A Characterization of the Permanent Function by the Binet-Cauchy Theorem*

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ABSTRACT

We prove that the well known Binet-Cauchy theorem for the permanent function characterizes the permanent. The corresponding result for the determinant was obtained by S. Kurepa in 1964.

1. INTRODUCTION

If $A=(a_{ij})$ is an $m\times n$ matrix over a commutative ring and $m\leqslant n$, then the permanent of A, $\operatorname{Per}A$, is the matrix function defined by

$$\operatorname{Per}A=\sum_{\sigma}a_{1\sigma(1)}\cdots a_{m\sigma(m)},$$

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where the summation is over all one-to-one functions $\sigma:\{1,\ldots,m\}\to\{1,\ldots,n\}$ [2, 8]. If $m=n$, i.e. A is a square matrix, then Per$A$ is denoted by per$A$. For the interesting history of this function consult [8]. Like its "signed" counterpart, the determinant function, the permanent function satisfies an identity which expresses the permanent of a product of two matrices, one $m\times n$, and the other $n\times m$, as the sum of products of permanents of $m\times m$ submatrices of the given matrices.

For the determinant function this identity is given by the following classical theorem.

THEOREM 1 (Binet-Cauchy for determinants). *If A is an $m\times n$ matrix and B is an $n\times m$ matrix, then for $m\leqslant n$*

$$\det AB=\sum_k \det[a_{k_1},\ldots,a_{k_m}]\det\langle b_{(k_1)},\ldots,b_{(k_m)}\rangle, \tag{1}$$

where the sum is over all strictly increasing m-tuples of integers $k=(k_1,\ldots,k_m)$ satisfying $1\leqslant k_1<\cdots<k_m\leqslant n$, a_j is the jth column of A, $b_{(i)}$ is the ith row of B, $[a_{k_1},\ldots,a_{k_m}]$ is an $m\times m$ matrix with the indicated columns, and $\langle b_{(k_1)},\ldots,b_{(k_m)}\rangle$ is an $m\times m$ matrix with the indicated rows [7, 6].

For the permanent function there is a similar expression involving a weight function $1/\mu(k)$, where

$$\mu(k)=\prod_{t=1}^{n}\nu_t(k)!.$$

$\nu_t(k)$ is the number of occurrences of t in $k=(k_1,\ldots,k_m)$ for $t=1,\ldots,n$, and $\sum_{t=1}^{n}\nu_t(k)=m$. The following theorem holds.

THEOREM 2 (Binet-Cauchy for permanents). *If A is an $m\times n$ matrix and B is an $n\times m$ matrix with $m\leqslant n$, then*

$$\operatorname{per}AB=\sum_k\frac{1}{\mu(k)}\operatorname{per}[a_{k_1},\ldots,a_{k_m}]\operatorname{per}\langle b_{(k_1)},\ldots,b_{(k_m)}\rangle \tag{2}$$

where the sum is over all weakly increasing m-tuples of integers satisfying $1\leqslant k_1\leqslant\cdots\leqslant k_m\leqslant n$.

REMARK 1. Note that

$$\frac{1}{\mu(k)} = \frac{1}{m!}\binom{m}{\nu},$$

where $\binom{m}{\nu}$ is the multinomial coefficient $m!/\nu_1!\cdots\nu_n!$ so the formula (2) can also be expressed by

$$\operatorname{per} AB = \frac{1}{m!}\sum_k \binom{m}{\nu(k)} \operatorname{per}\left[a_{k_1},\dots,a_{k_m}\right] \operatorname{per}\langle b_{(k_1)},\dots,b_{(k_m)}\rangle. \quad (3)$$

REMARK 2. It should also be noted that, since trivially $\det[a_{k_1},\dots,a_{k_m}] = 0$ for any m-tuple k with repetitions, (2) reduces to (1) if $\operatorname{per} A$ is replaced by $\det A$. Thus (2) [or (3)] is a "natural" extension of (1).

In the determinant case, if $m = n$ then the Binet-Cauchy formula (1) reduces to

$$\det AB = \det A \det B.$$

If $\mathbf{K}$ is a field of characteristic 0 and if $f: M_n(\mathbf{K}) \to \mathbf{K}$ satisfies

$$f(AB) = f(A)f(B), \quad (4)$$

it is well known that $f(A) = \lambda(\det A)$ where $\lambda:\mathbf{K}\to\mathbf{K}$ is an arbitrary multiplicative function on $\mathbf{K}$, i.e. $\lambda(xy) = \lambda(x)\lambda(y)$ for all $x, y \in \mathbf{K}$ [3, 1, 4, 5]. Easy examples show that if $m = n$ then in general $\operatorname{per} AB \neq \operatorname{per} A \operatorname{per} B$ (also see [2]).

S. Kurepa [6] showed that if $f: M_m(\mathbf{K}) \to \mathbf{K}$ satisfies the Binet-Cauchy theorem (1) for determinants with $m \leqslant n \leqslant m+1$, then $f(A) = \phi(\det A)$ where $\phi:\mathbf{K}\to\mathbf{K}$ is either identically zero or an isomorphism.

It is the intent of this paper to present a similar characterization of permanents. We will show that if a nonconstant $f: M_n(\mathbf{K}) \to \mathbf{K}$ satisfies (2) the Binet-Cauchy theorem for permanents with $m = n$ and if $f(E) \neq 0$, where $E = (1/n)$ is the doubly stochastic matrix with all entries $1/n$, then $f(A) = \phi(\operatorname{per} A)$, where $\phi:\mathbf{K}\to\mathbf{K}$ is an isomorphism of $\mathbf{K}$.

One of our major tools is a special inversion theorem of multinominal type.

2. NOTATION

Throughout the paper K will denote a field of characteristic 0, and $M_n(\mathrm{K})$ will denote the set of all $n \times n$ matrices over K.

