A NOTE ON THE PROBLEM OF k SAMPLES

A FAMILIAR problem in analytic statistics is to test the hypothesis whether k samples of sizes $n_1, n_2, \cdots, n_k$ have been drawn at random from the same unknown Normal Universe. In general, the k samples could have come from k different Normal Universes with means $\mu_1, \mu_2, \cdots, \mu_k$ and standard deviations $\sigma_1, \sigma_2, \cdots, \sigma_k$. The most common hypothesis tested is whether, given that $\sigma_1 = \sigma_2 = \cdots = \sigma_k = \sigma$, we can infer that $\mu_1 = \mu_2 = \cdots = \mu_k = \mu$, where σ and μ are unknown. The 'statistic' which has been considered appropriate for this problem is Fisher's Ratio of Variances, say V_b/V_w where

$$V_b = \sum_1^k n_i (\bar{x}_i - \bar{x})^2 / (k - 1);$$

$$V_w = \sum_{i:1}^{k} \sum_{j:1}^{n_i} (x_{ij} - \bar{x}_i)^2 / (N - k);$$

$$N = \sum_1^k (n_i) \text{ and } \bar{x} = \sum_1^k (n_i \bar{x}_i) / N.$$

When there are only two samples ($k = 2$) we have

$$V_b / V_w = (\bar{x}_1 - \bar{x}_2)^2 / V_w \left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

which is distributed like "Student's" t^2 with $n_1 + n_2 - 2$ degrees of freedom. Fisher has also shown that when $k > 2$, we may compare the differences between any two sample means, by means of the same t-test, calculating V_w from within all the k samples, with $N-k$ degrees of freedom. Thus

$$t^2_{ij} = (\bar{x}_i - \bar{x}_j)^2 / V_w \left(\frac{1}{n_i} + \frac{1}{n_j}\right).$$

The total number of such tests of pair comparisons is $k(k-1)/2$. Thus the single hypothesis $\mu_1 = \mu_2 = \cdots = \mu_k$ is being broken up into $k(k-1)/2$ separate tests involving the hypothesis $\mu_i = \mu_j$ for the particular pair of ith and jth samples.

The following identity is easy to prove

$$\Sigma n_i (\bar{x}_i - \bar{x})^2 \equiv \Sigma \Sigma n_i n_j (\bar{x}_i - \bar{x}_j)^2 / N.$$

Therefore

$$V_b/V_w = \Sigma\Sigma n_i n_j (\bar{x}_i - \bar{x}_j)^2 / N (k-1) V_w$$
$$= \Sigma\Sigma (n_i + n_j) t_{ij}^2 / \Sigma\Sigma (n_i + n_j).$$

The ratio of variances is therefore only a weighted mean of the $k(k-1)/2$ different values of t^2. This result brings out clearly the connection between the two tests of significance.

It may be recalled that a paper by P. V. Krishna Iyer[1] and also certain notes he[2] and the author[3,4] published in this journal some time ago dealt with the same problem of k samples. Mr. Krishna Iyer took the unweighted mean of all the values of t^2 and finding it different from the ratio of variances felt that the former would be the proper criterion for discriminating between the sample means. The form he obtained for the distribution of the unweighted mean of all the values of t^2 was found to be erroneous. In fact the true distribution does not come out in a suitable form for exact tests of significance. The relationship between the ratio of variances and the weighted mean of the different values of t^2, established above would indicate that there is no point in trying to develop a test based on the unweighted mean.

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1. P. V. Krishna Iyer, *Proc. Ind. Acad. Sci.*, 1937, 5, 528. 2. —, *Curr. Sci.*, 1938, 6, 392. 3. K. R. Nair, *Ibid.*, 1937, 6, 290. 4 —, *Ibid.*, 1938, 7, 21.