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ADDITIVE AND MULTIPLICATIVE HETEROSCEDASTICITY : A BAYESIAN ANALYSIS WITH AN APPLICATION TO AN EXPENDITURE MODEL

K. SUREKHA
Indian Statistical Institute, 8th Mile, Mysore Road, Bangalore 560059, India

and

W. E. GRIFFITHS
University of New England, Armidale, Australia 2351

Linear expenditure models with additive and multiplicative heteroscedastic variances are considered and Bayesian estimation techniques based on non-informative priors are suggested. Marginal and conditional posteriors for parameters of interest and Bayes' estimates for parameters of expenditure models are also obtained. A Bayesian comparison of the two heteroscedastic structures has also been carried out.

1. INTRODUCTION

In most econometric specifications of household expenditure relationships heteroscedasticity of expenditure observations is well recognised. Many different functional forms of heteroscedasticity have been suggested in this context (Theil 1951, Kempthorne 1952, Prais and Houthakker 1955, Jorgenson 1965, Battese and Bonyhady 1981, Surekha 1980, Surekha and Griffiths 1984b). In most cases the variance of the dependent expenditure observations is assumed to be a function of certain known exogenous variables.

Two important types of heteroscedastic specifications in a general linear model,

$$y_t = x_t\beta + u_t, \quad t = 1, 2, \ldots, T, \tag{1.1}$$

where y_t is the tth observation on a dependent variable, x_t is a k-dimensional row vector containing the tth observation on k explanatory variables, β is an unknown coefficient vector to be estimated, and the u_t are independent normal random variables with zero mean, are the following. First,

$$V(y_t) = V(u_t) = \sigma_t^2 = (z_t\alpha)^2, \tag{1.2}$$

where the z_t are $(1 \times p)$ vectors of observations on known exogenous variables with first element unity, and α is a $(p \times 1)$ vector of unknown parameters[1]; and second,

$$V(y_t) = V(u_t) = \sigma_t^2 = \exp(z_t\gamma), \tag{1.3}$$

where the z_t are as defined above and γ is a $(p \times 1)$ vector of unknown parameters. In both cases z_t may or may not be related to the regressor x_t and the order p may or may not be equal to k (generally $p \leqslant k$). The model with variance (1.2) will be referred to as the *additive model* and following Harvey (1976), the model with variance (1.3) will be referred to as the *multiplicative model.*

Sampling theory estimators have been suggested by Harvey (1974, 1976) for the additive and multiplicative models respectively. Special cases of the multiplicative model have been estimated by Geary (1966), Park (1966), Kmenta (1971), Goldfeld and Quandt (1972), whereas, for the additive model, Glejser (1969) has suggested a two step estimated generalised least squares estimator and Rutemiller and Bowers (1968) have suggested a maximum likelihood estimator based on the method of scoring. However, for the purpose of estimating expenditure relationships neither of the two variance specifications has been used. Although a special case of the additive model in which the variance of expenditure observations is assumed proportional to the square of its mean has been proposed and estimated from sampling theory point of view by Theil (1951), Prais and Houthakker (1955), Jorgenson (1965) and Battese and Bonyhady (1981).

The present study has two main objectives. The first is to suggest a Bayesian estimation technique for both types of models and illustrate it for an expendi-

1. For σ_t to be meaningful, α cannot be identically equal to zero and, also, we may need to impose certain other restrictions on z_t or α to ensure of positive σ_t.

ture model on the basis of the data obtained from the Macquari University Survey of Consumer Expenditures and Finances 1966-68. The reason for suggesting Bayesian estimation is two fold. The first of which is based on the results of a recent Monte Carlo study by Surekha and Griffiths (1984a) which suggests that the Bayes' estimators are more efficient in terms of their MSE than a wide range of classical estimators. Secondly, Bayesian posterior probability density functions (pdf) of parameters of interest are more informative than the point or interval estimates. This point will be more obvious from the empirical results we have presented in Section 4.

The second objective of this study is to consider a Bayesian comparison of the two models on the basis of their posterior odds using non-informative priors.

2. BAYESIAN ESTIMATION

In order to carry out Bayesian estimation we need to specify a prior probability density function which is combined with the likelihood function by means of Bayes' theorem to yield a posterior pdf for the parameters. The marginal posteriors are obtained by integrating the nuisance parameters out of the joint posterior. In most econometric studies prior information available on regression coefficients and other variance parameters of a heteroscedastic model is rather vague and this can be represented in the form of Jeffreys' non-informative priors (see, Jeffreys 1961). We use these prior pdfs in our study.