Let $Z_+ = \{0,1,2,\ldots\}$ be the set of nonnegative integers. If $\alpha = (\alpha_1,\ldots,\alpha_k) \in Z_+^k$, let $|\alpha| = \alpha_1 + \cdots + \alpha_k$. If α, $\beta \in Z_+^k$, let $\beta^\alpha = \beta_1^{\alpha_1} \cdots \beta_k^{\alpha_k}$. If $s = (s_1,\ldots,s_n) \in Z_+^n$ and $|s| = n$, let $s! = s_1! \cdots s_n!$. Then $\binom{n}{s}$ is the multinomial coefficient $n!/s_1! \cdots s_n! = n!/s!$. If $\beta = (\beta_1,\ldots,\beta_k) \in Z_+^k$ with $|\beta| = n$, then partition s into $s = (s(1),\ldots,s(k))$, where inside of each $s(i)$, $i = 1,2,\ldots,k$, the subscripts j of the s_j run from $\beta_1 + \cdots + \beta_{i-1} + 1$ to $\beta_1 + \cdots + \beta_i$. Then $|s(i)| = \alpha_i$ for each i, $\alpha = (\alpha_1,\ldots,\alpha_k) \in Z_+^k$, and $|s| = |\alpha| = n$.

In order to simplify our notation we will adopt a formal "product" notation for repeated adjacent identical terms inside n-tuples. Thus

$$\Big(\underbrace{x_1,\ldots,x_1}_{s_1},\ldots,\underbrace{x_n,\ldots,x_n}_{s_n}\Big)$$

will be denoted by $(x_1^{s_1},\ldots,x_n^{s_n})$ or (x^s), where s_i is the number of times that x_i appears together inside the n-tuple. If $s_i = 0$ then x_i does not appear. If $m \leqslant n$ and $k = (k_1,\ldots,k_m)$ is a weakly increasing m-tuple of integers with $1 \leqslant k_1 \leqslant \cdots \leqslant k_m \leqslant n$, then $\nu_t = \nu_t(k)$ for $t = 1,2,\ldots,n$ is the number of occurrences of t in k. Let $\nu = (\nu_1,\ldots,\nu_n)$; then $\nu \in Z_+^n$, $|\nu| = m$, since k has m terms, and $\nu! = \nu_1! \cdots \nu_n!$. With our product notation $k = (k_1,\ldots,k_m) = (1^{\nu_1},2^{\nu_2},\ldots,n^{\nu_n})$. If A is an $m \times n$ matrix and B is an $n \times m$ matrix, the jth column of A is denoted by a_j and the ith row of B is denoted by $b(i)$. Let $A^\nu = [a_{k_1},\ldots,a_{k_m}] = [a_1^{\nu_1},\ldots,a_n^{\nu_n}]$ denote the $m \times m$ matrix with the indicated columns, and let $B_\nu = \langle b_{(k_1)},\ldots,b_{(k_m)}\rangle = \langle b_{(1)}^{\nu_1},\ldots,b_{(n)}^{\nu_n}\rangle$ denote the $m \times m$ matrix with the indicated rows. If $m = n$, let $s_t = \nu_t$ for $t = 1,\ldots,n$; then $A^s = [a_1^{s_1},\ldots,a_n^{s_n}]$ and $B_s = \langle b_{(1)}^{s_1},\ldots,b_{(n)}^{s_n}\rangle$ where $|s| = n$. In this notation (1) is given by

$$\det(AB) = \sum_{|\nu| = m} \det A^\nu \det B_\nu, \tag{5}$$

where each ν_t is either 0 or 1, and (3) is given by

$$\operatorname{per}(AB) = \frac{1}{m!} \sum \binom{m}{\nu} \operatorname{per} A^\nu \operatorname{per} B_\nu. \tag{6}$$

Let J denote the $n\times n$ matrix with ones in all its entries then $E=(1/n)J$. It will be shown in section 4 that if $f: M_n(\mathbf{K})\to\mathbf{K}$ is a nonzero function satisfying (6) the Binet-Cauchy Theorem for permanents with $m=n$ and $f(E)\neq 0$ then $f(E)=n!/n^n$. A multiplicative function $m:\mathbf{K}\to\mathbf{K}$ will then be defined by

$$m(x)=\frac{f(xE)}{f(E)}=\frac{n^n}{n!}f(xE).$$

At times it will be necessary to consider column and row vectors with all ones or all zeros as entries. Let $\mathbf{e}$ denote a vector with all ones, and let $\mathbf{0}$ denote a vector with all zeros. It will be clear from context whether a column or row is intended. For example, let $U^{(i)}=[\mathbf{e}^i,\mathbf{0}^{n-i}]$, $i=1,2,\ldots,n-1$, be the matrices with all ones in the first i columns and all zeros in the last $n-i$ columns. For $i=0$ we have $U^{(0)}=0$, and for $i=n$ we have $U^{(n)}=J$. Let A^T denote the transpose of a matrix A. Then the $U^{(i)T}=\langle\mathbf{e}^i,\mathbf{0}^{n-i}\rangle$, $i=0,1,\ldots,n$, are matrices with all ones in the first i rows and all zeros in the last $n-i$ rows.

If $\sigma:\{1,\ldots,n\}\to\{1,\ldots,n\}$ is a permutation and if A is an $n\times n$ square matrix, let $A^\sigma=[a_{\sigma(1)},\ldots,a_{\sigma(n)}]$. Let $r_i=\sum_j a_{ij}$ be the sum of the entries in the ith row of A, and let $c_j=\sum_i a_{ij}$ be the sum of the entries in the jth column of A.

Let $\mathbf{e}_i$, $i=1,2,\ldots,n$, denote the columns or rows of the $n\times n$ identity matrix or the standard basis in $\mathbf{K}^n$.

3. AN INVERSION RELATION

Let $\mathbf{X}$ be a nonempty set, and let $\mathbf{V}$ be a vector space over $\mathbf{K}$. The following inversion theorem will be one of our major tools.

THEOREM 3. *Let $\Phi,\Psi:\mathbf{X}^n\to\mathbf{V}$ be functions satisfying*

$$\Psi(x_1,\ldots,x_n)=\sum_{|s|=n}\binom{n}{s}\Phi(x_1^{s_1},\ldots,x_n^{s_n})\tag{7}$$

for all $(x_1,\ldots,x_n)\in\mathbf{X}^n$. Then

$$\Phi(x_1,\ldots,x_n)=\sum_{|s|=n}c_s\Psi(x_1^{s_1},\ldots,x_n^{s_n})\tag{8}$$

for unique scalars $c_s\in\mathbf{K}$.

