J. Austral. Math. Soc. (*Series A*) 34 (1983), 7–15

PROJECTIVES IN SOME CATEGORIES OF HAUSDORFF SPACES

ASIT BARAN RAHA

(Received 24 July 1981; revised 22 September 1981)

Communicated by R. H. Street

Abstract

An investigation on the existence of projective objects and projective resolutions has been carried out in some categories more general than the category of compact spaces and continuous maps.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 18 G 05, 18 G 10.

1. Introduction

Gleason [1] has determined the projective spaces in the category of compact Hausdorff spaces and continuous maps and has proved the existence of essentially unique minimal projective resolutions. In this note we shall study the same problems in the category of paracompact spaces and perfect maps, and in other categories containing the category of compact Hausdorff spaces as a full subcategory. All topological spaces are assumed Hausdorff. Prerequisite category theory can be obtained from Mitchell [3].

2. Definitions

A continuous map $f: X \to Y$, where X and Y are arbitrary Hausdorff spaces, is called *perfect* if f is closed and the set $f^{-1}(y)$ is compact for each y in Y.

For a Tychonoff space X, we write βX for the Stone-Čech compactification and $\eta_X: X \to \beta X$ for the reflector map.

The following results of Henriksen and Isbell [2] will be used:

(a) *a continuous map $f\colon X \to Y$ between Tychonoff spaces is perfect if and only if its extension $F\colon \beta X \to \beta Y$ takes $\beta X - \eta_X(X)$ into $\beta Y - \eta_Y(Y)$;*

(b) *if $f\colon X \to Y$ is a perfect onto map between Tychonoff spaces, X is paracompact if and only if Y is paracompact.*

3. The categories P and T

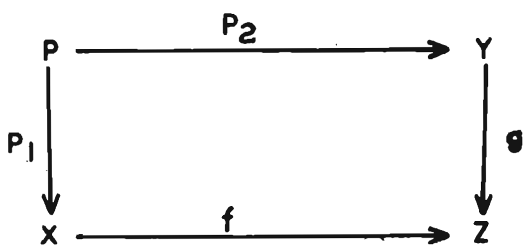
Let **P** be the category of all paracompact spaces and perfect maps and **T** be the category of all Tychonoff spaces and perfect maps. It is to be noted that both of these categories contain **C**, the category of compact spaces and continuous maps, as a full subcategory. **P** is also a full subcategory of **T**. Let us first examine whether these categories have pullbacks. Let us remember that in the above categories epimorphism, monomorphism and isomorphism stand for onto, one-one and homeomorphic maps respectively.

THEOREM 1. *The category* **P** *has pullbacks.*

PROOF. Let $f\colon X \to Z$ and $g\colon Y \to Z$ be two morphisms in the category **P** (that is, X, Y, Z be paracompact spaces and f, g are perfect maps). We have to show the existence of a pullback diagram for f and g. Let $P = \{(x, y) \in X \times Y\colon f(x) = g(y)\}$ and p_1 and p_2 be the projection on X and Y respectively. Suppose there exist $p_1'\colon P' \to X$ and $p_2'\colon P' \to Y$ such that $fp_1' = gp_2'$. Define $h\colon P' \to X \times Y$ as follows:

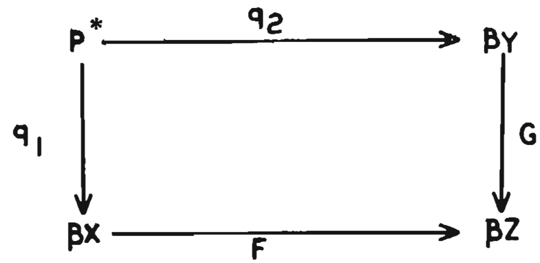
$$h(t) = (p_1'(t), p_2'(t)), \qquad t \in P'.$$

Since $fp_1' = gp_2'$, $h(t) \in P$ that is, $h\colon P' \to P$ such that $p_1 h = p_1'$ and $p_2 h = p_2'$. It is easy to see that the map h is unique. Thus the diagram



is a pullback for f and g. We show that this diagram belongs to **P**, that is, that the maps p_1 and p_2 are perfect.

