NOTES

A NOTE ON THE THEORY OF UNBIASSED ESTIMATION

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1. Summary. It is shown that even in very simple situations (like estimating the mean of a normal population) where a uniformly minimum variance unbiassed estimator of the unknown population characteristic is known to exist, no best (even locally) unbiassed estimator exists as soon as we alter slightly the definition of variance.

2. Introduction. Let $(\mathfrak{X}, \mathfrak{A})$ be an arbitrary measurable space (the "sample space") and let $\{P_\theta\}$, $\theta \varepsilon \Omega$, be a family of probability measures on $\mathfrak{A}$. A real-valued function $\mu = \mu_\theta$ of θ is "estimable" if it has an "unbiassed estimator". An unbiassed estimator of μ is a mapping $\eta = \eta_x$ of the "sample space" $\mathfrak{X}$ onto the space of all probability measures over the σ-field of all the Borel sets on the real line such that

(i) $T_x = \int_{-\infty}^{\infty} t \, d\eta_x$ is an $\mathfrak{A}$-measurable function of x,

(ii) $\mu_\theta \equiv \int_{\mathfrak{X}} T_x \, dP_\theta$ for all $\theta \varepsilon \Omega$.

If, for every $x \varepsilon \mathfrak{X}$, the whole probability mass of η_x is concentrated at one point, say T_x, then η_x (or equivalently T_x) is called a nonrandomized estimator. With reference to a given loss or weight function $w(t, \theta)$, which is a Borel-measurable function of the real variable t for every fixed $\theta \varepsilon \Omega$, an unbiassed estimator η_θ of μ_θ is better than an alternative unbiassed estimator η_x' at the point $\theta = \theta_0$ if

$$\int_{\mathfrak{X}} dP_{\theta_0} \int_{-\infty}^{\infty} w(t, \theta_0) \, d\eta_x < \int_{\mathfrak{X}} dP_{\theta_0} \int_{-\infty}^{\infty} w\,(t, \theta_0) \, d\eta_x'.$$

We consider only such estimators η_x for which $\int_{-\infty}^{\infty} w(t, \theta) \, d\eta_x$ is an $\mathfrak{A}$-measurable function of x for all $\theta \varepsilon \Omega$.

Hodges and Lehmann [2] noted that if, for every $\theta \varepsilon \Omega$, the loss function $w(t, \theta)$ is a convex (downwards) function of the variable t, then the class of nonrandomized estimators of μ is essentially complete. Barankin [1] and Stein [4] considered the particular case where $w = |t - \mu_\theta|^s$ for $s \geqq 1$ and proved, under a few regularity assumptions, that there always exists an unbiassed estimator which is locally the best at a given value of $\theta = \theta_0$. Simple examples may be

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given to show that there need not exist a uniformly best unbiassed estimator even in the simplest case of $s = 2$. If, however, there exists a complete sufficient statistic [3] for θ and if w is convex (downwards) for every fixed $\theta \varepsilon \Omega$, then there exists an essentially unique uniformly best unbiassed estimator for every estimable parametric function μ_θ. The convexity of the loss function is essential in the proofs of the above results. We demonstrate in the next section how a slight departure from the convexity of the loss function might destroy all these results.

3. Nonexistence of a best unbiassed estimator. Let us assume that $w(t, \theta) \geqq w(\mu_\theta, \theta) = 0$ for all t and θ. That is, we assume that the loss function is nonnegative and that there is no loss when the estimate hits the mark. Let U be the class of all unbiassed estimators η, of μ_θ for which the risk function

$$r(\theta \mid \eta) = E[w(t, \theta) \mid \eta, \theta] = \int_{\mathfrak{X}} dP_\theta \int_{-\infty}^{\infty} w(t, \theta)\, d\eta_x$$

is defined for all θ. We prove the following

THEOREM. *If for every fixed $\theta \varepsilon \Omega$ the loss function $w(t, \theta)$ is bounded in every finite interval $|t - \mu_\theta| \leqq A$, and is $o(|t - \mu_\theta|)$ as $|t - \mu_\theta| \to \infty$, then*

$$\inf_{\eta \epsilon U} r(\theta \mid \eta) \underset{\theta}{=} 0.$$

PROOF. Let $T = T_x$ be a nonrandomized unbiassed estimator of μ_θ. The existence of an unbiassed estimator clearly implies the existence of such a T_x. Consider now the randomized estimator $\eta^{(\delta)} = \eta_x^{(\delta)}$ which, for any $x \epsilon \mathfrak{X}$, has its entire probability mass concentrated at the two points μ_{θ_0} and $(T_x - \mu_{\theta_0})/\delta + \mu_{\theta_0}$ on the real line in the ratio $1 - \delta$ to δ, with $0 < \delta < 1$. It is easily verified that $\eta^{(\delta)}$ is an unbiassed estimator of μ_θ and that

$$\begin{aligned} r(\theta_0 \mid \eta^{(\delta)}) &= E[w(t, \theta_0) \mid \eta^{(\delta)}, \theta_0] \\ &= E[\delta w(H/\delta + \mu_{\theta_0}, \ \theta_0) \mid \theta_0], \qquad H = T_x - \mu_{\theta_0}. \end{aligned}$$