2.1 *Additive Model*

Let us first consider the additive model. Given independent normal random errors u_t, the likelihood function for α and β, in matrix notation, is

$$p(y/\beta, \alpha_0^2, \delta) = (2\pi)^{-T/2} \alpha_0^{-T} \mid D_A \mid^{1/2}.$$

$$\exp\left[-\frac{1}{2\alpha_0^2} \left\{ \nu s^2 + (\beta - \hat{\beta})' X' D_A X (\beta - \hat{\beta}) \right\} \right], \qquad (2.1.1)$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{p-1})'$,

$$\delta_j = \frac{\alpha_j}{\alpha_0}, \qquad j = 1, 2, \ldots, p-1$$

$$\delta = (\delta_1, \delta_2, \ldots, \delta_{p-1})',$$

$$D_A = \text{Diag}\left[\left(1 + \sum_{j=1}^{p-1} \delta_j z_{j1}\right)^{-2}, \left(1 + \sum_{j=1}^{p-1} \delta_j z_{j2}\right)^{-2}, \ldots, \left(1 + \sum_{j=1}^{p-1} \delta_j z_{jT}\right)^{-2}\right];$$

$$vs^2 = (Y - X\hat{\beta})'D_A(Y - X\hat{\beta}), \qquad v = T - k, \text{ and}$$
$$\hat{\beta} = (X'D_AX)^{-1}X'D_AY. \tag{2.1.2}$$

Since α and β are location and scale parameters respectively, we assume prior independence, and take the prior pdf[2] as $p(\alpha, \beta) = p(\alpha)p(\beta)$. Following Jeffreys' rule for multiparameter case prior pdfs for α and β taken are taken as proportional to the square root of the determinant of Fisher's information matrix. This yields

$$p(\alpha) \propto |\ \text{Inf}\ (\alpha)\ |^{1/2}, \quad \propto |\ Z'D_A^*Z\ |^{1/2};$$

$$p(\beta) \propto |\ \text{Inf}\ (\beta)\ |^{1/2}, \quad \propto |\ Z'D_A^*Z\ |^{1/2}, \quad \propto \text{constant};$$

hence $\quad p(\alpha, \beta) \propto |\ Z'D_A^*Z\ |^{1/2}, \qquad (2.1.3)$

where $\quad D_A^* = \text{diag}\ [(z_1\alpha)^{-2}, (z_2\alpha)^{-2}, \ldots, (z_T\alpha)^{-2}]. \qquad (2.1.4)$

It is easy to see that

$$p(\beta, \alpha_0^2, \delta) \propto \frac{1}{\alpha_0^2}\ |\ Z'D_AZ\ |^{1/2} \tag{2.1.5}$$

and that α_0 and δ are *a priori* independent (see, Wilks 1962 : 57-58). We find that $p(\alpha_0^2) \propto (1/\alpha_0^2)$ is the usual improper prior for the variance in the standard regression model whereas $p(\delta) \propto |\ Z'D_AZ\ |^{1/2}$ is a proper pdf[3]. Although $p(\delta)$ is a function of Z' matrix which is a part of the sample information, it is a noninformative prior. As it has been discussed by Jeffreys (1961), Box and

2. A uniform prior on β and α resulted in an improper posterior pdf for β and δ. See Surekha (1980).

3. For a proof see, for example, Surekha (1980) and Griffiths *et al.* (1979) which discuss the case $k = p = 2$ and $k = p = 3$ respectively.

Tiao (1973) that a noninformative prior does not imply no or zero information, rather it refers to little information relative to the information contained in the likelihood function (see, e.g., Surekha 1980 : Chapter 5).

By using techniques of integral calculus, the joint posterior pdf for $(\beta, \alpha_0^2, \delta)$ obtained from (2.1.1) and (2.1.5) by use of Bayes' Theorem is

$$p(\beta, \alpha_0^2, \delta/Y) \propto |Z'D_AZ|^{1/2} \, |D_A|^{1/2} \alpha_0^{-(T+1)} \cdot \exp\left[-\frac{1}{2\alpha_0^2}\left\{\nu s^2 + (\beta - \hat{\beta})'X'D_AX(\beta - \hat{\beta})\right\}\right]. \tag{2.1.6}$$

This joint posterior distribution in (2.1.6) is of more than two $[(k+p), k \geqslant 2, p \geqslant 2]$ dimensions as such it is hard to conceptualise. Therefore, to make inferences about the parameters of interest we proceed in the following manner to obtain marginal and/or conditional posteriors.

