SOME RESULTS ON QUADRICS IN FINITE PROJECTIVE GEOMETRY BASED ON GALOIS FIELDS

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1. Introduction. In a paper (5) published in the *Proceedings of the Cambridge Philosophical Society*, Primrose obtained the formulae for the number of points contained in a non-degenerate quadric in $PG(n, s)$, the finite projective geometry of n dimensions based on a Galois field $GF(s)$. In § 3 of the present paper the formulae for the number of p-flats contained in a non-degenerate quadric in $PG(n, s)$ are obtained. In § 4 an interesting property of a non-degenerate quadric in $PG(2k, 2^m)$ is proved. These properties of a quadric will be used in solving some combinatorial problems of statistical interest in a later paper.

In finite projective geometry $PG(n, s)$ of n dimensions based on Galois field $GF(s)$, where s is a prime power, the points can be taken as $(n + 1)$-tuples $x = (x_0, x_1, \ldots, x_n)$ where $x_0, x_1, \ldots, x_n$ are elements of $GF(s)$ and the $(n + 1)$-tuple $\rho x = (\rho x_0, \rho x_1, \ldots, \rho x_n)$ is regarded as the same point as x for any non-zero element ρ of $GF(s)$. The null $(n + 1)$-tuple $(0, 0, \ldots, 0)$ is not regarded as a point. The set of points x which satisfy an equation

$$\mathbf{x}C = 0$$

where C is a matrix of order $(\overline{n + 1} \times k)$ with elements in $GF(s)$ and has rank k, $k = 1, 2, \ldots \overline{n + 1}$, is taken as an $(n - k)$-flat. In what follows for any point x of $PG(n, s)$ we shall use $\mathbf{x}$ to denote a row vector arising from the co-ordinates of x. A quadric Q in $PG(n, s)$ is the set of all points x which satisfies an equation

$$\mathbf{x}A\mathbf{x}' = 0 \tag{1.1}$$

where A is a triangular matrix of order $(\overline{n + 1} \times \overline{n + 1})$ with elements in

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$GF(s)$ and $\mathbf{x}'$ is the transpose of the row vector $\mathbf{x}$. If the characteristic of the field $GF(s)$ is not equal to 2, then the equation of Q can be taken as

$$\mathbf{x}B\mathbf{x}' = 0 \tag{1.2}$$

where B is a symmetric matrix of order $(\overline{n+1} \times \overline{n+1})$. In this case the quadric Q can be regarded as the set of self-conjugate points of the polarity defined by the symmetric bilinear form

$$\mathbf{x}B\mathbf{y}' = 0 \tag{1.3}$$

where $\mathbf{y} = (y_0, y_1, y_2, \ldots, y_n)$.

However, when the characteristic of the field $GF(s)$ is 2, the equation of Q cannot always be written in the form (1.2) and hence Q cannot be regarded as the set of self-conjugate points of a polarity defined by a symmetric bilinear form. For this reason we shall use (1.1) as the equation of Q.

Any non-singular matrix B of order $(\overline{n+1} \times \overline{n+1})$ defines a mapping of the points of $PG(n, s)$ onto itself. Under the mapping induced by B, the point x is mapped into the point y where

$$\mathbf{y} = \mathbf{x}B.$$

Such a mapping will be called a non-singular mapping. A quadric Q in $PG(n, s)$ is said to be degenerate if there exists a non-singular mapping which takes Q into a quadric Q' with the following equation

$$\mathbf{x}C\mathbf{x}' = 0$$

where C is a triangular matrix with all elements in the last row and last column equal to zero. A quadric Q in $PG(n, s)$ is said to be non-degenerate if it is not degenerate. A point α is defined to be conjugate to a point β with respect to (w.r.t.) Q if

$$\alpha(A + A')\beta' = 0. \tag{1.4}$$

The relationship of conjugacy is symmetrical. The polar space $T(\alpha)$ of a point α with respect to Q is the set of all points which are conjugate to α w.r.t. Q. The polar of α is the $(n-1)$-flat determined by the equation

