ON A THEOREM IN METRIC SPACES

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0. Introduction. In his paper "On a class of probability spaces" ([1]), D. Blackwell observed that the class of Borel sets of a metric space may be a separable σ-field without the metric space being separable. However, in a subsequent letter to one of the authors, he stated that the question remained open. The object of the present note is to prove that the separability of the σ-field of Borel sets implies separability of the metric space, assuming the continuum hypothesis. What is actually used, is not the continuum hypothesis but the following proposition, which we will abbreviate as $\mathcal{P}$: If u is an uncountable cardinal, $2^u > c$ (the cardinal of the continuum). This is easily deduced from the continuum hypothesis and it seems to us that it has not so far been proved without the continuum hypothesis (cf. [4]). The main conclusion is as follows: A metric space is separable if and only if the cardinality of the Borel sets is $\leqq c$, provided we assume $\mathcal{P}$. It is also shown that the above theorem implies $\mathcal{P}$.

1. The main result. We introduce certain notations. X is a metric space and $\mathcal{B}$ is the σ-field generated by open subsets of X. Sets of $\mathcal{B}$ are called Borel sets of X. $\mathcal{B}$ is called separable if there is a sequence $\{A_n\}$ of sets of $\mathcal{B}$ generating it. In that case, cardinality of $\mathcal{B}$ is $\leqq c$ ([2]). Before proving the main result we prove an auxiliary result, interesting in itself.

THEOREM 1. *X is separable if and only if every disjoint family of nonempty open subsets of X is countable.*

Proof. If X is separable, its topology has a countable basis $G_1, G_2, \cdots$. Since any nonempty open set of X contains a nonempty G_i, the existence of an uncountable disjoint family of nonempty open subsets of X implies the existence of an uncountable disjoint family of nonempty G_i's, which is impossible. To prove the converse, let us suppose that every disjoint family of nonempty open subsets of X is countable. Let n be an integer $\geqq 1$ and let $\mathcal{K}_n$ be defined as follows: $\mathcal{K}_n = \{A: A \subset X; x, y \varepsilon A \Rightarrow d(x, y) > 1/n\}$. Elements of $\mathcal{K}_n$ are subsets of X and are partially ordered by the relation of set inclusion. Further, every linearly ordered sub-family of $\mathcal{K}_n$ has a supremum in the family (namely, the set-union) and hence, by Zorn's lemma, there are maximal elements containing any element of $\mathcal{K}_n$, in particular any point of X. Let A_n be one such nonempty maximal element. Maximality of A_n implies that if $y \varepsilon X - A_n$, $d(y, x) \leqq 1/n$ for some $x \varepsilon A_n$. Further, each A_n must be countable, as otherwise, the spheres with centres at the points of A_n and radii $(1/2n)$ will be an uncountable disjoint family of nonempty open subsets of X.

Let now n run over $1, 2, \cdots$ and set $A = \cup_n A_n$. A is countable and for any $y \varepsilon X - A$ and any positive integer n, there is an $x_n \varepsilon A_n$ such that $d(y, x_n) \leqq$

Received November 8, 1957.

$1/n$. This shows that A is dense in X. Since A is countable, this completes the proof that X is separable.

Remark. This result need not be true if X is not metric. See, for example, [3].

We now prove our main result.

THEOREM 2. (*Under assumption* $\mathcal{P}$) *X is separable if and only if cardinality of* $\mathcal{B} \leqq c$. *In particular, X is separable if and only if $\mathcal{B}$ is separable.*

Proof. If X is separable, its topology has a countable base $G_1, G_2, \cdots$ which generates $\mathcal{B}$ and hence cardinality of $\mathcal{B} \leqq c$. Conversely let cardinality of $\mathcal{B}$ be $\leqq c$. If $\{A_\alpha\}_{\alpha \epsilon I}$ is any disjoint family of nonempty open subsets of X, then, every subunion of the A_α's is open and hence $\epsilon\ \mathcal{B}$. There are 2^u such subunions where u is the cardinal of I and since cardinality of $\mathcal{B}$ is $\leqq c$, we have $2^u \leqq c$. This however implies (in virtue of assumption $\mathcal{P}$) that $u \leqq \aleph_0$. Theorem 1 now applies and proves that X is separable. This completes the proof.

Remark. We can show that Theorem 2 implies $\mathcal{P}$. To see this, let X be an uncountable set with cardinal u. Give X the discrete topology so that $\mathcal{B}$ is the class of all subsets of X. Cardinality of $\mathcal{B}$ is thus 2^u and since X is not separable, Theorem 2 implies that $2^u > c$. This is precisely assumption $\mathcal{P}$.

REFERENCES

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[3] J. L. KELLEY, *General Topology*, D. Van Nostrand, 1955, p. 166, ex. 0, (f).

[4] W. SIERPINSKI, "Sur l'hypothèse du continuum," *Fundamenta Mathematica*, Vol. 5, p. 182.