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E-OPTIMAL BLOCK AND ROW-COLUMN DESIGNS WITH UNEQUAL NUMBER OF REPLICATES

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SUMMARY. It is well known that in experimental settings where v treatments are being tested in b blocks of size k, a balanced incomplete block design and a group divisible design having parameters $\lambda_2 = \lambda_1 + 1$ is E-optimal among all possible competing designs. In this paper, we show that under certain conditions, the E-optimal designs mentioned in the previous sentence can be used to construct E-optimal block and row-column designs with unequal replicates to handle experimental situations in which heterogeneity is to be eliminated in either one or two directions.

1. Introduction

In the usual setting of block designs, let v denote the number of treatments, b, the number of blocks and k, the number of units per block. Any allocation of v treatments to the bk experimental units is a block design. Under the usual fixed effects additive model with homoscedasticity and independence, the coefficient matrix of the reduced normal equations for estimating linear functions of treatment effects, using a block design d with parameters v, b, k, is given by

$$\boldsymbol{C}_d = \boldsymbol{R}_d - k^{-1} \boldsymbol{N}_d \boldsymbol{N}_d', \qquad \ldots \quad (1.1)$$

where $\boldsymbol{R}_d = \text{diag}\,(r_{d1}, \ldots, r_{dv})$, r_{di} is the replication of the i-th treatment in d and $\boldsymbol{N}_d = ((n_{dij}))$ is the $v \times b$ incidence matrix of the design d.

The row-column designs considered here have bk experimental units arranged in a rectangular array of b columns and k rows such that each unit receives only one of the v treatments being studied. For an arbitrary row-column design d, the 'C-matrix', under an appropriate model is given by

$$\begin{aligned} \boldsymbol{C}_d^{(RC)} &= \boldsymbol{R}_d - k^{-1} \boldsymbol{N}_d \boldsymbol{N}_d' - b^{-1} \boldsymbol{M}_d \boldsymbol{M}_d' + (bk)^{-1} \boldsymbol{r}_d \boldsymbol{r}_d' \\ &= \boldsymbol{R}_d - k^{-1} \boldsymbol{N}_d \boldsymbol{N}_d' - b^{-1} \boldsymbol{M}_d (I - k^{-1} \mathbf{1}\,\mathbf{1}') \boldsymbol{M}_d', \qquad \ldots \quad (1.2) \end{aligned}$$

where $\boldsymbol{R}_d$ is as defined earlier, $\boldsymbol{r}_d = (r_{d1}, \ldots, r_{dv})'$, $\mathbf{N}_d$ and $\boldsymbol{M}_d$ are the $v \times b$ treatment-column and $v \times k$ treatment-row incidence matrices, respectively, $\boldsymbol{I}$ is an identity matrix (of appropriate order) and $\mathbf{1}$, a column vector of unities.

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It is known that C_d as in (1.1) and $C_d^{(RC)}$ as in (1.2) are symmetric, non-negative definite matrices, with zero row sums. A block (resply. row-column) design d is called connected if and only if Rank $(C_d) = v-1$ (Rank $(C_d^{(RC)}) = v-1$). Henceforth, only connected designs are considered.

For given positive integers v, b, k $D_0(v, b, k)$ will denote the class of all connected block designs with v treatments, b blocks and block size k. Similarly, $D(v, b, k)$ will denote the class of all connected row-column designs with v treatments, k rows and b columns.

For a block design $d \epsilon D_0(v, b, k)$, let $0 = z_{d0}^N < z_{d1}^N \leqslant z_{d2}^N \leqslant \ldots \leqslant z_{d,v-1}^N$, denote the eigenvalues of C_d. Similarly, let $0 = z_{d0} < z_{d1} \leqslant z_{d2} \leqslant \cdots \leqslant z_{d,v-1}$ denote the eigenvalues of $C_d^{(RC)}$ for $d \epsilon D(v, b, k)$.