$$\alpha(A + A')\mathbf{x}' = 0. \tag{1.5}$$

If the quadric Q is non-degenerate, (1.5) will determine an $(n-1)$-flat. The polar space $T(\Sigma_p)$ of a p-flat Σ_p w.r.t. a quadric Q is the set of all points which are conjugate to every point of Σ_p w.r.t. Q. Two flats Σ_p and Σ_q are said to be mutually conjugate w.r.t. Q if every point of Σ_p is conjugate to every point of Σ_q w.r.t. Q. If α and β are two points which are mutually conjugate w.r.t. Q and α' and β' are images of α and β under a non-singular mapping B, then α' and β' are mutually conjugate w.r.t. Q' where Q' is the image of Q under the mapping B. It should be noticed that with our definition of conjugacy, in $PG(n, 2^m)$ all points are self conjugate and in $PG(n, s)$, s odd, only the points on the quadric are self conjugate.

It has been shown by Primrose **(5)** that every non-degenerate quadric in $PG(2k, s)$ contains linear spaces of dimensionality $(k - 1)$ and does not contain any linear space of higher dimensionality. So with respect to the maximum dimensionality of a linear space contained in the quadric, the non-degenerate quadrics in $PG(2k, s)$ belong to only one type. However, the non-degenerate quadrics in $PG(2k - 1, s)$ belong to two different types, hyperbolic or elliptic. If a non-degenerate quadric in $PG(2k - 1, s)$ contains $(k - 1)$-flats and does not contain any linear space of higher dimensionality, then the quadric is said to be a hyperbolic non-degenerate quadric. If a non-degenerate quadric in $PG(2k - 1, s)$ contains $(k - 2)$-flats and does not contain any linear space of higher dimensionality, then the quadric is said to be elliptic. Primrose **(5)** uses the words unruled and ruled quadric, for elliptic and hyperbolic quadrics. Tallini **(8)** uses the names elliptic and hyperbolic quadrics.

2. Some results on the polar spaces are stated below in the form of lemmas and theorems, for convenience of reference. These results are either well known or can easily be proved.

LEMMA 2.1. *If a point α is conjugate to the points $\beta_1, \beta_2, \ldots, \beta_p$ w.r.t. a quadric Q, then α is conjugate to the linear flat determined by the points $\beta_1, \beta_2, \ldots, \beta_p$.*

LEMMA 2.2. *The polar space of a p-flat Σ_p is the intersection of the polar spaces of $\alpha_0, \alpha_1, \ldots, \alpha_p$ where $\alpha_0, \alpha_1, \ldots, \alpha_p$ are $(p + 1)$ independent points in Σ_p.*

LEMMA 2.3. *Let $\alpha_0, \alpha_1, \ldots, \alpha_p$ be independent points on a quadric Q in $PG(n, s)$. Then the p-flat Σ_p determined by these points is contained in Q if and only if the $(p + 1)$-points are pairwise conjugate* w.r.t. *Q.*

THEOREM 2.1. *Let Σ_k be a k-flat contained in a non-degenerate quadric Q_n in $PG(n, s)$. Let Σ_{n-k-1} be an $(n - k - 1)$-flat not intersecting Σ_k. Then*

(a) *$T(\Sigma_k)$ is an $(n - k - 1)$-flat.*

(b) *$T(\Sigma_k) \cap \Sigma_{n-k-1}$ is an $(n - 2k - 2)$-flat and $Q_n \cap T(\Sigma_k) \cap \Sigma_{n-k-1}$ is a non-degenerate quadric Q_{n-2k-2} on the $(n - 2k - 2)$-flat $T(\Sigma_k) \cap \Sigma_{n-k-1}$ which is elliptic or hyperbolic according as Q_n is elliptic or hyperbolic.*

THEOREM 2.2. *Let Σ_k be a k-flat contained in a non-degenerate quadric Q_n in $PG(n, s)$. Let Σ_p be any linear flat which is contained in Q_n and contains Σ_k. Then Σ_p is contained in $T(\Sigma_k)$, the polar space of Σ_k.*

Proof. Let Σ_k be determined by the $(k + 1)$-independent points $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k$ and Σ_p be determined by the $(p + 1)$ independent points $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_p$. Any point α of Σ_p can be represented as $\sum_{i=0}^{p} \lambda_i \alpha_i$ where $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_p$ are elements of $GF(s)$.