Consider the pullback diagram



for the maps F: $\beta X \to \beta Z$ and G: $\beta Y \to \beta Z$ where F and G are the extensions of the map f and g onto βX and βY respectively (βX, βY and βZ are the Stone-Čech compactifications and η_X, η_Y and η_Z are the reflector maps of X, Y and Z respectively). We have $F\eta_X = \eta_Z f$, $G\eta_Y = \eta_Z g$ and $P^* = \{(x^*, y^*) \in \beta X \times \beta Y\text{: } F(x^*) = G(y^*)\}$. q_1 and q_2 are projections of P^* to βX and βY respectively. Again, let p_1^*: $\beta P \to \beta X$, p_2^*: $\beta P \to \beta Y$ be the extensions of p_1 and p_2 onto βP. Hence $\eta_X p_1 = p_1^* \eta_P$, $\eta_Y p_2 = p_2^* \eta_P$. Since $fp_1 = gp_2$, $\eta_Z fp_1 = \eta_Z gp_2$. Note that $Fp_1^*\eta_P = F\eta_X p_1 = \eta_Z fp_1$ and $Gp_2^*\eta_P = G\eta_Y p_2 = \eta_Z gp_2$. Therefore, $Fp_1^*\eta_P = Gp_2^*\eta_P$. Since $\eta_P(P)$ is dense in βP we have $Fp_1^* = Gp_2^*$ on βP. From the definition of pullback there exists a (unique) mapping h: $\beta P \to P^*$ such that $p_1^* = q_1 h$ and $p_2^* = q_2 h$. Again, for the maps $\eta_X p_1$: $P \to \beta X$ and $\eta_Y p_2$: $P \to \beta Y$ we have $F\eta_X p_1 = G\eta_Y p_2$ (this equality is already noted earlier).

From the definition of pullback once again we get a map k: $P \to P^*$ such that

$$\eta_X p_1 = q_1 k \quad \text{and} \quad \eta_Y p_2 = q_2 k.$$

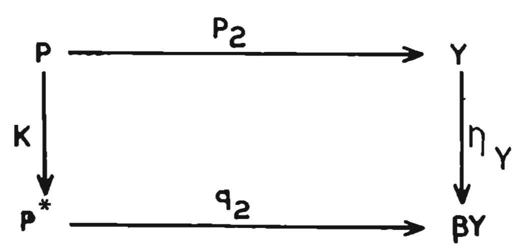
It is easy to see that the map k is as follows:

$$k(x, y) = (\eta_X p_1(x, y), \eta_Y p_2(x, y)) = (\eta_X(x), \eta_Y(y)), \qquad (x, y) \in P.$$

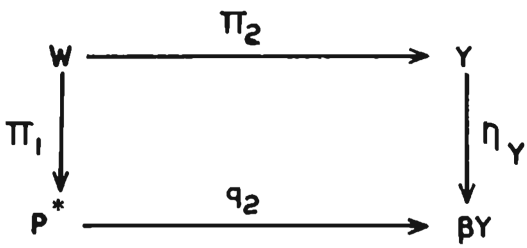
k clearly turns out to be a homeomorphism into P^*. Moreover it is not difficult to notice that $k = h\eta_P$. Now k is a homeomorphism of P onto $k(P) \subset P^*$. From the property of Stone-Čech compactification it follows that

(1) $$h(\beta P - \eta_P(P)) \subset \overline{k(P)} - k(P) \subset P^*.$$

Now $q_2 k = \eta_Y p_2$, that is,



is a commutative diagram. So we consider the pullback diagram for q_2: $P^* \to \beta Y$ and ηY: $Y \to \beta Y$ say



where W is given by $\{(s, y) \in P^* \times Y$: $q_2(s) = \eta_Y(y)\}$ and π_1 and π_2 are the respective projections to P^* and Y.

Since $q_2(s) = q_2(x^*, y^*) = y^*$, $q_2(s) = \eta_Y(y)$ implies $y^* = \eta_Y(y)$. Consequently, $W = \{((x^*, \eta_Y(y)), y) \in P^* \times Y$: $\eta_Y(y) = y^*\} = \{((x^*, \eta_Y(y)), y) \in (\beta X \times \beta Y) \times Y$: $F(x^*) = G(\eta_Y(y))\}$.

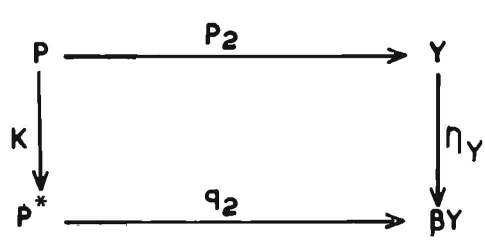
If $F(x^*) = G(\eta_Y(y))$ then $F(x^*) = G(\eta_Y(y)) = \eta_Z g(y)$. Since f is a perfect map, $F(\beta X - \eta_X(X)) \subset \beta Z - \eta_Z(Z)$. As a consequence, $x^* \in \eta_X(X)$, that is, $x^* = \eta_X(x)$ for some $x \in X$. So we have $W = \{((\eta_X(x), \eta_Y(y)), y) \in (\beta X \times \beta Y) \times Y$: $F(\eta_X(x)) = G(\eta_Y(y))\}$. Again $\eta_Z g(y) = G(\eta_Y(y)) = F(\eta_X(x)) = \eta_Z f(x)$ and this naturally implies $f(x) = g(y)$. We then get,