With each $d \epsilon D(v, b, k)$ we associate the block design $d^N \epsilon D_0(v, b, k)$ obtained from d by considering {columns} of d as blocks and ignoring row effects. We denote the usual C-matrix of d^N by C_d^N. Clearly from (1.2)

$$C_d^{(RC)} = C_d^N - b^{-1} M_d (I - k^{-1} \mathbf{1}\, \mathbf{1}') M_d' \qquad \ldots \quad (1.3)$$

and since the second matrix on the r.h.s. of (1.3) is non-negative definite, we have

$$z_{d1} \leqslant z_{d1}^N \qquad \ldots \quad (1.4)$$

where z_{d1}^N represents the minimum non-zero eigenvalue of C_d^N.

The optimality criterion considered here for selecting optimal designs in $D_0(v, b, k)$ or in $D(v, b, k)$ is the E-optimality criterion introduced by Ehrenfeld (1955). This criterion chooses those designs in $D_0(v, b, k)$ and $D(v, b, k)$ whose C-matrices have minimal non-zero eigenvalues of maximum size, and is equivalent to finding designs which minimize the maximum variance of the best linear unbiased estimator (BLUE) for treatment constrasts of the form $\sum_{i=1}^{v} l_i t_i$ where $\sum_{i=1}^{v} l_i^2 = 1$ (A treatment constrast is any linear combination $\sum_{i=1}^{v} l_i t_i$ of the treatment effects t_i $(i = 1, \ldots, v)$ where $\sum_{i=1}^{v} l_i = 0$).

We shall denote a Balanced Incomplete Block (BIB) design with parameters v, b, r, k, λ by BIB (v, b, r, k, λ). Note for a BIB (v, b, r, k, λ), $bk = vr$ and $r(k-1) = (v-1)\lambda$. Also, a Group Divisible (GD) design with parameters $v = mn$, b, r, k, λ_1, λ_2, $m \geqslant 2$, $n \geqslant 2$ will be denoted by $GD(v\ b, r, k\ \lambda_1, \lambda_2, m, n)$

A number of results are already known concerning the E-optimality of certain equireplicate designs in classes $D_0(v, b, k)$ and $D(v, b, k)$, e.g., see Takeuchi (1961, 1963), Kiefer (1958, 1975), Cheng (1978, 1980), Constantine (1982), Jacroux (1980a, 1985, 1986). In exploratory experiments aimed at providing as much information as possible on the effects of the treatments being studied and where heterogeneity needs to be eliminated in one or two directions, use of an equireplicate block or row-column design may mean wasting some of the available experimental units. Here we consider the problem of determining unequally replicated E-optimal designs in classes $D_0(v, b, k)$ and $D(v, b, k)$ where bk/v is not an integer. The only results concerning this problem are those obtained by Jacroux (1980b, 1983), Constantine (1981), Sathe and Bapat (1985) and Bagchi (1988) for block designs and those by Jacroux (1982, 1990) and Das and Dey (1989) for row-column designs. In this paper we prove the E-optimality in respective classes $D_0(v, b, k)$ and $D(v, b, k)$ of several different types of block and row-column designs that have unequally replicated treatments. Such unequally replicated designs can maximize the information on treatment effects without wasting units. In Section 2, we define extended quotient designs and study certain eigenvalue properties of the C-matrices of these designs. In Section 3 E-optimal block designs derived from BIB designs or GD designs with $\lambda_2 = \lambda_1+1$ are obtained. Some series of such E-optimal designs are also given. In Section 4 results similar to those in Section 3 are derived for various row-column designs. Finally in Section 5 we tabulate the parameter sets of E-optimal block and row-column designs obtained in Sections 3 and 4.

2. Preliminaries

In this section, certain definitions and results are given, as we shall have occasion to refer to these in the sequel.