By Lemma 2.3 α_i is conjugate to each of the points $\alpha_0, \alpha_1, \ldots, \alpha_p$. Therefore, we have

$$\alpha(A + A')\alpha_i' = \left(\sum_{i=0}^{p} \lambda_i \alpha_i \right) (A + A') (\alpha_i') = 0, \; i = 0, 1, 2, \ldots, k.$$

So α is conjugate to every point α_i, $i = 0, 1, \ldots, k$. By Lemma 2.1 α is conjugate to Σ_k, the k-flat determined by the points $\alpha_0, \alpha_1, \ldots, \alpha_k$. Hence $\alpha \in T(\Sigma_k)$ and

$$\Sigma_p \subset T(\Sigma_k).$$

3. A projection and its use in determining the number of p-flats contained in a non-degenerate quadric in PG(n,s). Let O be a point in $PG(n, s)$ and π be an $(n - 1)$-flat not passing through 0. Let P be a point other than 0. The line OP intersects π at a point P'. P' is called the projection of P on π through 0. The projection of a set A in $PG(n, s)$ on π through O is defined to be the set of all points which are projections of the points of A on π through O. The projection of A on π through O will be denoted by $S_{0,\pi}(A)$. If O and π are assumed to be fixed, $S_{0,\pi}(A)$ will be written as $S(A)$. If C is a set of points containing O, then the projection of the set $C - \{O\}$ through O will be written as $S(C)$ for convenience.

LEMMA 3.1. *Let P be a point on a non-degenerate quadric Q_n in $PG(n, s)$ and T be the tangent space at P and π be an $(n - 1)$-flat not passing through P. In the following any projection is on π through P. Then*

(a) $S(Q_n \cap T)$ is Q_{n-2}, a non-degenerate quadric on the $(n - 2)$-flat $T \cap \pi$.

(b) *If Σ_p is a p-flat containing P and contained in Q_n, then*

$$S(\Sigma_p) = \Sigma_{p-1}, \text{ a } (p - 1)\text{-flat}$$

and

$$S(\Sigma_p) \subset Q_{n-2}.$$

(c) *If Σ_p is a p-flat not containing P and contained in $Q_n \cap T$, then*

$$S(\Sigma_p) = \Sigma_p', \text{ a } p\text{-flat}$$

and

$$S(\Sigma_p) \subset Q_{n-2}.$$

(d) *If Σ_p is a p-flat contained in Q_n but not in T, then*

$$S(\Sigma_p) = \Sigma_p', \text{ a } p\text{-flat}$$

and

$$S(\Sigma_p) \not\subset Q_{n-2}.$$

Proof. (a) By Theorem 2.1 $Q_n \cap T \cap \pi$ is a non-degenerate quadric Q_{n-2} in $PG(n - 2, s)$. Hence it will be sufficient to show that

$$S(Q_n \cap T) = Q_n \cap T \cap \pi. \tag{3.1}$$

(3.1) follows immediately from the definition of projection.

(b) Since $P \in \Sigma_p$ and

$$\Sigma_p \subset Q_n,$$

by Theorem 2.2

$$\Sigma_p \subset T.$$

So we have

$$\Sigma_p \subset Q_n \cap T.$$

It follows that

$$S(\Sigma_p) \subset S(Q_n \cap T) = Q_{n-2}.$$

Now to prove (b) it is sufficient to show that

$$S(\Sigma_p) = \Sigma_{p-1}, \text{ a } (p-1)\text{-flat.} \tag{3.2}$$

It is easy to check that

$$S(\Sigma_p) = \Sigma_p \cap \pi.$$

Hence (3.2) follows from the fact that Σ_p is not contained in π.