$$
\begin{aligned}
W &= \{((\eta_X(x), \eta_Y(y)), y) \in (\beta X \times \beta Y) \times Y: f(x) = g(y)\} \\
&= \{(k(x, y), y): (x, y) \in P \text{ and } p_2(x, y) = y\}
\end{aligned} \tag{2}
$$

Since $\eta_Y p_2 = q_2 k$ there exists a unique map j: $P \to W$ as follows:

$$j(x, y) = (k(x, y), p_2(x, y)), \qquad (x, y) \in P.$$

Easy to see from (2) that $j(P) = W$. In fact j is a homeomorphism of P and W. Now W is, by construction, a closed subset of $P^* \times Y$ which is paracompact (as P^* is compact and Y is paracompact). As a result W is paracompact. This makes P paracompact and J is an *isomorphism* of P and W in the category **P**. We then obtain that the diagram



is a pullback diagram. Note that η_Y is a one-one map, that is, η_Y is a monomorphism. From the definition of inverse image we see that $P = q_2^{-1}(Y)$ as a subobject of P^*. In terms of sets this means that $k(P) = q_2^{-1}(\eta_Y(Y))$. As a result $q_2(P^* - k(P)) \subset \beta Y - \eta_Y(Y)$. We know from (1) that $h(\beta P - \eta_P(P)) \subset \overline{k(P)} - k(P) \subset P^* - k(P)$, so that $p_2^*(\beta P - \eta_P(P)) = q_2 h(\beta P - \eta_P(P)) = q_2[h(\beta P - \eta_P(P))] \subset q_2(P^* - k(P)) \subset \beta Y - \eta_Y(Y)$. Hence, by the characterisation of Henriksen and Isbell mentioned at the beginning, p_2 is a perfect map. Similarly, p_1 is a perfect map.

The proof of Theorem 1 also yields the following theorem.

THEOREM 2. *The category* **T** *has pullbacks.*

Let us observe that each of the two categories **P** and **T** satisfies the following three conditions:

(a) all admissible maps are continuous;

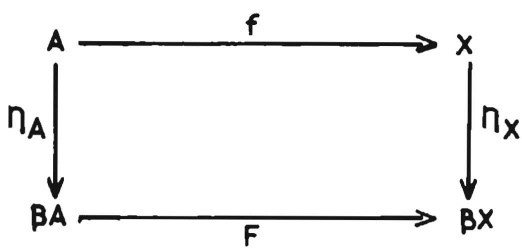
(b) if A is an admissible space and $\{p, q\}$ is a two-element space, then $AX\{p, q\}$ and the projection map of this space onto A are admissible;

(c) if A is an admissible space and B is a closed subspace of A, then B and the inclusion map of B into A are admissible. Now from Theorem 1.2 of Gleason [1] it follows:

THEOREM 3. *A projective space in either of the categories* **P** *or* **T** *is extremally disconnected.*

THEOREM 4. *Let X be any extremally disconnected object from the category* **P**. *Any perfect mapping f: $A \to X$ of another object A onto X is a retraction.*

PROOF. We have f: $\to X$ onto. Then we can draw the following diagram



where F is the unique continuous extension of f onto βA taking values in βX. Since f is a surjection, F is also onto. But βX is extremally disconnected and F is an onto map. Since βX is *projective* in the category **C** (see Gleason [1]), F is a retraction, that is, there exists a mapping $g: \beta X \to \beta A$ such that $Fg = 1_{\beta X}$ = the identity map on βX. Since f is a perfect map, $F(\beta A - \eta_A(A)) = \beta X - \eta_X(X)$. Therefore, $g(\eta_X(X)) \subset \eta_A(A)$. Put $h = \eta_A^{-1} g \eta_X$: $X \to A$. Now $fh(x) = f\eta_A^{-1} g \eta X(x)$. But $F(g\eta_X(x)) = \eta_X(x)$ and $g(\eta_X(x)) \in \eta_A(A)$, that is, $g(\eta_X(x)) = \eta_A(a)$ for some $a \in A$. Therefore, $\eta_X(x) = F(\eta_A(a)) = \eta_X f(a)$. So, $a = \eta_A^{-1}(\eta_A(a)) = \eta_A^{-1} g \eta_X(x)$ and $x = f(a)$ and hence, $f(a) = f\eta_A^{-1} g\eta_X(x) = x$. Consequently $fh(x) = x$ for each $x \in X$, that is, $fh = 1_X$. Naturally f is a retraction.