Definition 2.1. Let d_1 be a given block (row-column) design with parameters v^*, b, k, and let $V^* = \{1, 2, \ldots, v^*\}$ be the set of treatments of d_1. Denote by $W = \{w_1, w_2, \ldots, w_v\}$ a partition of V^* into $v(= v^*-p, p \geqslant 1)$ nonempty classes w_i, $1 \leqslant i \leqslant v$; i.e., $V^* = \bigcup_{i=1}^{v} w_i$ and $w_i \cap w_j = \phi$ for $i \neq j$. The design d_2, called the *quotient* design of d_1 has v treatments, $w_1, \ldots, w_v$. For each block (column) $(i_1, i_2, \ldots, i_k)$ of d_1, a corresponding block (column) of d_2 is obtained as follows : Since each treatment of d_1 belongs to a unique class w_j, there are uniquely determined classes $w_{j_1}, w_{j_2}, \ldots, w_{j_k}$ in W such that $i_1 \in w_{j_1}, i_2 \in w_{j_2}, \ldots, i_k \in w_{j_k}$. The block (column) of d_2 then has its contents $(w_{j_1}, w_{j_2}, \ldots, w_{j_k})$.

Informally, in a quotient design the treatments in class w_i of the partition are collapsed.

Definition 2.2. Let d_1 be a given block (row-column) design with parameters v, b^*, k. The design d_2 with parameters v, $b = b^*+x$, k obtained by adding (juxtaposing) $x\,(\geqslant 1)$ arbitrary blocks (columns) of size k each to d_1 is called an *extended* design.

Definition 2.3. Let d_1 be a given block (row-column) design with parameters v^*, b^*, k. The design d_2 with parameters v, b, k ($v = v^*-p$, $p \geqslant 0$, $b = b^*+x$, $x \geqslant 0$) is called an *Extended Quotient* (EQ) design where d_2 is obtained through extending d_1 by x blocks (columns) and then taking the quotient or vice-versa i.e., first taking quotient of d_1 and then extending by x blocks (columns).

Note that for an EQ design, p and x are not simultaneously equal to zero. Also an EQ design with $p = 0$ reduces to an extended design and that with $x = 0$ reduces to a quotient design.

Bagchi (1988) while dealing with quotient block designs has shown that $z^N_{d_1 1} \leqslant z^N_{d_2 1}$. Also Jacroux (1982) while proving optimality of certain extended row-column designs has shown that $z_{d_1 1} \leqslant z_{d_2 1}$ for some particular types of extended designs d_2 obtained from specific designs d_1 by addition of only disjoint columns. These results were proved using a different technique. We now give a more general result regarding $z^N_{d_1 1}$ ($z_{d_1 1}$) and $z^N_{d_2 1}$ ($z_{d_2 1}$) for *EQ* block (row-column) designs.

Theorem 2.1. *Let d_2 be a quotient design in $D_0(v, b, k)$ ($D(v, b, k)$) of a design d_1 in D_0 (v^*, b, k) ($D(v^*, b, k)$). then*

$$z^N_{d_1 1} \leqslant z^N_{d_2 1}(z_{d_1 1} \leqslant z_{d_2 1}).$$

Proof. Let θ be any normalized treatment contrast in d_2. Let $\hat{\theta}_{d_2}$ denote its BLUE. Let ϕ be a contrast in d_1 obtained by replacing in θ every t_i of d_2 by any of the t_j's which are mapped into t_i. Let $\hat{\phi}_{d_1}$ be the BLUE of ϕ in d_1. Clearly $E(\hat{\phi}_{d_1}\,|\,d_2) = \theta$ and hence $V(\hat{\theta}_{d_2}\,|\,d_2) \leqslant V(\hat{\phi}_{d_1}\,|\,d_2) = V(\hat{\phi}_{d_1}\,|\,d_1)$. Thus, for every normalized treatment contrast in d_2 we have one in d_1 which is estimated with larger variance. This implies that the smallest non-zero eigenvalue of C_{d_2} (or $C^{(RC)}_{d_2}$) is not less than the smallest non-zero eigenvalue of C_{d_1} (or $C^{(RC)}_{d_1}$). This completes the proof.