(c) Let Σ_{p+1} be the $(p+1)$-flat determined by P and Σ_p and $\Sigma_p' = \Sigma_{p+1} \cap \pi$. It is easy to check that

$$S(\Sigma_p) = \Sigma_p'.$$

Since

$$\Sigma_p \subset T \quad \text{and} \quad \Sigma_{p+1} \subset Q_n$$

by Theorem 2.2

$$\Sigma_{p+1} \subset Q_n \cap T.$$

So

$$\Sigma_p' \subset Q_n \cap T \cap \pi = Q_{n-2}.$$

(d) Let Σ_{p+1} and Σ_p' be defined as in (c). In this case Σ_p cannot contain P. If possible, suppose Σ_p contains P. Then Σ_p is a p-flat contained in Q_n and contains P. So by Theorem 2.2 Σ_p must be contained in T, the polar space of P. But this contradicts our hypothesis. Now, as in (c), we have

$$S(\Sigma_p) = \Sigma_p'$$

To show that $S(\Sigma_p) \subset Q_{n-2}$ it is sufficient to show that

$$\Sigma_p' \not\subset T \cap \pi. \tag{3.3}$$

(3.3) follows immediately from the fact that

$$\Sigma_p \not\subset T.$$

Projection of a class of sets. Let $\mathfrak{A}$ be a class of sets in $PG(n, s)$. Let P be a given point and π be an $(n-1)$-flat not passing through P. The projection of the class $\mathfrak{A}$ on π through 0 is defined to be the class consisting of the sets which are the projections on π through P of the sets of $\mathfrak{A}$ and is denoted by $S(\mathfrak{A})$.

LEMMA 3.2. *Let* $\mathfrak{A}$ *be a class of distinct p-flats passing through a point P in* $PG(n, s)$ *and* π *be an* $(n-1)$*-flat not passing through P. Then there exists a one-to-one correspondence between the two classes* $\mathfrak{A}$ *and* $S(\mathfrak{A})$.

Proof. Let Ω be any p-flat in the class $\mathfrak{A}$. Let Ω be made to correspond to $S(\Omega)$. We shall show that this correspondence is one to one. It will be sufficient to show that for any two different sets Ω and Ω' of the class $\mathfrak{A}$

$$S(\Omega) \neq S(\Omega'). \tag{3.4}$$

If possible, suppose (3.4) is not true. Then

$$S(\Omega) = S(\Omega').$$

Since $\Omega \neq \Omega'$ there exists a point R belonging to Ω but not belonging to Ω'. Let $R' = S(R)$, the projection of R. Then $R' \in S(\Omega')$, $P \in \Omega'$. So the line PR' is contained in Ω'. Obviously R is a point on the line PR'. Hence R is a point of the p-flat Ω' which is a contradiction.

THEOREM 3.1. *Let P be a point of a non-degenerate quadric* Q_n *in* $PG(n, s)$, $T(P)$ *be the tangent space at P and* π *be an* $(n-1)$*-flat not passing through P. Let* $\mathfrak{C}_{n,p}$ *denote the class of p-flats contained in* Q_n *and passing through P and* $\mathfrak{A}_{n,p}$ *be the class of all p-flats of* Q_n. *Then there exists a one-to-one correspondence between the classes* $\mathfrak{C}_{n,p}$ *and* $\mathfrak{A}_{n-2,p-1}$ *and hence the number of elements in each class is the same.*

