0 & \geqslant 0 & \leqslant 0 & \geqslant 0 \\ \geqslant 0 & \geqslant 0 & \geqslant 0 & \leqslant 0 \end{bmatrix}.$$

Since the value of J_G at $\bar{x}$ is positive and since $|J_G| > 0$, it follows that the principal minor of order 2×2 (omitting the first two rows and two columns) is negative at $\bar{x}$. In fact value of this 2×2 matrix is positive. Let $\bar{G} = (g_3, g_4)$ and $\bar{\Omega} = \{(x_3, x_4)$: $(a_1, a_2, x_3, x_4) \in \Omega\}$. Since $\bar{G}$ is differentiable and since $\bar{x}_3 > a_3$, $\bar{x}_4 > a_4$ we can construct for small positive t, $x_3 + tu_3 > a_3$ and $\bar{x}_4 + tu_4 > a_4$ where $u_3 < 0$, $u_4 < 0$ with $J_{\bar{G}} u < 0$ where $u = \binom{u_3}{u_4}$. [Existence of the vector u is a consequence of the fact that the value of the 2×2 matrix at $\bar{x}$ is positive.] For a suitable choice of $t > 0$, (see case (i), p. 10 in [13])

$$g_3(a_1, a_2, y_3, y_4) < g_3(a_1, a_2, \bar{x}_3, \bar{x}_4) \quad \text{and} \quad g_4(a_1, a_2, y_3, y_4) < g_4(a_1, a_2, \bar{x}_3, \bar{x}_4)$$

$$\text{where } y_3 = \bar{x}_3 + tu_3, \quad y_4 = \bar{x}_4 + tu_4.$$

Since $y_3 < \bar{x}_3$ and $y_4 < \bar{x}_4$, from the sign structure of the Jacobian of the map G, it follows that

$$g_1(a_1, a_2, y_3, y_4) \leqslant g_1(a_1, a_2, \bar{x}_4, \bar{x}_4) \quad \text{and} \quad g_2(a_1, a_2, y_3, y_4) \leqslant g_2(a_1, a_2, \bar{x}_3, \bar{x}_4)$$

In other words $G(a_1, a_2, y_3, y_4) \leq G(a_1, a_2, \bar{x}_3, \bar{x}_4)$ it follows that $(a_1, a_2, y_3, y_4) \in Y$ contradicting the minimality of $(a_1, a_2, \bar{x}_3, \bar{x}_4)$. Thus the case $\bar{x}_1 = a_1, \bar{x}_2 = a_2$ and $\bar{x}_i > a_i$, $i = 3, 4$, is impossible.

Suppose $\bar{x}_1 = a_1$, $\bar{x}_2 = a_2$, $\bar{x}_3 = a_3$ and $\bar{x}_4 > a_4$. If any one of the first three entries in the last column of J_G at $\bar{x}$ is positive we will arrive at a contradiction to the fact that $G(\bar{x}) \leq G(a)$. If all the first three entries in the last column of J_G at $\bar{x}$ are zero, the fourth entry in that column must be negative. This will mean that the value of the matrix game J_G^t at $\bar{x}$ will be less than or equal to zero contradicting our hypothesis namely value of J_G^t is positive throughout Ω. This completes the proof of $G(x) \leq G(a)$ and $x \geq a \Rightarrow x = a$. The rest of the proof can be repeated verbatim as the proof of Theorem 1. This terminates the proof of Theorem 2.

REMARK 1. We are unable to prove theorem 2 when $|J| < 0$ throughout Ω. The main trouble arises due to the failure of lemma 2 when $|J| < 0$ if we want to imitate the proof given here.

REMARK 2. Original proof of Gale-Nikaido's theorem uses induction argument (see Theorem 3, p. 85 in [2]), but the proof of Theorem 2 (as well as Theorem 1) of the present paper uses the sign structure of the Jacobian.

REMARK 3. Gale-Nikaido proves univalence in R^n when Ω is an open rectangular region and when the Jacobian is a weak P-matrix (that is $|J| > 0$ and proper principal minors are nonnegative). Compared to this result, Theorem 2 has limited scope as we shall see below.

Now one can raise the following question:

Is it possible to extend Theorem 1 as well as Theorem 2 to higher dimensions?

We are unable to say anything as far as Theorem 1 is concerned, but we have a counterexample in R^6 for Theorem 2. This example is based on an example given in [14]. We do not know the answer when $n = 5$ for Theorem 2.