Since $w = o(|t - \mu_{\theta_0}|)$ as $|t - \mu_{\theta_0}| \to \infty$, given $\epsilon > 0$ we can determine A so large that

$$w(t, \theta_0) \leqq \epsilon |t - \mu_{\theta_0}|, \qquad |t - \mu_{\theta_0}| \geqq A.$$

Let $B = \sup_{|t - \mu_{\theta_0}| < A} w(t, \theta_0) < \infty$. Then

$$\begin{aligned} r(\theta_0 \mid \eta^{(\delta)}) &= \left\{ \int_{|H| < \delta A} + \int_{|H| \geqq \delta A} \right\} \delta w(H/\delta + \mu_{\theta_0}, \ \theta_0)\, dP_{\theta_0} \\ &\leqq \delta B + \epsilon E(|H| \mid \theta_0). \end{aligned}$$

Since ϵ and δ are arbitrary and B depends only on ϵ, it follows that $\inf_{\eta \epsilon U} r(\theta_0 \mid \eta) = 0$. Since θ_0 is arbitrary, the theorem is proved.

Now, if $w(t, \theta_0) > 0$ for $t \neq \mu_{\theta_0}$, then $r(\theta_0 \mid \eta)$ can be zero only if η_x gives probability one to μ_{θ_0} for almost all x with respect to the measure P_{θ_0}. In the usual circumstances, η_x then would not be an unbiassed estimator of μ_θ.

Thus, this theorem shows that if we work with a loss function satisfying the conditions of the theorem, even locally best unbiassed estimators would not exist in all the familiar situations in which we are interested. In particular estimation problems, it will be easy to see that the theorem holds even in the restricted class U^* of all nonrandomized estimators of μ. In the next section we consider the classical problem of estimating the mean of a normal population, but with a slightly altered definition of variance.

4. The case of the normal mean. Let $x = (x_1, x_2, \cdots, x_n)$ be a random sample from $N(\theta, 1)$. The problem is to get a good unbiassed estimator of θ with the loss function

$$w(t, \theta) = \begin{cases} (t - \theta)^2, & |t - \theta| \leqq a, \\ a^{3/2} |t - \theta|^{1/2}, & |t - \theta| > a, \end{cases}$$

where a is an arbitrarily large constant.

Let $\bar{x}$ and s^2 be the sample mean and variance, respectively, and let c_δ be the upper 100δ per cent point of the probability distribution of s^2, where $0 > \delta > 1$. Consider the nonrandomized estimator

$$T^{(\delta)} = T_x^{(\delta)} = \begin{cases} \theta_0, & s^2 \leqq c_\delta, \\ (\bar{x} - \theta_0)/\delta + \theta_0, & s^2 > c_\delta. \end{cases}$$

Since the distribution of s^2 is independent of θ and $\bar{x}$, it follows that $T^{(\delta)}$ is a function of x and δ alone and that $T^{(\delta)}$, for every fixed δ with $0 < \delta < 1$, is an unbiassed estimator of θ. Also

$$\begin{aligned} r(\theta_0 \mid T^{(\delta)}) &= E[\delta w\{(\bar{x} - \theta_0)/\delta + \theta_0, \ \theta_0\} \mid \theta_0] \\ &= \int_{|\bar{x}-\theta_0| \leqq a\delta} \delta \left(\frac{\bar{x} - \theta_0}{\delta}\right)^2 \phi(\bar{x})\, d\bar{x} + \int_{|\bar{x}-\theta_0| > a\delta} \delta a^{3/2} \left|\frac{\bar{x} - \theta_0}{\delta}\right|^{1/2} \phi(\bar{x})\, d\bar{x} \\ &< \delta a^2 + \delta^{1/2} a^{3/2} E(|\bar{x} - \theta_0|^{1/2} \mid \theta_0), \end{aligned}$$

where $\phi(\bar{x})$ is the frequency function of $\bar{x}$ when $\theta = \theta_0$. Thus $r(\theta_0 \mid T^{(\delta)}) \rightarrow 0$ as $\delta \rightarrow 0$. Therefore

$$\inf_{T \epsilon U^*} r(\theta \mid T^{(\delta)}) \equiv 0, \qquad -\infty < \theta < \infty,$$

where U^* is the class of all nonrandomized unbiassed estimators of θ.

When the constant a is very large, the modification to the usual definition of variance apparently is very negligible, yet this slight change of variance completely wrecks the theory of unbiassed estimation. Not even locally best unbiassed estimators exist, let alone a uniformly best one.

In the construction of $T^{(\delta)}$, the independence of s^2 and $\bar{x}$ is not essential. As a matter of fact, we can replace s^2 by any real-valued statistic Y whose conditional distribution, given $\bar{x}$, is continuous. We then replace c_δ by $c_\delta'(\bar{x})$, where $c_\delta'(\bar{x})$ is, say, the upper 100δ per cent point of the conditional distribution of Y given $\bar{x}$. From the sufficiency of $\bar{x}$ it follows that $c_\delta'(\bar{x})$ is independent of δ, and the rest of the proof follows through. Under similar circumstances the general theorem proved earlier will remain true in the restricted class U^* of all nonrandomized unbiassed estimators of μ_θ.

REFERENCES

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