Proof. Since each p-flat of $\mathfrak{C}_{n,p}$ passes through P, owing to Lemma 2.2, it will be sufficient to show that

$$S(\mathfrak{C}_{n,p}) = \mathfrak{A}_{n-2,p-1}. \tag{3.5}$$

We shall show that (3.5) is true if for Q_{n-2} we take the non-degenerate quadric $Q_n \cap T(P) \cap \pi$ in $PG(n-2, s)$.
Let $\Sigma_p \in \mathfrak{C}_{n,p}$ and $S(\Sigma_p) = \Omega$, $\Omega \in S(\mathfrak{C}_{n,p})$. By part (b) of Lemma 3.1

$$\Omega = \Sigma_{p-1} \subset Q_{n-2}.$$

Hence $\Omega \in \mathfrak{A}_{n-2,p-1}$. It follows that

$$S(\mathfrak{C}_{n,p}) \subset \mathfrak{A}_{n-2,p-1}. \tag{3.6}$$

Conversely let $\Sigma_{p-1} \in \mathfrak{A}_{n-2,p-1}$. Let Σ_p be the p-flat determined by P and Σ_{p-1}. Then using Lemma 1.3 it can easily be seen that $\Sigma_p \subset Q_n$ and

$$S(\Sigma_p) = \Sigma_{p-1}.$$

Hence

$$\Sigma_{p-1} \in S(\mathfrak{C}_{n,p}) \quad \text{and} \quad \mathfrak{A}_{n-2,p-1} \subset S(\mathfrak{C}_{n,p}). \tag{3.7}$$

(3.5) follows from (3.6) and (3.7).

THEOREM 3.2. *Let* $N(p, n)$ *denote the number of different p-flats contained in a non-degenerate quadric* Q_n *in* $PG(n, s)$. *Then*

$$N(p, n) = \begin{cases} \Phi(p, k), & \textit{for } n = 2k,\ p < k - 1, \\ \Phi_1(p, k), & \textit{for } n = 2k - 1,\ Q_n \textit{ elliptic and } p < k - 2, \\ \Phi_2(p, k), & \textit{for } n = 2k - 1,\ Q_n \textit{ hyperbolic and } p < k - 1, \end{cases}$$

where

$$\Phi(p, k) = \prod_{r=0}^{p} \frac{(s^{2(k-p+r)} - 1)}{(s^{p+1-r} - 1)}, p < k - 1,$$

$$\Phi_1(p, k) = \prod_{r=0}^{p} \frac{(s^{n-2p+2r} + s^{k-p+r-1} - s^{k-p+r} - 1)}{(s^{p+1-r} - 1}, p < k - 2,$$

$$\Phi_2(p, k) = \prod_{r=0}^{p} \frac{(s^{n-2p+2r} - s^{k-p+r-1} + s^{k-p+r} - 1)}{(s^{p+1-r} - 1)}, p < k - 1.$$

The expressions for $N(0, n)$ were obtained by Primrose **(5)**.

Proof. First we shall establish the following equation.

$$N(p, n) = \frac{N(p - 1, n - 2)N(0, n)(s - 1)}{(s^{p+1} - 1)}. \tag{3.8}$$

Let P be a point of Q_n. From Theorem 3.1 it follows that the number of p-flats contained in Q_n and passing through P is $N(p - 1, n - 2)$. Let us count the points in the p-flats contianed in Q_n. Every p-flat contributes $(s^{p+1} - 1)/(s - 1)$ points and the number of p-flats contained in Q_n is $N(p, n)$. Hence this collection of p-flats contains

$$N(p, n)\frac{s^{p+1} - 1}{s - 1}$$

points which are not all different. In this collection every point will be repeated as many times as there are p-flats of Q_n passing through a point. Through every point of Q_n there pass $N(p - 1, n - 2)$ p-flats and the number of points of Q_n is $N(0, n)$. Hence the collection of p-flats of Q_n contains $N(0, n)$ $N(p - 1, n - 2)$ points. Hence (3.8) follows.