Integrating α_0^2 out of (2.1.6) yields

$$p(\beta, \delta/Y) \propto |D_A|^{1/2} \, |Z'D_AZ|^{1/2}[\nu s^2 + (\beta - \hat{\beta})'X'D_AX(\beta - \hat{\beta})]^{-T/2}. \tag{2.1.7}$$

The conditional distribution of β, $p(\beta/\delta, Y)$ conditional on δ, is a multivariate t-distribution. Using the properties of the t-distribution we can derive the marginal posterior pdf for a single $\beta_i (i = 0, 1, \ldots, k-1)$ by integrating the rest of β_i's out. This gives

$$p(\beta_i, \delta/Y) \propto s^{-(\nu+1)} \, |D_A|^{1/2} c_{ii}^{-1/2} \, |X'D_AX|^{-1/2} \, |Z'D_AZ|^{1/2} \cdot \left[1 + \frac{(\beta_i - \hat{\beta}_i)^2}{\nu s^2 c_{ii}}\right]^{-(\nu+1)/2}, \tag{2.1.8}$$

where c_{ii} is the ith diagonal element of $(X'D_AX)^{-1}$. Further, we can obtain

$$p(\delta/Y) \propto s^{-\nu} \, |D_A|^{1/2} \, |X'D_AX|^{-1/2} \, |Z'D_AZ|^{1/2}. \tag{2.1.9}$$

If $k = p = 2$ or $k = p = 3$, it can be easily shown that the joint posterior pdf $p(\delta/Y)$ is a proper density but its moments do not exist irrespective of $X = Z$ or not. In such a situation δ can be integrated out of (2.1.8) to obtain the marginal posterior for each of β_i, $i = 0, \ldots, k-1$. Next, from (2.1.6), integrating out β yields

$$p(\alpha_0^2, \delta/Y) \propto (\alpha_0^2)^{-(\nu+2)/2} \, |D_A|^{1/2} \, |X'D_AX|^{-1/2} \, |Z'D_AZ|^{1/2} \exp\left[-\frac{\nu s^2}{2\alpha_0^2}\right]. \tag{2.1.10}$$

This indicates that the conditional distribution of $[\alpha_0^2/\nu s^2]$ given ν is an inverted χ^2-distribution and its moments exist providing $\nu > 2k$. Also by transformation of variables, we get

$$p(\alpha_j, \delta/Y) \propto \alpha_j^{-(\nu+k-1)} \delta_j^{(\nu+k)} \mid D_A \mid^{1/2} \mid X'D_AX \mid^{-1/2} \mid Z'D_AZ \mid^{1/2}.$$

$$\exp\left[-\frac{\nu s^2 \delta_j^2}{2\alpha_j^2}\right] \text{ for } j = 1, 2, \ldots, p-1. \tag{2.1.11}$$

If the order p is only 3, it is obvious from (2.1.6) to (2.1.9) that the marginal posterior for a β_i, δ_j or an α_j can be obtained by at most 3 dimensional numerical integration. However, if $p > 3$, to reduce the computational burden there is a need to look for some kind of method for approximation of the posteriors (for example, Johnson 1967, 1970, Box and Hill 1974, Zellner and Rossi 1982 and Surekha and Griffiths 1986).

2.2 *Multiplicative Model*

Given the model in (1.3), the likelihood function for β and γ is

$$p(Y/\beta, \gamma) = (2\pi)^{-T/2} \prod_t \left\{\exp\left[-\frac{z_t\gamma}{2}\right]. \right.$$

$$\left.\exp\left[-\frac{1}{2}\sum_t e^{-Z_t\gamma}\cdot(y_t - x_t\beta)^2\right]\right\}. \tag{2.2.1}$$

Proceeding as in the case of the additive model, a noninformative prior for β and γ is obtained by assuming prior independence, i.e. $p(\beta, \gamma) = p(\beta)\, p(\gamma)$, where

$$p(\beta) \propto \mid \text{Inf}(\beta) \mid^{1/2} = \mid \sum_t e^{-Z_t\gamma} x_t' x_t \mid^{1/2}, \quad \propto \text{constant} \tag{2.2.2}$$

and

$$p(\gamma) \propto \mid \text{Inf}(\gamma) \mid^{1/2} = \mid \sum_t z_t' z_t \mid^{1/2} \propto \text{constant}, \tag{2.2.3}$$

and thus from (2.2.2) and (2.2.3)

$$p(\beta, \gamma) \propto \text{constant}. \tag{2.2.4}$$

Note that Jeffreys' uniform prior assuming that each of the β_i's, γ_i's are mutually independent and that $-\infty < \beta_i, \gamma_j < \infty$, $i = 0, 1, \ldots, p-1$, $j = 0, 1, \ldots, p-1$, will be identical with the Invariance prior as specified in (2.2.4). Note that this prior in (2.2.4) is an improper density.