REMARKS. Let us first observe that Theorem 1 is true in the category **T** as well and the proof is identical. In fact Theorem 4 is valid in any category which is a subcategory of **T** and which contains **C** as a full subcategory. The role of perfect maps is clearly brought out through the proof of Theorem 1. This is primarily the reason behind our choice of perfect maps as morphisms in **P** and **T**.

THEOREM 5. *Projective objects of* **P** *are the objects for which perfect maps onto them are retractions.*

PROOF. The result follows from Proposition 14.2 [3, page 70] as **P** has pullbacks.

REMARK. Because of Theorem 2, the assertion of Theorem 5 holds true in the category **T** also.

THEOREM 6. *In the category* **P**, *the projective objects are precisely the extremally disconnected paracompact spaces.*

PROOF. Follows from Theorems 3, 4 and 5.

THEOREM 7 (Flachsmeyer [6]). *In the category* **T**, *the projective objects are precisely the extremally disconnected ones.*

PROOF. Follows from Theorem 3 and remarks following Theorems 4 and 5.

THEOREM 8. *The category* **P** *has projectives, that is any paracompact space is the perfect image of a projective object. In fact, for every object X there is a projective object P and an onto perfect mapping p_1: $P \to X$ such that p_1 maps no proper closed subspace of P onto X. For any other such object P' and p_1': $P' \to X$ there is an isomorphism e: $P \to P'$ such that $p_1 = p_1' e$.*

Proof. Let X be any object of **P**. Look at βX, the Stone-Čech compactification of X. There exists an extremally disconnected compact space Y and a continuous onto map f: $Y \to \beta X$ such that $f(S) \neq \beta X$ for any proper closed subspace S of Y. (see Theorem 3.2 of [1]). Consider the pull-back diagram

$$\begin{array}{ccc} P & \xrightarrow{\;p_2\;} & Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{\;\eta_X\;} & \beta X \end{array}$$

for the morphisms η_X: $X \to \beta X$ and f: $Y \to \beta X$, where $P = \{(x, y) \in X \times Y: \eta_X(x) = f(y)\}$ and p_1 and p_2 are projections to X and Y respectively. We do not claim that this is a pullback in **P**. Clearly, $\eta_X p_1 = fp_2$. Since η_X is a monomorphism, p_2 is a monomorphism. Since f is onto, p_1 is onto. Again, P is a closed subset of $X \times Y$ and the latter is paracompact. P is, hence, paracompact. p_1 is also closed so that p_1 becomes a perfect map. $fp_2 = \eta_X p_1 \Rightarrow fp_2(P) = \eta_X(X)$. Let $W = p_2(P)$. Since f is a closed map, $f(\overline{p_2(P)}) = f(\overline{W}) = \beta X$. Observe that $\overline{W}$ is a closed subset of Y and $f(\overline{W}) = \beta X$. From the choice of Y it follows that $\overline{W} = Y$, that is, $W = p_2(P)$ is dense in Y. Y is extremally disconnected rendering W extremally disconnected. Now it is not very difficult to see that p_2 is a perfect map onto W. Since P is paracompact and p_2 is a perfect map onto W, W is a paracompact. By Theorem 4, p_2 is a retraction. Since p_2 is a monomorphism and a retraction also, it is an isomorphism, that is, p_2 is a homeomorphism of P and W. Thus P is an extremely disconnected paracompact space. So P is projective due to Theorem 6. Since p_1 is a *perfect map* of P onto X, X is a *perfect image of a projection object*. Let Q be a proper closed subset of P. Then $p_2(Q)$ is a proper closed subset of $p_2(P) = W$. Write $p_2(Q) = W \cap F$ where F is a closed subset of Y. Since $p_2(Q)$ is a proper closed subset of W, F is a proper closed subset of Y. If $p_1(Q) = X$ then

$$\eta_X(X) = \eta_X p_1(Q) = fp_2(Q) = f(W \cap F) \subset f(F).$$

Since f is a closed map of X onto βX, $f(F)$ is closed and hence equals βX. This is a contradiction. Consequently P enjoys the property that no proper closed subspace of P is mapped onto X by p_1.