Proof. For any $a \in \mathbf{X}$ set $(x_1,\dots,x_n)=(a,\dots,a)=(a^n)$ in (7) to obtain

$$\Psi(a^n)=\left[\sum_{|s|=n}\binom{n}{s}\right]\Phi(a^n)=n^n\Phi(a^n),$$

which determines $\Phi(a^n)$ as $\Psi(a^n)/n^n$. For our induction assume that $\Phi(a_1^{\alpha_1},\dots,a_r^{\alpha_r})$ is always uniquely expressible as a linear combination of the $\Psi(a_1^{\alpha_1},\dots,a_r^{\alpha_r})$ for $|\alpha|=n$ and $1\leqslant r\leqslant k-1$. Then for distinct $b_1,\dots,b_k\in\mathbf{X}$ consider

$$\Psi\left(b_1^{\beta_1},\dots,b_k^{\beta_k}\right)=\sum_{|s|=n}\binom{n}{s}\Phi\left(b_1^{|s(1)|},\dots,b_k^{|s(k)|}\right), \tag{9}$$

where $s(1)=(s_1,\dots,s_{\beta_1}),\dots,s(k)=(s_{n-\beta_k+1},\dots,s_n)$, all the $\beta_i>0$, and $|s(i)|=\alpha_i$, $i=1,2,\dots,k$. Rewriting (9), we obtain

$$\begin{aligned}\Psi(b^\beta)&=\Psi\left(b_1^{\beta_1},\dots,b_k^{\beta_k}\right)\\&=\sum_{|\alpha|=n}\frac{n!}{\alpha_1!\cdots\alpha_k!}\sum_{|s(i)|=\alpha_i}\frac{\alpha_1!}{s(1)!}\cdots\frac{\alpha_1!}{s(k)!}\Phi\left(b_1^{\alpha_1},\dots,b_k^{\alpha_k}\right)\\&=\sum_{|\alpha|=n}\binom{n}{\alpha}\beta_1^{\alpha_1}\cdots\beta_k^{\alpha_k}\Phi\left(b_1^{\alpha_1},\dots,b_k^{\alpha_k}\right)=\sum_{|\alpha|=n}\binom{n}{\alpha}\beta^\alpha\Phi(b^\alpha).\end{aligned} \tag{10}$$

The induction hypothesis applies to each α in the sum having some term $\alpha_i=0$. If α and β have strictly positive entries and $|\alpha|=|\beta|=n$, then according to Theorem 1 of [9] the matrix

$$\left(\binom{n}{\alpha}\beta^\alpha\right)$$

is invertible. Rewriting (10) we obtain

$$\Psi(b^\beta)-\sum_{|\alpha|=n}{}'\binom{n}{\alpha}\beta^\alpha\Phi(b^\alpha)=\sum_{|\alpha|=n}{}''\binom{n}{\alpha}\beta^\alpha\Phi(b^\alpha),$$

where the left-hand sum is over α's with $\alpha_i = 0$ for some i and the right-hand sum is over α's with strictly positive entries $\alpha_i > 0$. This can be solved uniquely for $\Phi(b^\alpha)$ as a linear combination over K of terms $\Psi(b^\gamma)$ with $|\gamma| = n$. ■

According to the multinomial theorem

$$(x_1 + x_2 + \cdots + x_n)^n = \sum_{|s|=n} \binom{n}{s} x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n} \qquad [2].$$

By taking $\Phi(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n$ and $\Psi(x_1, x_2, \ldots, x_n) = (x_1 + x_2 + \cdots + x_n)^n$ in Theorem 3, it then follows that

$$x_1 x_2 \cdots x_n = \sum_{|s|=n} c_s (s_1 x_1 + \cdots + s_n x_n)^n,$$

where the c_s are the coefficients in (8).

4. MAIN RESULT

We now show that the Binet-Cauchy theorem for permanents of square matrices characterizes the permanent function up to an isomorphism provided the function is nonconstant and nonzero on E.

THEOREM 4. *If $f: M_n(\mathrm{K}) \to \mathrm{K}$ satisfies*

$$f(AB) = \frac{1}{n!} \sum_{|s|=n} \binom{n}{s} f(A^s) f(B_s) \tag{11}$$

with $f(E) \neq 0$, then either $f(A) = n!/n^n$ or else $f(A) = \phi(\operatorname{per} A)$, where ϕ is an isomorphism of K.

The essential parts of the theorem are separated into six lemmas.

Proof. First consider the case where f is a constant function with $f(A) = c$ for all A. Then from (11) it follows that

$$c = \frac{1}{n!}\left[\sum_{|s|=n}\binom{n}{s}\right]c^2 = \left(\frac{n^n}{n!}\right)c^2.$$

Since $c = f(E) \neq 0$, it follows that $c = n^n/n!$.

Next consider the zero matrix and let $B = 0$ then $AB = A0 = 0$ for all A. Then from (11) it follows that

$$f(0) = \frac{1}{n!}\sum_{|s|=n}\binom{n}{s}f(A^s)f(0).$$

If $f(0) \neq 0$ then

$$1 = \frac{1}{n!}\sum\binom{n}{s}f(A^s), \qquad \text{or} \quad n! = \sum\binom{n}{s}f(A^s).$$

But now by Theorem 3 with $\Psi(a_1, \ldots, a_n) = n!$ and $\Phi(a_1, \ldots, a_n) = f(A)$ it follows that f is a constant function with $f(A) = f(0) = n!/n^n$. Thus if f isn't a constant function, then $f(0)$ must equal 0. It remains to solve (11) for a nonconstant f.

At this point it will be convenient to introduce a multiplicative function $m(x)$. Consider the matrix J. Since $J^2 = nJ$, it follows that $E^2 = E$. But then from (11) it follows that $f(E) = (n^n/n!)f(E)^2$, and from $f(E) \neq 0$ it follows that $f(E) = n!/n^n$. Now since $xyE = xEyE$, it follows from (11) that $f(xyE) = (n^n/n!)f(xE)f(yE)$, or

$$\frac{n^n}{n!}f(xyE) = \left(\frac{n^n}{n!}f(xE)\right)\left(\frac{n^n}{n!}f(yE)\right).$$

Let $m(x) = f(xE)n^n/n! = f(xE)/f(E)$. Then $m: \mathbf{K} \to \mathbf{K}$ is multiplicative, i.e. $m(xy) = m(x)m(y)$. Note that we already have $m(0) = f(0)/f(E) = 0$ and $m(1) = f(E)/f(E) = 1$.

The following results can be proved using Theorem 3.