If we let $\gamma = (\gamma_0, \gamma^{*\prime})'$, then

$$V(y_t) = k^* e^{z_t^{*} \gamma^*}, \tag{2.2.5}$$

where $k^* = e^{\gamma_0}$ and z_t^* is z_t without the first element unity. The joint posterior $p(\beta, k^*, \gamma^*/Y)$ can be obtained from (2.2.1) and (2.2.4) and simplified as in the case of the additive model. For the parameters of interest we obtain the following posterior pdfs :

$$p(\beta, \gamma^*/Y) \propto | D_A |^{1/2} [\nu s^2 + (\beta - \hat{\beta})' X' D_M X (\beta - \hat{\beta})]^{-T/2}. \tag{2.2.6}$$

This yields

$$p(\beta/\gamma^*, Y) \propto s^{-k} | X' D_M X |^{1/2} \left[1 + \frac{(\beta - \hat{\beta})' X' D_M X (\beta - \hat{\beta})}{\nu s^2} \right]^{-(\nu+k/2)}, \tag{2.2.7}$$

which is a multivariate t-distribution with parameters ν and k;

$$p(r^*/Y) \propto s^{-\nu} | X' D_M X |^{-1/2} | D_M |^{1/2}, \tag{2.2.8}$$

where

$$D_M = \text{diag}\,[e^{-z_1^* \gamma^*}, e^{-z_2^* \gamma^*}, \ldots, e^{-z_T^* \gamma^*}],$$

$$\nu s^2 = (Y - X\hat{\beta})' D_M (Y - X\hat{\beta}), \qquad \nu = T - k,$$

and

$$\hat{\beta} = (X' D_M X)^{-1} X' D_M Y. \tag{2.2.9}$$

Properties of multivariate t-distribution can be used to obtain

$p(\beta_i, \gamma^*/Y)$ from $p(\beta, \gamma^*/Y)$.

That is,

$$p(\beta_i, \gamma^*/Y) \propto s^{-(\nu+k+1)} | D_M |^{1/2} | X' D_M X |^{-1/2} c_{ii}^{-1/2} \cdot$$
$$\left[1 + \frac{(\beta - \hat{\beta}_i)^2}{\nu s^2 c_{ii}} \right]^{-(\nu+k+1/2)}, \tag{2.2.10}$$

where c_{ll} is the lth diagonal element of $(X'D_M X)^{-1}$. From the joint posterior $p(\beta, k^*, \gamma^*/Y)$, β can be integrated out, i.e.,

$$p(k^*, \gamma^*/Y) \propto |D_M|^{1/2} \, |X'D_M X|^{-1/2} (k^*)^{-T/2} \exp\left[-\frac{\nu s^2}{2k^*}\right], \quad (2.2.11)$$

$$p\left(\frac{k^*}{\nu s^2}\Big/ Y, \gamma^*\right) \propto \left(\frac{k^*}{\nu s^2}\right)^{-(\nu+2/2)} \exp\left[-\frac{\nu s^2}{2k^*}\right], \quad (2.2.12)$$

which is an inverted χ^2-distribution with $T - k$ degrees of freedom (Box and Tiao 1973 : 89). In this case also, as in the case of the additive model, for $p = 3$ the marginal posteriors for each of the parameters can be obtained by at most trivariate numerical integration of the posteriors given in (2.2.10) to (2.2.12). However, if $p > 3$, we must use some method of approximating the joint posterior.