Construction of counterexample. Define $f(u, v) = (F_1(u, v), F_2(u, v))$ where $F_1(u, v) = e^{2u} - v^2 + 3$ and $F_2(u, v) = 4ve^{2u} - v^3$. Observe that $|J_f| > 0$ and $f(0, \pm 2) = (0, 0)$—this map was constructed by Gale and Nikaido. Next define a smooth map F of R^3 into itself by $F(u, v, w) = (F_1(u, v), F_2(u, v), F_3(u, v, w))$ where $F_3(u, v, w) = (10 + e^{2u})(e^v + e^{-v})(e^{100w} - e^{-100w})$. Note that $F(u, v, 0) = (F_1(u, v), F_2(u, v), 0)$ and therefore $F(0, \pm 2, 0) = (0, 0, 0)$. Now take a linear mapping A of R^3 into itself given by the matrix

$$A = \begin{bmatrix} 1/3 & -2/3 & 1/3 \\ 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

Now define the mapping G of R^3 into itself by $G(x) = A^{-1} \circ F \circ A(x)$. Note that G is not one-one and all the partial derivatives are positive [14]. Define a mapping H of R^3 to R^3 by $H(y) = (-y_1, -y_2, -y_3)$ where $y = (y_1, y_2, y_3)$. Now the required map is the following map L from R^6 to R^6:

$$L(y, x) = (-G(x), H(y)) \quad \text{where } y, x \in R^3.$$

Note that G is independent of the first three coordinates and H is independent of the last three coordinates. One can verify that the Jacobian of the map L is of Leontief type and $|J| > 0$ throughout R^6. In other words the map L satisfies the conditions of Theorem 2 but clearly L is not one-one since G is not one-one. Thus Theorem 2 fails in higher dimensions. However we do not know the answer to Theorem 2 when $n = 5$. However one can prove partial results with some additional assumptions. These results are stated in the next section.

We close this section by giving an example to show that a matrix game has the value greater than zero but a principal subgame (omitting a row and the corresponding column from the given matrix) has the value 0. For example take

$$\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

As such results obtained by Charnes et al. [1] are not applicable to our situation.

3. Further extensions, remarks and open problems. One can extend Theorem 1 partially for $n \geqslant 5$ as follows.

THEOREM 3. *Let F: $\Omega \subset R^n \to R^n$ be a $C^{(1)}$ differentiable map with $n \geqslant 5$ and Ω a rectangular region in R^n. Suppose the following conditions are met throughout Ω:*

(i) *Jacobian J is nonsingular.*

(ii) *J has strictly positive diagonal entries and strictly negative off-diagonal entries.*

(iii) *Every principal minor of the Jacobian of order $K \times K$ is positive where $K = 1, 2, \ldots [n/2]$ where $[n/2]$ denotes the integral part of $n/2$. Then F is globally one-one in Ω.*

Essential ideas of the proof are already contained in the proof of Theorem 1 and as such we will not make an attempt to prove Theorem 3. Now the following question arises:

In Theorem 3, will (i) and (ii) alone imply global univalence?

Answer is yes for $n \leqslant 4$, but for general $n \geqslant 5$, it is not known.

In fact Theorem 1 can be proved under slightly weaker hypothesis.

THEOREM 1′. *Let F: $\Omega \subset R^4 \to R^4$ be a C^1 differentiable map where Ω is a rectangular region in R^4. Suppose the Jacobian J of F is a nonsingular matrix with the following sign structure throughout Ω:*

$$\begin{bmatrix} \geqslant 0 & - & - & - \\ - & \geqslant 0 & - & - \\ - & - & \geqslant 0 & - \\ - & - & - & \geqslant 0 \end{bmatrix}.$$

Then F is globally one-one in Ω.

Note that Theorem 1′ is not included in Theorem 2 since $|J| < 0$ is possible by our assumptions in Theorem 1′. Next theorem includes Theorem 1 as well as Theorem 1′.

It was pointed out in the previous section that we do not know whether global univalence prevails in Theorem 2 if we assume $|J| < 0$ (instead of $|J| > 0$) throughout Ω. However it can be proved under additional assumptions.

Call a matrix A *decomposable* if we can find two permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

otherwise A is said *to be indecomposable*. In particular a reducible matrix (where $P = Q$) is decomposable.

THEOREM 2′. *Let F: $\Omega \subset R^4 \to R^4$ be a C^1 differentiable function where Ω is a rectangular region. Suppose the Jacobian J is indecomposable, nonsingular and has the* *following sign structure*:

$$J = \begin{bmatrix} \geq 0 & \leq 0 & \leq 0 & \leq 0 \\ \leq 0 & \geq 0 & \leq 0 & \leq 0 \\ \leq 0 & \leq 0 & \geq 0 & \leq 0 \\ \leq 0 & \leq 0 & \leq 0 & \geq 0 \end{bmatrix}$$

throughout Ω. *Then* F *is globally one-one in* Ω.

Indecomposability makes the value of the game J to be nonzero and as such same proof goes through as that of Theorem 2.

Does Theorem 2′ remain valid if we replace the assumption of indecomposability by irreducibility?

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