Theorem 2.2. *Let d_2 be an extended design in $D_0(v, b, k)$ ($D(v, b, k)$) obtained from a design d_1 in $D_0(v, b^*, k)$ ($D(v, b^*, k)$). Then*

$$z^N_{d_1 1} \leqslant z^N_{d_2 1}(z_{d_1 1} \leqslant z_{d_2 1}).$$

Proof. The set of linear functions of observations free from block (or row and column) effects in the design d_1 is a subset of the corresponding set for the design d_2. Hence $C_{d_2} \geqslant C_{d_1}$ (or $C_{d_2}^{(RC)} \geqslant C_{d_1}^{(RC)}$) and the result follows.

As a result of Theorems 2.1 and 2.2 the following hold.

Theorem 2.3. *Let d_2 be an EQ design in $D_0(v, b, k)$ $(D(v, b, k))$ obtained from a design d_1 in $D_0(v^*, b^*, k)$ $(D(v^*, b^*, k))$. Then*

$$z_{d_1 1}^N \leqslant z_{d_2 1}^N (z_{d_1 1} \leqslant z_{d_2 1}).$$

We now state a result due to Jacroux (1983). Let $\bar{r} = [bk/v]$ and $\bar{\lambda} = [\bar{r}(k-1)/(v-1)]$ where $[m]$ denotes the largest integet not exceding m.

Theorem 2.4. *Let $D_0(v, b, k)$ be a class of block designs such that $bk = v\bar{r}+s$, $0 \leqslant s < v$, $\bar{r}(k-1) = (v-1)\ \bar{\lambda}+t$, $0 \leqslant t < v-1$, and with $v \leqslant (v-s)\ (v-t)$. Then for $d \in D_0(v, b, k)$*

$$z_{d1}^N \leqslant (\bar{r}(k-1)+\bar{\lambda})/k.$$

3. E-OPTIMAL BLOCK DESIGNS

This section establishes the E-optimality of certain unequally replicated block designs. Let us first consider the EQ design d_2 (with v treatments in b blocks each of size k) of a BIB design d_1 with parameters $v^* = v+p$, $b^* = b-x$, r, k, λ.

Theorem 3.1. *Let d_1 be a BIB $(v^*, b^*, r, k, \lambda)$ and d_2 an EQ design with parameters $v = v^*-p$, $b = b^*+x$, k obtained from d_1. Then d_2 is E-optimal in $D_0(v, b, k)$ provided the parameters satisfy the following conditions :*

(*a*) $v-pr-xk \geqslant 1$

(*b*) $v-p\lambda \geqslant 2$

(*c*) $v \leqslant (v-pr-xk)\ (v-p\lambda)$.

Proof. First observe that $z_{d_1 1}^N = \lambda v/k = (r(k-1)+\lambda)/k$. *Therefore from Theorem* 2.3 $(r(k-1)+\lambda)/k \leqslant z_{d_2 1}^N$. *Also from Theorem* 2.4 *for any design* $d \in D_0(v, b, k)\, z_{d1}^N \leqslant (\bar{r}(k-1)+\bar{\lambda})/k$ and d_2 is E-optimal in $D_0(v, b, k)$ provided

$$z_{d_2 1}^N = (\bar{r}(k-1)+\bar{\lambda})/k \qquad \dots \quad (3.1a)$$

and

$$v \leqslant (v-s)\ (v-t), \qquad \dots \quad (3.1b)$$

where $bk = v\bar{r}+s$, $0 \leqslant s < v$, $\bar{r}(k-1) = (v-1)\bar{\lambda}+t$, $0 \leqslant t < v-1$.