3. A BAYSESIAN COMPARISON OF THE ADDITIVE AND THE MULTIPLICATIVE MODEL

Let H_A and H_M represent the hypothesis that the underlying model is an additive or a multiplicative model respectively. A Bayesian comparison of these two hypotheses entails a comparison of their respective posterior probabilities $p(H_i/Y)$, $i = A, M$, on the basis of their predictive pdfs

$p(Y/H_i)$, $i = A, M$. That is,

$$K_{A,M} = \frac{p(H_A/Y)}{p(H_M/Y)} = \left[\frac{p(Y/H_A)}{p(Y/H_M)}\right]\left[\frac{p(H_A)}{p(H_M)}\right]. \quad (3.1)$$

where

$$p(H_i/Y) = \frac{p(Y/H_i)\, p(H_i)}{\sum\limits_{l=A,M} p(Y/H_l)\, p(H_l)}, \quad (3.2)$$

$p(H_i)$, $i = A, M$ is the prior probability for the hypothesis H_i, $i = A, M$. The predicative pdfs $p(Y/H_i)$, $i = A, M$ are given as

$$p(Y/H_A) = \iint \int p(Y/\beta, \alpha_0^2, \delta, H_A)\, p(\beta, \alpha_0^2, \delta/H_A)\, d\alpha_0^2, d\beta, d\delta, \quad (3.3)$$

and

$$p(Y/H_M) = \iint \int p(Y/\beta, \gamma_0, \gamma^*, H_M)\, p(\beta, \gamma_0, \gamma^*/H_M)\, d\gamma_0, d\beta, d\gamma^*. \quad (3.4)$$

The first factor in (3.1) is called the Bayes factor and the second factor is referred to as the prior odds. The computation of posterior odds $K_{A,M}$ in (3.1) gives only an indication of which hypothesis is preferred and does not necessarily involve a decision to accept or reject a particular hypothesis. For example, if the prior odds equal one and $p(Y/H_A) > p(Y/H_M)$ or $K_{A,M}$ is greater than one, then we can simply say that the data favours H_A relative to H_M.

The predictive pdfs for the two models as in (3.3) and (3.4) can be computed by the similar procedure as used in Section 2.1 and 2.2. However, a major point of difference is that non-informative prior pdfs for the parameters which are used for the computation of posterior odds must be proper (see, e.g. Leamer 1977, Surekha and Griffiths 1984b). Therefore, we restrict the range of β_i, $\log \alpha_0^2$, γ_0 and γ^* to be finite. Without loss of generality we also assume that the range of β_i is same in both hypothesis and also range of $\log \alpha_0^2$ is same as that of γ_0. Further, for simplicity assume β, α, and γ are 2 dimensional vectors. That is,

$$
\begin{aligned}
& -M_i \leqslant \beta_i \leqslant M_i \qquad i = 0, 1, \\
& -N < \log \alpha_0^2, \gamma_0 \leqslant N, \\
& -P < \gamma^* < P; \qquad (3.5)
\end{aligned}
$$

where M_1, M_2, N and P are sufficiently large so that they are outside the region for which the likelihood functions are appreciable. Noting that $p(\delta) = 1/R \mid Z'D_AZ|^{1/2} d\delta$, the joint prior pdf for the two models can be written as follows:

$$p(\beta, \alpha_0^2, \delta/H_A) = \frac{1}{8M_0 NR\alpha_0^2} \left| Z'D_AZ \right|^{1/2} \qquad (3.6)$$

and

$$p(\beta, \gamma_0, \gamma^*/H_M) = \frac{1}{16M_0M_1 NP}. \qquad (3.7)$$

Following as in (2.1.6) to (2.1.9)

$$
\begin{aligned}
p(Y/H_A) &= \frac{\Gamma v/2\ (\pi)^{-(T+2/2)} \int_\delta p(\delta/Y, H_A)\, d\delta}{2.8\ M_0 M_1 NR} \\
&= \frac{\Gamma v/2 (\pi)^{-(T+2/2)}}{2.8\ M_0 M_1 NR} \int_\delta s^{-v} \mid D_A|^{1/2} \mid X'D_AX|^{-1/2} \mid Z'D_AZ|^{1/2}\, d\delta
\end{aligned}
\qquad (3.8)
$$

(see, Box and Tiao 1973 : Chapter 2).