Primrose **(5)** has obtained the following formulae:

$$\begin{aligned} \Phi(0, k) &= \frac{s^{2k} - 1}{s - 1} \\ \Phi_1(0, k) &= \frac{(s^{2k-1} + s^{k-1} - s^k + 1}{s - 1} \\ \Phi_2(0, k) &= \frac{(s^{2k-1} - s^{k-1} + s^k - 1}{s - 1}. \end{aligned} \tag{3.9}$$

Applying the difference equation (3.8) repeatedly and using the formulae (3.9), we get the required expressions for $\Phi(p, k)$, $\Phi_1(p, k)$ and $\Phi_2(p, k)$.

THEOREM 3.3. *The number of p-flats contained in a non-degenerate quadric*

Q_n in $PG(n, s)$ which pass through a given k-flat Σ_k contained in Q_n is $N(p-k-1, n-2k-2)$, where $N(p, n)$ denotes the number of p-flats contained in a non-degenerate quadric of the type (elliptic or hyperbolic) of Q_n.

Proof. Let $T(\Sigma_k)$ denote the polar space of Σ_k and Σ_{n-k-1} be an $(n-k-1)$-flat which does not intersect Σ_k. Let $\mathfrak{C}_{k,p}$ denote the class of p-flats contained in Q_n and passing through Σ_k. Let $\mathfrak{D}_{k,p}$ denote the class of $(p-k-1)$-flats contained in $Q_n \cap T(\Sigma_k) \cap \Sigma_{n-k-1}$. By Theorem 2.1 it is known that $Q_n \cap T(\Sigma_k) \cap \Sigma_{n-k-1}$ is a non-degenerate quadric Q_{n-k-2} in $PG(n-2k-2, s)$. Hence to prove the theorem it will be sufficient to show that there is a one-to-one correspondence between the classes $\mathfrak{C}_{k,p}$ and $\mathfrak{D}_{k,p}$ of p-flats.

Let $\Sigma_p \in \mathfrak{C}_{k,p}$. Then

$$\Sigma_k \subset \Sigma_p. \tag{3.10}$$

$$\Sigma_k \cap \Sigma_{n-k-1} = \Phi, \text{ the null set.} \tag{3.11}$$

From (3.10) and (3.11) it follows that

$$\Sigma_p \not\subset \Sigma_{n-k-1}.$$

So $\Sigma_p \cap \Sigma_{n-k-1}$ has dimensionality at least equal to $(p-k-1)$. Since

$$\Sigma_k \cap \Sigma_{n-k-1} = \Phi, \text{ the null set,}$$

the dimensionality of $\Sigma_p \cap \Sigma_{n-k-1}$ cannot exceed $(p-k-1)$. Hence $\Sigma_p \cap \Sigma_{n-k-1}$ is a $(p-k-1)$-flat Σ_{p-k-1}. Since

$$\Sigma_k \subset \Sigma_p \subset Q_n,$$

by Theorem 2.1 $\Sigma_p \subset T(\Sigma_k)$. So

$$\Sigma_p \cap \Sigma_{n-k-1} \subset Q_n \cap T(\Sigma_k) \cap \Sigma_{n-k-1} = Q_{n-2k-2}.$$

So

$$\Sigma_{p-k-1} \in \mathfrak{D}_{k,p}.$$

Let us make Σ_p of $\mathfrak{C}_{k,p}$ correspond to Σ_{p-k-1} of $\mathfrak{D}_{k,p}$. It can easily be seen that the correspondence is one to one.