Lemma 1. *Let $A \in M_n(\mathbf{K})$. Then*

(i) $f(xA) = m(x)f(A)$ *for all* $x \in \mathbf{K}$;
(ii) $f(A^\sigma) = f(A)$ *for any permutation* σ *of the columns of* f;
(iii) $f(A^T) = f(A)$, *where* A^T *is the transpose of* A.

Proof. (i): Put $A(xE)=(xA)E$ into (11) for each $x\in\mathbf{K}$ to obtain

$$f(A(xE))=\frac{f(xE)}{n!}\sum_{|s|=n}\binom{n}{s}f(A^s)$$

$$=\frac{f(E)}{n!}\sum_{|s|=n}\binom{n}{s}f((xA)^s)=f((xA)E),$$

or

$$\sum_{|s|=n}\binom{n}{s}m(x)f(A^s)=\sum_{|s'|=n}\binom{n}{s'}f((xA)^{s'}).$$

From Theorem 3 we can conclude that $m(x)f(A)=f(xA)$.

(ii): If $r_i=\sum_j a_{ij}=\sum_j a_{i\sigma(j)}$ is the common sum of the entries in the ith rows of A and A^σ, then

$$AJ=\begin{pmatrix} r_1 & \cdots & r_1\\ \vdots & & \vdots\\ r_n & \cdots & r_n\end{pmatrix}=A^\sigma J.$$

Put them into (11) to obtain

$$\frac{f(J)}{n!}\sum_{|s|=n}f(A^s)=\frac{f(J)}{n!}\sum_{|s'|=n}f((A^\sigma)^{s'}).$$

Cancel the $f(J)/n!$ from both sides and use Theorem 3 to conclude that $f(A^\sigma)=f(A)$.

(iii): Note that for any $(c_1,\dots,c_n)\in\mathbf{K}^n$ we have

$$(c_1+\cdots+c_n)J=J\begin{pmatrix} c_1 & \cdots & c_1\\ \vdots & & \vdots\\ c_n & \cdots & c_n\end{pmatrix}=\begin{pmatrix} c_1 & \cdots & c_n\\ \vdots & & \vdots\\ c_1 & \cdots & c_n\end{pmatrix}J.$$

Then from (11) it follows that

$$\frac{f(J)}{n!}\sum_{|s|=n}\binom{n}{s}f\left(\begin{pmatrix} c_1 & \cdots & c_1\\ \vdots & & \vdots\\ c_n & \cdots & c_n\end{pmatrix}_s\right)$$

$$=\frac{f(J)}{n!}\sum_{|s'|=n}\binom{n}{s}f\left(\begin{pmatrix} c_1 & \cdots & c_n\\ \vdots & & \vdots\\ c_1 & \cdots & c_n\end{pmatrix}^{s'}\right).$$

Cancel $f(J)/n!$ from both sides. Then by Theorem 3 we have

$$f\begin{pmatrix} c_1 & \cdots & c_1 \\ \vdots & & \vdots \\ c_n & \cdots & c_n \end{pmatrix} = f\begin{pmatrix} c_1 & \cdots & c_n \\ \vdots & & \vdots \\ c_1 & \cdots & c_n \end{pmatrix}.$$

But now for any $A \in M_n(\mathbb{K})$ it follows that

$$JA = \begin{pmatrix} c_1 & \cdots & c_n \\ \vdots & & \vdots \\ c_1 & \cdots & c_n \end{pmatrix} \quad \text{and} \quad A^T J = \begin{pmatrix} c_1 & \cdots & c_1 \\ \vdots & & \vdots \\ c_n & \cdots & c_n \end{pmatrix}$$

where c_j is the sum of the entries in the jth column of A. Thus we have $f(JA) = f(A^T J)$, so

$$\frac{f(J)}{n!} \sum_{|s|=n} \binom{n}{s} f(A_s) = \frac{f(J)}{n!} \sum_{|s'|=n} \binom{n}{s'} f((A^t)^{s'}).$$

Cancel $f(J)/n!$ from both sides and observe that both sums are functions of the rows of A, i.e. a column of A^T is a transposed row of A. Then apply Theorem 3 to obtain $f(A^T) = f(A)$. ■

Henceforth we will use the result of this lemma without specifically mentioning its use.

The next lemma involves the matrices $U^{(i)} = [\mathrm{e}^i, 0^{n-i}]$ defined in Section 2.

Lemma 2.

(i) $f(U^{(i)}) = 0$, $i = 1, 2, \ldots, n-1$.
(ii) $m(i) = i^n$, $i = 0, 1, \ldots, n$.
(iii) $f(J) = n!$.

Proof. First note that $U^{(i)} J = iJ$. Then from (11)

$$\frac{f(J)}{n!} \sum_{|s|=n} \binom{n}{s} f(U^{(i)s}) = m(i) f(J)$$

implies that

$$m(i) = \frac{1}{n!} \sum_{|s|=n} \binom{n}{s} f(U^{(i)s})$$

$$= \frac{1}{n!} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} \sum_{\substack{|s(1)|=j \\ |s(2)|=n-j}} \frac{j!}{s(1)!} \frac{(n-j)!}{s(2)!} f(U^{(j)})$$

$$= \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} i^j (n-i)^{n-j} f(U^{(j)})$$

$$= \frac{i^n}{n!} f(J) + \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} i^j (n-1)^{n-j} f(U^{(j)}).$$

At this point, for convenience write

$$m(i) - \frac{i^n}{n!} f(J) = \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} i^j (n-i)^{n-j} f(U^{(j)}). \tag{12}$$

Next we note that $U^{(i)}U^{(i)T} = iJ$. Then from (11)

$$m(i) f(J) = \frac{1}{n!} \sum_{|s|=n} \binom{n}{s} [f(U^{(i)s})]^2$$

$$= \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j} i^j (n-i)^{n-j} [f(U^{(j)})]^2$$

$$= \frac{i^n}{n!} f(J)^2 + \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} i^j (n-i)^{n-j} [f(U^{(j)})]^2.$$

Writing

$$\left(m(i) - \frac{i^n}{n!} f(J)\right) f(J) = \frac{1}{n!} \sum_{j=1}^{n-1} \binom{n}{j} i^j (n-i)^{n-j} [f(U^{(j)})]^2. \tag{13}$$