Similarly, following as in (2.2.6) to (2.2.7),

$$p(Y/H_M) = \frac{\Gamma(\nu/2)\,(\pi)^{-(T+2)/2}}{16.\,M_0\,M_1\,N.P} \int_{\gamma^*} p(\gamma^*/Y,\,H_M)\,d\gamma^*$$

$$= \frac{\Gamma(\nu/2)\,(\pi)^{-(T+2)/2}}{16.\,M_0\,M_1\,N.P} \int_{-P}^{P} s^{-\nu}\,|\,X'D_MX\,|^{-1/2}\,|\,D_M\,|^{1/2}\,d\gamma^* \qquad (3.9)$$

Assuming equal prior odds, i.e. $p(H_A) = p(H_M)$ the posterior odds are given by

$$K_{A,M} = \frac{p(Y/H_A)}{p(Y/H_M)} = \frac{P\int_{\delta} s^{-\nu}\,|\,D_A\,|^{1/2}\,|\,X'D_AX\,|^{-1/2}\,|\,Z'D_AZ\,|^{1/2}\,d\delta}{R\int_{-P}^{P} s^{-\nu}\,|\,X'D_MX\,|^{-1/2}\,|\,D_M\,|^{1/2}\,d\gamma^*}$$

(3.10)

Although M_0, M_1, N all cancel out and R is well-defined, however, a small difficulty remains because of the unavoidable arbitrariness in the choice of the limits of integration ($-P$ to P) for γ^* as the relative weight on H_M can be made as small as one pleases by choosing P sufficiently large (see, e.g. Surekha and Griffiths 1984a). We can partly overcome this problem by choosing the limits of integration so as to include the effective range of integration with respect to the posterior density of γ^* and also large enough to include the total range over which likelihood function is dominant.

4. EMPIRICAL RESULTS—APPLICATION TO AN EXPENDITURE MODEL

Assume the model in (1.1) represents an expenditure function

$$y_t = \beta_0 + \beta_1 x_t + u_t, \qquad (4.1)$$

where y_t is the annual expenditure on food for the tth household and x_t is the total annual household expenditure on all items. The u_t are normal random unobservable error terms with mean zero and heteroscedastic variances. We will consider both the additive heteroscedastic model

$$V(y_t) = V(u_t) = \sigma_0^2\,(1 + \delta x_t)^2, \qquad (4.2)$$

where $\delta = \alpha_1/\alpha_0$, and the multiplicative heteroscedastic model

$$V(y_t) = V(u_t) = k^* e^{\gamma^* z_t}, \tag{4.3}$$

where $k^* = e^{\gamma_0}$. Note that in (4.2) and (4.3), it is assumed that z_t is equal to our only regressor x_t. For the analysis of the expenditure data it seems a reasonable assumption. The data were taken for the city of Perth (W. Australia) and we chose households of size 3.[4] All observations on x_t and y_t were divided by a constant = 1000 for the ease of handling data.

By using Simpsons rule for numerical integration[5], marginal posteriors for β_0 and β_1 have been obtained from equations (2.1.8) and (2.2.10) for the additive (A) and multiplicative (M) models, respectively. Marginal posteriors for δ, α_0, α_1 of the additive model were obtained from the equations (2.1.9), (2.1.10) and (2.1.11), respectively. For the multiplicative model, $p(k^*/y)$ and $p(\gamma^*/y)$ were obtained from (2.2.11).

The posteriors $p(\beta_0/Y)$ and $p(\beta_1/Y)$ in Figures 1 and 2 for the additive model

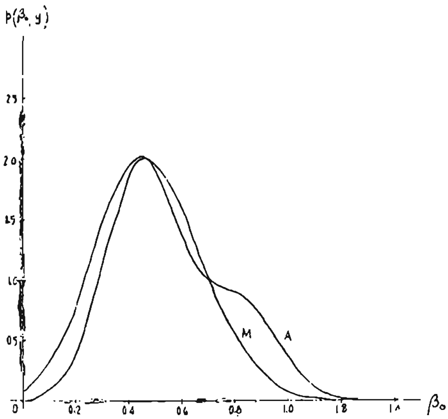

Figure 1. Marginal posterior pdf for β_0 : additive and multiplicative models.

4 Similar analysis can be carried out for data on any other city and for another household size also.

5. For details see Zellner, A. (1971), *An Introduction to Bayesian Inference in Econometrics*, New York, Wiley, Appendix C.

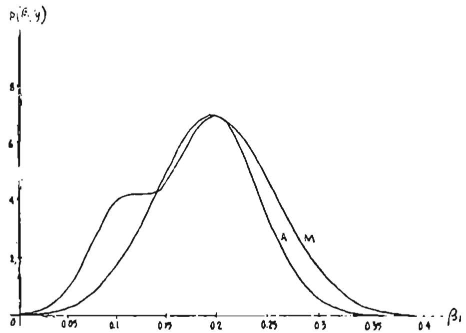

Figure 2. Marginal posterior pdf for β_1 : additive and multiplicative models.