4. Nucleus of polarity of a quadric

LEMMA 4.1. *Let α and β be two points of $PG(2k, 2^m)$ not lying on a non-degenerate quadric Q_n in $PG(n, 2^m)$. The line $\alpha\beta$ intersects the quadric in a single point if and only if the points α and β are mutually conjugate.*

Proof. Sufficiency. Assume that α and β are mutually conjugate. Any point on the line $\alpha\beta$, the line determined by the points α and β, other than α, can be represented as $\beta + \lambda\alpha$ where λ is an element of $GF(2^m)$. The number of points at which the line $\alpha\beta$ intersects Q_n is equal to the number of solutions in λ of the equation.

$$(\beta + \lambda\alpha)A(\beta' + \lambda\alpha') = 0 \tag{4.1}$$

where the equation of Q_n is

$$\mathbf{x}A\mathbf{x}' = 0. \tag{4.2}$$

Since α and β are mutually conjugate points not belong to Q_n, we have

$$\begin{cases} \alpha(A + A')\beta' = 0, \\ \alpha A\alpha' \neq 0 \text{ and} \\ \beta A\beta' \neq 0. \end{cases} \tag{4.3}$$

Using (4.3) and the fact that every element in $GF(2^m)$ has a unique square root, we can see that (4.1) has exactly one solution in λ.

Necessity. Assume that $\alpha\beta$ intersects Q_n at the single point γ_1. Suppose α and γ_1 are mutually conjugate. Then γ_1 belongs to $T(\alpha)$. Also, since the characteristic of the field $GF(2^m)$ is 2, α is self conjugate and belongs to $T(\alpha)$. So the line $\alpha\gamma_1$ belongs to $T(\alpha)$ and β, being a point on the line $\alpha\gamma_1$ belongs to $T(\alpha)$. But this contradicts our hypothesis that α and β are not mutually conjugate. So α and γ_1 are not mutually conjugate. Since α and γ_1 are not mutually conjugate, α is not a point of Q_n and γ_1 is a point of Q_n, we have

$$\begin{cases} \alpha A\alpha' \neq 0, \\ \gamma_1 A\gamma_1 = 0 \text{ and} \\ \alpha(A + A')\gamma_1' \neq 0. \end{cases} \tag{4.4}$$

Using (4.4), we can see that the point $\gamma_2 = \alpha + \lambda\gamma_1$ is a second point at which the line $\alpha\beta$ intersects Q_n where

$$\lambda = \frac{\alpha A\alpha'}{\alpha(A + A')\gamma_1'}.$$

But this contradicts our hypothesis. Hence α and β must be mutually conjugate.

THEOREM 4.1. *For every non-degenerate quadric Q_{2k} in $PG(2k, 2^m)$ there exists a point S not lying on the quadric such that every line through S intersects the quadric Q_{2k} in a single point. The point S is called the nucleus of polarity of Q_{2k}.*

Proof. Let Q_{2k}' be a non-degenerate quadric in $PG(2k, 2^m)$. Then according to Dickson (4) there exists a non-singular mapping which transforms Q_{2k}' to Q_{2k} with the equation

$$x_0^2 + x_1x_2 + x_3x_4 + \ldots + x_{2k-1}x_{2k} = 0.$$

Since the incidence properties in a projective geometry are invariant over non-singular mappings, it will be sufficient to prove the theorem for Q_{2k}. Let

$$S = (1\,0\,0\,\ldots\,0).$$

We shall show that S possesses the required properties with respect to Q_{2k}. Obviously S is not a point of Q_{2k}. Let R be any other point not in Q_{2k}. It is

easily seen that S and R are mutually conjugate. Then by Lemma 4.1 the line SR intersects the quadric in a single point. Let R' be a point of the quadric. It is easy to see that S and R' are mutually conjugate. If possible, suppose the line SR' intersects the quadric in another point R'' of Q_{2k}. Since S and R' are mutually conjugate, the point S occurs in $T(R')$, the polar space at R'. Also $T(R')$ contains R'. So the line SR' is contained in $T(R')$. Hence R'' occurs in $T(R')$ and R' and R'' are mutually conjugate. R' and R'' are points of the quadric and are mutually conjugate. So by Lemma 2.3 the line $R''R'$ is contained in Q_{2k}. So S is a point of Q_{2k} which is a contradiction.
For the case $k = 1$, Theorem 4.1 was obtained by Qvist (**6**) and Bose (**3**, pp. 158).

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