We obtain from [9]

$$[f(U^{(j)})]^2 = f(U^{(j)})f(J), \qquad j = 1,2,\ldots,n-1. \tag{14}$$

Now we note that $[e^{i_1}, e^{i_2}, 0^{i_3}]\langle e^{i_1}, 0^{i_2}, 0^{i_3}\rangle = i_1 J = U^{(i_1+i_2)}U^{(i_1)T}$. Partitioning the s in (11) into three parts $(s(1), s(2), s(3))$ with $|s(i)| = r_i$, $i = 1,2,3$, we have

$$m(i_1)f(J) = \frac{1}{n!}\sum_{|s(1)|+|s(2)|+|s(3)|=n}\binom{n}{s}f([e^{|s(1)|}, e^{|s(2)|}, 0^{|s(3)|}])$$

$$\times f([e^{|s(1)|}, 0^{|s(2)|}, 0^{|s(3)|}])$$

$$= \frac{1}{n!}\sum_{|r|=n}\binom{n}{r}\sum_{|s(i)|=r_i}\frac{r_1!}{s(1)!}\frac{r_2!}{s(2)!}\frac{r_3!}{s(3)!}f(U^{(r_1+r_2)})f(U^{(r_1)})$$

$$= \frac{1}{n!}\sum_{|r|=n}\binom{n}{r}i_1^{r_1}i_2^{r_2}i_3^{r_3}f(U^{(r_1+r_2)})f(U^{(r_1)})$$

$$= \frac{1}{n!}\sum_{r_1=0}^{n}\sum_{r_2=0}^{n-r_1}\frac{n!}{r_1!r_2!(n-r_1-r_2)!}i_1^{r_1}i_2^{r_2}(n-i_1-i_2)^{n-r_1-r_2}$$

$$\times f(U^{(r_1+r_2)})f(U^{(r_1)}).$$

Therefore,

$$f(J)\left(m(i_1) - \frac{i_1^n}{n!}f(J)\right)$$

$$= \frac{1}{n!}\sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1}\frac{n!}{r_1!r_2!(n-r_1-r_2)!}i_1^{r_1}i_2^{r_2}$$

$$\times (n-i_1-i_2)^{n-r_1-r_2}f(U^{(r_1)})f(U^{(r_1+r_2)})$$

$$+ \frac{1}{n!}\sum_{r_1=1}^{n-1}\frac{n!}{r_1!(n-r_1)!}i_1^{r_1}(n-i_1-i_2)^{n-r_1}[f(U^{(r_1)})]^2. \tag{15}$$

Next we consider $U^{(i_1)}U^{(i_1)T}=i_1J$ again and note that from (13)

$$f(J)\left(m(i_1)-\frac{i_1^n}{n!}f(J)\right)$$

$$=\frac{1}{n!}\sum_{r_1=1}^{n-1}\frac{n!}{r_1!(n-r_1)!}i_1^{r_1}(n-i_1)^{n-r_1}\left[f(U^{(r_1)})\right]^2$$

$$=\frac{1}{n!}\sum_{r_1=0}^{n-1}\sum_{r_2=0}^{n-r_1}\frac{n!}{r_1!(n-r_1)!}i_1^{r_1}\frac{(n-r_1)!}{r_2!(n-r_1-r_2)!}i_2^{r_2}$$

$$\times(n-i_1-i_2)^{n-r_1-r_2}\left[f(U^{(r_1)})\right]^2$$

$$=\frac{1}{n!}\sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1}\frac{n!}{r_1!r_2!(n-r_1-r_2)!}i_1^{r_1}i_2^{r_2}(n-i_1-i_2)^{n-r_1-r_2}\left[f(U^{(r_1)})\right]^2$$

$$+\frac{1}{n!}\sum_{r_1=1}^{n-1}\frac{n!}{r_1!(n-r_1)!}i_1^{r_1}(n-i_1-i_2)^{n-r_1}\left[f(U^{(r_1)})\right]^2. \tag{16}$$

From (15) and (16) we have

$$\sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1}\frac{n!}{r_1!r_2!(n-r_1-r_2)!}i_1^{r_1}i_2^{r_2}(n-i_1-i_2)^{n-r_1-r_2}f(U^{(r_1)})^2$$

$$=\sum_{r_1=1}^{n-1}\sum_{r_2=1}^{n-r_1}\frac{n!}{r_1!r_2!(n-r_1-r_2)!}i_1^{r_1}i_2^{r_2}$$

$$\times(n-i_1-i_2)^{n-r_1-r_2}f(U^{(r_1+r_2)})f(U^{(r_1)}),$$

and again applying [9], it follows that for $i\leqslant i+j\leqslant n$,

$$\left[f(U^{(i)})\right]^2=f(U^{(i)})f(U^{(i+j)}). \tag{17}$$

Now, (17) implies that

$$f(U^{(i)})^2=f(U^{(i)})f(U^{(i+1)})=\cdots=f(U^{(i)})f(U^{(n-1)})=f(U^{(i)})f(J).$$

Therefore, either $f(U^{(i)})=0$, or if $f(U^{(i)})\neq 0$ then $f(U^{(i)})=f(J)$ for $i\leqslant j\leqslant n$. If k is the least integer such that $f(U^{(k)})\neq 0$ [or $f(U^{(k)})=f(J)$], then $f(U^{(j)})=f(J)$ for $k\leqslant j\leqslant n$.

Next we note that $[e^i,0^{n-i}]\langle 0^i,e^{n-i}\rangle = 0$. It follows that

$$0=\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}i^j(n-i)^{n-j}f(U^{(j)})f(U^{(n-j)}),$$

or

$$0=\sum_{j=1}^{n-1}\binom{n}{j}i^j(n-i)^{n-j}f(U^{(j)})f(U^{(n-j)}),\qquad i=1,2,\ldots,n,$$

and once again by Theorem 3 we have

$$f(U^{(j)})f(U^{(n-j)})=0\qquad\text{for}\quad j=1,2,\ldots,n-1. \tag{18}$$

If $n=2k$ by (18) we would have $f(U^{(k)})^2=0$, which would contradict the definition of k. Accordingly, $n-k<k$, or $0<2k-n$. For $n=2$ we have $k=n=2$. For $3\leqslant n$ assume that $k<n$. Setting $i=1$ in (13), we obtain

$$f(J)\left(1-\frac{f(J)}{n!}\right)=\frac{1}{n!}\left(\sum_{j=k}^{n-1}\binom{n}{j}1^j(n-1)^{n-j}\right)f(J)^2. \tag{19}$$