have slightly unusual bumps. One possible explanation for this is the sensitivity of $p(\beta_0/Y)$ and $p(\beta_1/Y)$ to certain values of δ. We obtained conditional posterior pdfs and posterior means of β_0 and β_1 for a large number of δ values and we found that for a set of values of δ (between .01 and 1.0) there were very sharp changes in the values of the conditional means of β_0 and β_1. These were found to be in the range which is reflected in the bumps in the figures (see Table 1). For the multiplicative model, $p(\beta_0/Y)$ and $p(\beta_1/Y)$ are approximately normal as is to be expected with a non-informative prior and a large sample-size. The prior and posterior pdfs for δ, given in Figure 3, indicate how sample information has revised our prior beliefs about δ. Similarly, the marginal posterior pdfs for α_0, α_1, k^* and γ^* were derived and relative to the priors, they were quite informative. As the range of $p(\gamma^*/Y)$ is of interest in determining posterior odds, it is presented in Figure 4.

If one was interested in obtaining some summary statistics of the marginal posterior pdfs then one can obtain Bayes' estimates[6] for all parameters of interest. These are given in Table 2. Note that for δ, it is the median of the posterior $p(\delta/Y)$ which has been taken as the Bayes' estimate of δ because, as mentioned earlier, the posterior mean does not exist. Similar situation was encountered in Griffiths *et al.* (1979). The estimate of γ_0 was obtained using the

6. If the loss function is quadratic, posterior mean is the Bayes' estimate and if loss function is absolute, posterior median is taken as the Bayes' estimate.

Table 1

Posterior Means for the Additive Model

	β_0	β_1
Unconditional	0.557	0.175
Conditional on δ =		
0.0	0.819	0.104
0.00005	0.819	0.104
0 0005	0.818	0.104
0.01	0.802	0.108
0.05	0.747	0.123
0.1	0.698	0.135
1.0	0.477	0.196
1.5	0.448	0.204
2.0	0.432	0.208
5.0	0.397	0.218
10.0	0.384	0.222
15.0	0.397	0.224

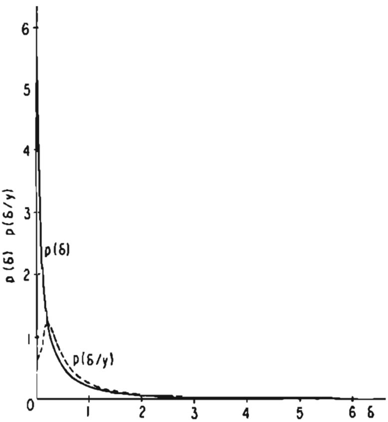

Figure 3. Marginal prior and posterior pdf for δ.

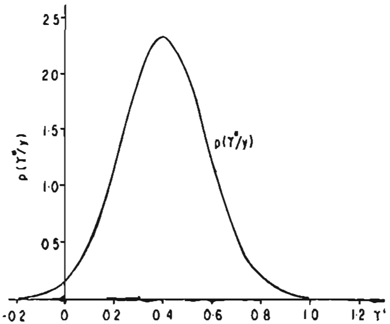

Figure 4. Marginal posterior pdf for γ^*.

Table 2

Bayes' Estimates for Parameters of the Additive and Multiplicative Models

Model	β_0	β_1	α_0	δ	α_1	k^*	γ_0	γ^*
A	0.557	0.175	0.133	0.4	0.06			
M	0.481	0.200				0.035	−3.36	0.404

relationship $\gamma_0 = \log k^*$. For the remaining parameters posterior mean was taken as the Bayes' estimate. Since the data set considered is only illustrative of the technique suggested in this paper, no significant conclusions can be drawn about the expenditure behaviour of the City of Perth on the basis of the sign and values of the parameter estimates, β's, α_0, δ, γ_0 and γ^*.

On the basis of these marginal posteriors as well as Bayes' estimates for β's and the variance parameters α_0, δ, γ_0 and γ^* no conclusions can be reached about the type of expenditure function to be chosen. In order to compare these two models on the basis of this given data we compute posterior odds from (3.10). The range of $\gamma^*(-P$ to $P)$ which is wide enough to cover the effective range of the posterior density of γ^* was found to be $(-1$ to $1)$ (See Figure 4).

Assuming equal prior odds, the posterior odds as in (3.10) was found to be

$$K_{A,M} = \frac{p(Y/H_A)}{p(Y/H_M)} = \frac{P\,0.12255128 \times 10^{20}}{R\,0.62591508 \times 10^{18}},$$

where $R = \int_{\delta} p(\delta)\, d\delta = 0.1696545 \times 10^{8}$

hence $K_{A,M} = 1.15$, (4.4)

which suggest that odds are slightly higher in favour of the additive model.