If possible let P' be a projective paracompact space with a perfect map p_1': $P' \to X$ such that $p_1'(P') = X$ and if Q is any proper closed subspace of P' then $p_1'(Q) \neq X$. Then there exist a morphism e: $P \to P'$ and a morphism e': $P' \to P$ such that $p_1 = p_1' e$ and $p_1' = p_1 e'$. Then $p_1(P) = X = p_1'(P') \Rightarrow p_1' e(P) = X = p_1 e'(P')$. Naturally, e and e' are onto; we shall show that $e'e = 1_P$, that is, e is a coretraction. If $e'e \neq 1_P$, by Lemma 1 of Rainwater [5], there exists a proper closed subset S of P such that

$$d^{-1}(S) \cup S = P \quad \text{where } d = e'e.$$

Obviously, $d(d^{-1}(S)) \subset S$ whence $p_1 d(d^{-1}(S)) \subset p_1(S)$. But $p_1 d = p_1 e'e = p_1' e = p_1$, hence $p_1(S) \supset p_1 d(d^{-1}(S)) = p_1(d^{-1}(S))$; so that $p_1(S) = p_1(P) = X$, a contradiction as S is a proper closed subset of P. We thus conclude that e is a coretraction. Already e is a retraction; hence e is an isomorphism, that is, e is a homeomorphism of P onto P'.

Remarks. In the proof of Theorem 8 the use of Theorem 6 in order to demonstrate that P is projective can be avoided as follows. Let Q and R be objects of **P** such that there exist an onto morphism $g: Q \to R$ and a morphism $h: P \to R$. Let $G: \beta Q \to \beta R$, $H: \beta P \to \beta R$ be the respective Stone-Čech extensions. As P is extremally disconnected βP is extremally disconnected. βP is hence a projective object in **C** (see Gleason [1]) and G is onto. Naturally there exists a cont. map $K: \beta P \to \beta Q$ such that $H = GK$. Since g and h are perfect maps. $G(\beta Q - \eta_Q(Q)) = \beta R - \eta_R(R)$ and $H(\beta P - \eta_P(P)) \subset \beta R - \eta_R(R)$. Hence $K(\beta P - \eta_P(P)) \subset \beta Q - \eta_Q(Q)$ and $K(\eta_P(P)) \subset \eta_Q(Q)$. Naturally we can define a cont. map $k: P \to Q$ such that $K \circ \eta_P = \eta_Q \circ k$. This map k is a perfect map such that $h = gk$. As a result P is projective.

Theorem 8 provides a specific projective resolution for each object of **P**.

Incidentally, an *alternative proof of Theorem* 6 can now be obtained through Theorems 3, 4, 8 and Proposition 14.2 of [3, page 70]. Theorem 8 is valid also for the category **T**.

4. Some other categories

According to the terminology of Henriksen and Isbell, we say that a class **K** of Tychonoff spaces is *perfect* if for every two Tychonoff spaces X and Y for which there exists a perfect mapping $f: X \to Y$ onto Y, the conditions $X \in \mathbf{K}$ and $Y \in \mathbf{K}$ are equivalent.

It is known that the classes of compact spaces, locally compact spaces, regular Lindelöf spaces, countably compact Tychonoff spaces, countably paracompact Tychonoff spaces and spaces complete in the sense of Čech are perfect (see Engelking [4]).

We can also consider the following full subcategories of **T** in addition to **C**, **P** and **T**:

(i) the category of locally compact spaces and perfect maps;

(ii) the category of regular Lindelöf spaces and perfect maps;

(iii) the category of countably compact Tychonoff spaces and perfect maps;

(iv) the category of countably paracompact Tychnoff spaces and perfect maps;

(v) the category of spaces complete in the sense of Čech.

In each of the above categories the statements corresponding to Theorems 1, 4, 5, 7 and 8 are true. The proofs are omitted. Gleason [1] obtained these results for the category of locally compact spaces and perfect maps.

References

[1] A. M. Gleason, 'Projective topological spaces', *Illinois J. Math.* **2** (1958), 482–489.

[2] M. Henriksen and J. R. Isbell, 'Some properties of compactifications', *Duke Math. J.* **25** (1958), 88–106.

[3] Barry Mitchell, *Theory of categories* (Academic Press, New York, 1965).

[4] R. Engelking, *Outline of general topology* (North-Holland, Amsterdam and Polish Scientific Publishers, Warsaw, 1968).

[5] J. Rainwater, 'A note on projective resolutions', *Proc. Amer. Math. Soc.* **10** (1959), 734–735.

[6] J. Flachsmeyer, 'Topologische Projektivräume', *Math. Nachr.* **26** (1963), 57–66.

Indian Statistical Institute
203, Barrackpore Trunk Road
Calcutta 700035
India