Rewriting (19), we obtain the following for $i=1,2,\ldots,n-2$:

$$\begin{aligned}
f(J)&\left(1-\frac{f(J)}{n!}\right)\\
&=\frac{1}{n!}\left(\sum_{j_1=k}^{n-1}\sum_{j_2=0}^{n-j_1}\frac{n!}{j_1!j_2!(n-j_1-j_2)!}1^{j_1}i^{j_2}(n-1-i)^{n-j_1-j_2}\right)[f(J)]^2\\
&=\frac{f(J)^2}{n!}\sum_{j_1=k}^{n-1}\sum_{j_2=1}^{n-j_1}\frac{n!i^{j_2}(n-1-i)^{n-j_1-j_2}}{j_1!j_2!(n-j_1-j_2)!}\\
&\quad+\frac{f(J)^2}{n!}\sum_{j_1=k}^{n-1}\frac{n!}{j_1!(n-j_1)!}(n-1-i)^{n-j_1}.
\end{aligned}$$

Finally, note that $[e^1,e^i,0^{n-1-i}]\langle e^1,0^i,e^{n-1-i}\rangle = J$. Partitioning the s in

(11) into three parts $(s(1), s(2), s(3))$ with $|s(i)| = r_i$, $i = 1,2,3$, we have

$$f(J) = \frac{1}{n!} \sum_{|s(1)|+|s(2)|+|s(3)|=n} \binom{n}{s} f([e^{|s(1)|}, e^{|s(2)|}, 0^{|s(3)|}])$$

$$\times f([e^{|s(1)|}, 0^{|s(2)|}, e^{|s(3)|}])$$

$$= \frac{1}{n!} \sum_{r_1+r_2+r_3=n} \frac{n!}{r_1! r_2! r_3!} 1^{r_1} i^{r_2} (n-1-i)^{n-r_1-r_2} f(U^{(r_1+r_2)}) f(U^{(n-r_2)})$$

$$= \frac{f(J)^2}{n!} + \frac{1}{n!} \sum_{r_2=0}^{n} \frac{n!}{r_2!(n-r_2)!} i^{r_2} (n-1-i)^{n-r_2} f(U^{(r_2)}) f(U^{(n-r_2)})$$

$$+ \frac{1}{n!} \sum_{r_1=1}^{n-1} \sum_{r_2=1}^{n-r_1} \frac{n!}{r_1! r_2! (n-r_1-r_2)!} i^{r_2} (n-1-i)^{n-r_1-r_2} f(U^{(r_1+r_2)})$$

$$\times f(U^{(n-r_2)})$$

$$+ \frac{1}{n!} \sum_{r_1=1}^{n-1} \frac{n!}{r_1!(n-r_1)!} (n-1-i)^{n-r_1} f(U^{(r_1)}) f(J).$$

Now, $f(J) = f(U^{(j)})$ for $k \leqslant j \leqslant n$ and $f(U^{(j)}) = 0$ for $j < k$ implies that

$$f(J)\left(1 - \frac{f(J)}{n!}\right) = \frac{f(J)^2}{n!} \left(\sum_{r_1=1}^{n-1} \sum_{r_2=1}^{n-r_1} \frac{n! i^{r_2} (n-1-i)^{n-r_1-r_2}}{r_1! r_2! (n-r_1-r_2)!} \right)$$

$$+ \frac{f(J)^2}{n!} \left(\sum_{r_1=k}^{n-1} \frac{n!}{r_1!(n-r_1)!} (n-1-i)^{n-r_1} \right), \quad (20)$$

where $k \leqslant r_1 + r_2$ and $k \leqslant n - r_2$.

Now the equality of (19) and (20) implies that

$$\sum_{j_1=k}^{n-1} \sum_{j_2=1}^{n-j_1} \frac{n! i^{j_2} (n-1-i)^{n-j_1-j_2}}{j_1! j_2! (n-j_1-j_2)!} = \sum_{j_1=1}^{n-1} \sum_{j_2=1}^{n-j_1} \frac{n! i^{j_2} (n-1-i)^{n-j_1-j_2}}{j_1! j_2! (n-j_1-j_2)!}, \quad (21)$$

where $k \leqslant j_1 + j_2 \leqslant n$ and $k \leqslant n - j_2$ on the right-hand side. Equivalently,

$$\sum_{j_1=2k-n}^{k-1} \sum_{j_2=k-j_1}^{n-k} \frac{n!}{j_1! j_2! (n-j_1-j_2)!} i^{j_2} (n-1-i)^{n-j_1-j_2} = 0. \quad (22)$$

But $1 \leqslant n-k < 2k$ implies $0 < 3k-n$, and $3 \leqslant n < 2k$ implies $2 \leqslant k$ or $1 \leqslant k-1$. An isomorphic copy of the integers is contained in $\mathbb{K}$, and the left-hand side is a positive integer which can't be zero in a field of characteristic 0. Thus $k \leqslant n-1$ is impossible. Therefore we must have $k=n$ and we conclude that $f(U^{(i)})=0$ for $i=0,1,\ldots,n-1$. ∎

In the next step we consider matrices with blocks of zero columns on the right.

LEMMA 3. *If f satisfies* (11), *then*

$$f\begin{pmatrix} x_1 & \cdots & x_{n-1} & 0 \\ \vdots & & \vdots & \vdots \\ x_1 & \cdots & x_{n-1} & 0 \end{pmatrix} = 0.$$

Proof. Note that

$$\Bigg(\underbrace{\begin{matrix} x_1 & \cdots & x_i \\ \vdots & & \vdots \\ x_1 & \cdots & x_i \end{matrix}}_{i} \quad \underbrace{\begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix}}_{n-i}\Bigg) \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = (x_1 + \cdots + x_i)J,$$

where the second matrix on the right is $\langle e^i, 0^{n-i}\rangle$, which has i rows of all one entries, $i=1,2,\ldots,n$. Accordingly,

$$m(x_1+\cdots+x_i)f(J) = \frac{1}{n!}\sum_{|s|=n}\binom{n}{s} f\big(\big[(x_1 e)^{s_1},\ldots,(x_i e)^{s_i},0^{n-s_1-\cdots-s_i}\big]\big)$$
$$\times f(U^{(s_1+\cdots+s_i)}). \tag{23}$$

But the only nonzero terms on the right hand side correspond to $s_1+\cdots+s_i=n$ and $U^{(n)}=J$. Therefore,

$$m(x_1+\cdots+x_i) = \frac{1}{n!}\sum_{s_1+\cdots+s_i=n}\frac{n!}{s_1!\cdots s_i!} f\big((x_1 e)^{s_1},\ldots,(x_i e)^{s_i}\big) \tag{24}$$

for $1 \leqslant i \leqslant n$.