5. CONCLUSION

We have illustrated how Bayesian methods can be easily applied in the analysis of simple expenditure functions with two types of heteroscedastic variance structures. Our numerical example has been restricted to the case where there is only one exogenous variable $X_t = Z_t$ which also influences the variance of the error term. An additional exogenous variable could be added at the expense of trivariate rather than bivariate numerical integration, but for more than two variables in the variance function we would need to resort to some methods for approximating posteriors. The same problem does not arise with the numbers of exogenous variables which appear in the mean function (x_i's), because the β_i's, can be integrated out analytically.

REFERENCES

BATTESE, G. E. and BONYHADY, B. P. (1981). Estimation of household expenditure functions : An application of a class of heteroscedastic regression models. *Economic Record* 57 : 80-85.

BOX, G. E. P. and HILL, W. J. (1974). Correcting inhomogeneity of variance with power transformation weighting. *Technometrics* 16 : 385-389.

BOX, G. E. P. and TIAO, G. C. (1973). *Bayesian Inference in Statistical Analysis*. Addison-Wesley, Mass.

GEARY, R. C. (1966). A note on residual heterovariance and estimation efficiency in regression. *American Statistician* 20(4) : 30-31.

GLEJSER, H. (1969). A new test for heteroscedasticity. *Journal of American Statistical Association* 64 : 316-323.

GOLDFELD, S. M. and QUANDT, R. E. (1972). *Non-linear Methods in Econometrics*. North Holland, Amsterdam.

GRIFFITHS, W. E., DRYNAN, R. G. and SUREKHA, K. (1979). Bayesian estimation of a random coefficient model. *Journal of Econometrics* 10 : 201-220.

HARVEY, A. C. (1974). Estimation of parameters in a heteroscedastic regression model. Paper presented at the European Meeting of the Econometric Society, Grenoble.

_____ (1976). Estimating regression models with multiplicative heteroscedasticity. *Econometrica* 44 : 460-465.

JEFFREYS, H. (1961). *Theory of Probability*, Ed. 3. Clarendon, Oxford.

JOHNSON. R. A. (1967). An asymptotic expansion for posterior distributions. *The Annals of Mathematical Statistics* 38 : 1899-1906.

_____ (1970). Asymptotic expansions associated with posterior distributions. *The Annals of Mathematical Statistics* 41 : 851-864.

JORGENSON, E (1965). *Income-Expenditure Relations of Danish Wage and Salary Earners*. The Statistical Department, Copenhagen.

Kempthorne, O. (1952). *The Design and Analysis of Experiments*. Wiley, New York.

KMENTA, J. (1971). *Elements of Econometrics*. The Macmillan Company, New York.

LEAMER, E. E. (1978). *Specification Searches : Ad-hoc Inference with Non-Experimental Data*. Wiley, New York.

PRAIS, S. J. and HOUTHAKKER, H. S. (1955). *The Analysis of Family Budgets*. Cambridge University Press, London.

RUTEMILLER. H. C. and BOWERS, D. A. (1968). Estimation in a heteroscedastic regression model. *Journal of the American Statistical Association* 63 : 552-557.

SUREKHA, K. (1980). Contributions to Bayesian Analysis in Heteroscedastic Models. Unpublished *Ph.D. Thesis*, University of New England, Armidale, Australia.

SUREKHA, K. and GRIFFITHS, W. E. (1984a). A Monte Carlo comparison of some Bayesian and sampling theory estimators in two heteroscedastic error models. *Communications in Statistics*, B 13 : 85-105.

_____ and _____ (1984b). A Bayesian Comparison of Distributional Assumptions in Heteroscedastic Error Models. Paper presented at the 29th NBER-NSF seminar on Bayesian Statistics and Econometrics at Rutgers, New Brunswick, U.S.A. Forthcoming, *Statistica*.

_____ and _____ (1986). A note on posterior approximation in a heteroscedastic model. *Economics Letters* 19 : 249-252.

THEIL, H. (1951). Estimates and their variance of parameters of certain heteroscedastic disturbances. *Review of International Statistical Institute* 19 : 141-147.

ZELLNER, A. and ROSSI, P. (1982). Bayesian Analysis of Dichotomous Quantal Response Models (mimeo). Graduate School of Business, University of Chicago, U.S.A.