Now note that

$$\underbrace{\begin{pmatrix} x_1 & \cdots & x_i \\ \vdots & & \vdots \\ x_1 & \cdots & x_i \end{pmatrix}}_{i}\!\!\underbrace{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{n-i} J=(x_1+\cdots+x_i)J,$$

$i=1,2,\ldots,n$. Then

$$\begin{aligned} m(x_1+\cdots+x_n) &= \frac{1}{n!}\sum_{p=0}^{n}\sum_{\substack{|s(1)|=p\\ |s(2)|=n-p}}\frac{n!}{s(1)!s(2)!}f\left((x_1e)^{s_1},\ldots,(x_ie)^{s_i},0^{n-p}\right)\\ &=\frac{1}{n!}\sum_{p=1}^{n-1}\frac{n!}{p!(n-p)!}\sum_{s_1+\cdots+s_i=p}\frac{p!}{s_1!\cdots s_i!}(n-i)^{n-p}\\ &\quad\times f\left(\left[(x_1e)^{s_1},\ldots,(x_ie)^{s_i},0^{n-p}\right]\right)\\ &\quad+\frac{1}{n!}\sum_{s_1+\cdots+s_i=n}\frac{n!}{s_1!\cdots s_i!}f\left((x_1e)^{s_1},\ldots,(x_ie)^{s_i}\right). \end{aligned} \tag{25}$$

But the second term on the right-hand side equals $m(x_1+\cdots+x_i)$ itself, by (24). Consequently, for $1\leqslant i\leqslant n-1$ we have

$$\begin{aligned} 0&=\frac{1}{n!}\sum_{p=1}^{n-1}\frac{n!}{p!(n-p)!}i^p(n-i)^p\\ &\times\left(\sum_{s_1+\cdots+s_i=p}\frac{p!}{s_1!\cdots s_i!}\frac{1}{i^p}f\left(\left[(x_1e)^{s_1},\ldots,(x_ie)^{s_i},0^{n-p}\right]\right)\right), \end{aligned}$$

and so by [9] it follows that

$$0=\sum_{s_1+\cdots+s_i=p}\frac{p!}{s_1!\cdots s_i!}\frac{1}{i^p}f\left(\left[(x_1e)^{s_1},\ldots,(x_ie)^{s_i},0^{n-p}\right]\right)$$

or by Theorem 3

$$0=\frac{1}{i^p}f\left(\left[x_1e,\ldots,x_ie,0^{n-i}\right]\right).$$ ∎

If A_0 is any matrix in $M_n(\mathbb{K})$ with a zero column, we can prove that $f(A_0)=0$. By Lemma 1 it suffices to show this for a matrix A_0 with a zero last column.

LEMMA 4. *If A_0 is any matrix in $M_n(\mathbb{K})$ with a zero last column, i.e. A_0 is of the form*

$$A_0=\begin{pmatrix} a_{11} & \cdots & a_{1n-1} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn-1} & 0 \end{pmatrix},$$

then $f(A_0)=0$.

Proof. Since

$$JA_0=\begin{pmatrix} c_1 & \cdots & c_{n-1} & 0 \\ \vdots & & \vdots & \vdots \\ c_1 & \cdots & c_{n-1} & 0 \end{pmatrix},$$

where $c_j=\sum_i a_{ij}$, then by Lemma 3 and (11) it follows that

$$\frac{f(J)}{n!}\sum_{|s|=n}\binom{n}{s}f((A_0)_s)=f\left(\begin{pmatrix} c_1 & \cdots & c_{n-1} & 0 \\ \vdots & & \vdots & \vdots \\ c_1 & \cdots & c_{n-1} & 0 \end{pmatrix}\right)=0.$$

Therefore,

$$0=\sum_{|s|=n}\binom{n}{s}f((A_0)_s).$$

Now if $\phi=f((A_0)_s)$ is a function of the n rows of A_0, then by Theorem 3 it follows that $f(A_0)=0$. ∎

The next lemma is concerned with diagonal matrices.

LEMMA 5. *If $D=[x_1e_1,\dots,x_ne_n]$ is any diagonal matrix, then $f(DA)=f(D)f(A)=f(AD)$.*

Proof. Using (11) we have

$$f(DA) = \frac{1}{n!} \sum_{|s|=n} \binom{n}{s} f\left(\left[(x_1 e_1)^{s_1}, \ldots, (x_n e_n)^{s_n}\right]\right) f(A_s).$$

But $f([(x_1e_1)^{s_1}, \ldots, (x_ne_n)^{s_n}]) = 0$ if any $s_i > 1$, since then some other $s_j = 0$, which implies the jth row is zero. Therefore,

$$f(DA) = \frac{1}{n!} \frac{n!}{1! \cdots 1!} f\left(\left[x_1 e_1, \ldots, x_n e_n\right]\right) f(A) = f(D) f(A).$$

Since $(AD)^T = D^T A^T = DA^T$, it also follows that $f(AD) = f(D)f(A)$. ■

As a consequence of Lemma 5 we have

$$f(D) = f\begin{pmatrix} x_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} f\begin{pmatrix} 1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$\times \cdots \times f\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 & \\ 0 & & & x_n \end{pmatrix}$$

$$= f\begin{pmatrix} x_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} f\begin{pmatrix} x_2 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

$$\times \cdots \times f\begin{pmatrix} x_n & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

after appropriate permutations. Let $\phi : \mathbb{K} \to \mathbb{K}$ be defined by

$$\phi(x) = f\begin{pmatrix} x & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}.$$

We have just shown that $\phi(xy) = \phi(x)\phi(y)$ and that $\phi(D) = \phi(x_1 x_2 \cdots x_n) = \phi(\operatorname{per} D)$. Since $xIJ = xJ$, we have $m(x)f(J) = f(xI)f(J) = \phi(x^n)f(J) = [\phi(x)]^n f(J)$ or $m(x) = [\phi(x)]^n = \phi(x^n)$. Therefore, $\phi(1) = m(1) = 1$ and $\phi(x) \neq 0$ for $x \neq 0$. Let $T(x)$ be defined by

$$T(x) = \left(\begin{array}{cc|c} 1 & x & 0 \\ 0 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right);$$

then $T(x)T(y) = T(x+y)$.

LEMMA 6. *For the above* $T(x)$,

(i) $f(T(x))f(T(y)) = f(T(x+y))$;
(ii) $f(T(x)) = 1$.

Proof. (i): Writing $T(x) = [e_1, xe_1 + e_2, \dots, e_n]$ and $T(y) = \langle e_1 + ye_2, e_2, e_3, \dots, e_n \rangle$, we have from (11) that

$$f(T(x+y)) = \frac{1}{n!} \sum_{|s|=n} \binom{n}{s} f\left(\left[e_1^{s_1}, (xe_1 + e_2)^{s_2}, e_3^{s_3}, \dots, e_n^{s_n}\right]\right)$$
$$\times f\left(\left[(e_1 + ye_2)^{s_1}, e_2^{s_2}, \dots, e_n^{s_n}\right]\right), \tag{26}$$

using $f(A^T) = f(A)$. In this sum, if any $s_i = 0$, then one of the two corresponding matrices appearing in (26) will have a zero row and hence the corresponding term will be 0. But since each $s_i \geq 1$ and $s_1 + s_2 + \cdots + s_n = n$, it follows that the only remaining term is

$$f(T(x+y)) = \frac{1}{n!}\frac{n!}{1! \cdots 1!} f(T(x))f(T(y)) = f(T(x))f(T(y)).$$

(ii): Note that

$$\left(\begin{array}{cc|c} 1 & x & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)\left(\begin{array}{cc|c} x & 0 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)\left(\begin{array}{cc|c} x^{-1} & 0 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)=\left(\begin{array}{cc|c} 1 & x & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)$$

for any nonzero x. Then

$$f(T(x))=\phi(x^{-1})f(T(x))\phi(x)$$

$$=f\left(\left(\begin{array}{cc|c} x^{-1} & 0 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)\left(\begin{array}{cc|c} 1 & x & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)\left(\begin{array}{cc|c} x & 0 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)\right)$$

$$=f\left(\begin{array}{cc|c} 1 & 1 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)=f(T(1)).$$

But $T(1)T(1) = T(2)$. Therefore, $f(T(1)) = f(T(2)) = f(T(1))f(T(1)) = f(T(1))f(T(-1)) = f(T(1)T(-1)) = f(T(0)) = f(I) = \phi(1) = 1 = f(T(x))$ for all x. ■

We now show that if f satisfies (11) then $\phi(x+y)=\phi(x)+\phi(y)$. First note that

$$\left(\begin{array}{cc|c} x & y & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)=\left(\begin{array}{cc|c} 1 & y & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)\left(\begin{array}{cc|c} x & 0 & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right).$$

Then apply f on both sides to get

$$f\left(\begin{array}{cc|c} x & y & \\ 0 & 1 & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)=f(T(y))\phi(x)=\phi(x).$$

Next note that

$$\left(\begin{array}{cc|c} x & y & 0 \\ 0 & 1 & \\ \hline 0 & & I_{n-2} \end{array}\right) J = \begin{pmatrix} x+y & \cdots & x+y \\ 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

$$= \begin{pmatrix} x+y & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} J.$$

Again applying f on both sides, we obtain

$$\phi(x+y)f(J) = \frac{f(J)}{n!} \sum_{|s|=n} \binom{n}{s} f\left([(xe_1)^{s_1}, (ye_1+e_2)^{s_2}, e_3^{s_3}, \ldots, e_n^{s_n}]\right),$$

or

$$\phi(x+y) = \frac{1}{n!} \sum_{|s|=n} \binom{n}{s} f\left([(xe_1)^{s_1}, (ye_1+e_2)^{s_2}, e_3^{s_3}, \ldots, e_n^{s_n}]\right)$$

$$= \frac{1}{n!}\left(n! f(xe_1, ye_1+e_2, e_3, \ldots, e_n) + \frac{n!}{2!} f\left((ye_1+e_2)^2, e_3, \ldots, e_n\right)\right).$$

Thus,

$$\phi(x+y) = \phi(x) + \tfrac{1}{2} f\left(\begin{array}{cc|c} y & y & 0 \\ 1 & 1 & \\ \hline 0 & & I_{n-2} \end{array}\right)$$

$$= \phi(x) + \tfrac{1}{2}\phi(y) f\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & \\ \hline 0 & & I_{n-2} \end{array}\right). \qquad (27)$$

Since $\phi(x+y)$ is a symmetric function of x and y, it follows that

$$\phi(x)+\tfrac{1}{2}\phi(y)f\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)=\phi(y)+\tfrac{1}{2}\phi(x)f\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right),$$

and consequently $\phi(x)c=\phi(y)c$, where

$$c=1-\tfrac{1}{2}f\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right).$$

For any $x \neq 0$ set $y=0$. Then $c=0$, since $\phi(x)\neq 0$ for $x \neq 0$. Therefore,

$$f\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right)=2,$$

and from (27) we conclude that $\phi(x+y)=\phi(x)+\phi(y)$. Thus $\phi:\mathbf{K}\to\mathbf{K}$ is an isomorphism.

We are now at the final step in the proof of Theorem 4. Note that if $A\in M_n(\mathbf{K})$ then

$$AJ=\begin{pmatrix} r_1 & \cdots & r_1 \\ \vdots & & \vdots \\ r_n & \cdots & r_n \end{pmatrix}=\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}J$$

Where $r_i=\sum_j a_{ij}$, $i=1,2,\ldots,n$. Then

$$\operatorname{per}(AJ)=(r_1\cdots r_n)n!=\sum_{|s|=n}\binom{n}{s}\operatorname{per}A^s$$

Applying the isomorphism ϕ to both sides, we obtain

$$\phi(r_1\cdots r_n)n!=\sum_{|s|=n}\binom{n}{s}\phi(\operatorname{per}A^s). \tag{28}$$

But we also have

$$\phi(r_1 \cdots r_n)n! = f\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix} f(J) = f(AJ)$$

$$= \sum_{|s'|=n} \binom{n}{s'} f(A^{s'}).$$

It follows from Theorem 3 that $f(A) = \phi(\mathrm{per}(A))$. ∎

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