Now since $(r(k-1)+\lambda)/k \leqslant z^N_{d_2 1} \leqslant (\bar{r}(k-1)+\bar{\lambda})/k$, (3.1$a$) holds if $r = \bar{r}$ and $\lambda = \bar{\lambda}$. Now $r = \bar{r}$ implies $s = bk-vr = xk+pr$ and since $s < v$, we get condition (a). Again $r = \bar{r}$ and $\lambda = \bar{\lambda}$ implies $t = r(k-1)-\lambda(v-1) = \lambda p$ and since $t < v-1$, we get condition (b). Finally from (3.1b) we get condition (c). This completes the proof.

The results of Bagchi (1988) and Constantine (1981) follow as corollaries to the above theorem when $x = 0$ and $p = 0$ respectively.

Corollary 3.1. *Let d_1 be a BIB (v^*, b, r, k, λ) and d_2 quotient design with parameters $v = v^*-p, b, k$ obtained from d_1. Then d_2 is E-optimal in $D_0(v, b, k)$ provided the parameters satisfy*

(*a*) $v-pr \geqslant 2$

(*b*) $v \leqslant (v-pr)\ (v-p\lambda)$.

Proof. Putting $x = 0$ in Theorem 3.1, we get the following conditions under which d_2 is E-optimal in $D_0(v, b, k)$. (i) $v-pr \geqslant 1$, (ii) $v-p\lambda \geqslant 2$ and (iii) $v \leqslant (v-pr)(v-p\lambda)$. Now if $v-pr = 1$, then condition (iii) is not true Hence for d_2 to be E-optimal, we have a modified condition (i)$'$ $v-pr \geqslant 2$. Furthermore, since $r > \lambda$ condition (i)$'$ $\Rightarrow$ (ii) and the result follows.

Corollary 3.2. *Let d_1 be a BIB (v, b^*, r, k, λ) and d_2 an extended design with parameters v, $b = b^*+x$, k obtained from d_1. Then d_2 is E-optimal in $D_0(v, b, k)$ provided $x \leqslant (v-1)/k$.*

Remark 3.1. The EQ design with $p > 0$ obtained from a BIB design is necessarily non-binary. Therefore the E-optimal designs of Theorem 3.1 and Corollary 3.1 are non-binary. There may be a binary E-optimal design in $D_0(v, b, k)$ along with the non-binary E-optimal design. But this need not be so in all situations. Shah and Das (1992) has shown that the class of binary designs is *not* essentially complete w.r.t. the E-optimality criterion. For example consider the quotient design d^* with parameters $v = 6$, $b = 7$, $k = 3$ of a BIB (7, 7, 3, 3, 1).

$$d^* = \left| \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 6 \\ 2 & 3 & 4 & 5 & 6 & 6 & 1 \\ 4 & 5 & 6 & 6 & 1 & 2 & 3 \end{array} \right.$$

Using Corollary 3.1, d^* is E-optimal in $D_0(6, 7, 3)$. Shah and Das (1992) shows that d^* is E-better than any binary design in $D_0(6, 7, 3)$. In that context, though for $p > 0$ designs in Theorem 3.1 and Corollary 3.1 are non-binary, the result may be considered important from optimality point of view.

Example 3.1. Consider the following design $d \epsilon D_0(12, 14, 4)$

$$d = \left| \begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 12 & 1 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 12 & 1 & 2 \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 12 & 1 & 2 & 3 & 3 \\ 10 & 11 & 12 & 12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 4 \end{array} \right.$$

This design d is an EQ design with $p = 1$, $x = 1$ obtained from BIB (13, 13, 4, 4, 1) by collapsing treatment number 13 with 12 and adding a block (1, 2, 3, 4). It is easy to verify that d satisfies the conditions in Theorem 3.1 and hence d is E-optimal in $D_0(12, 14, 4)$. Note that though we have added the block (1, 2, 3, 4) to obtain d from BIB (13, 13, 4, 4, 1), the design obtained by adding any arbitrary block would also be E-optimal.

In particular, the EQ designs with $p > 0$, derivable from the following series of BIB designs satisfy the requirements of Theorem 3.1, and hence lead to E-optimal block designs.

(*i*) $v^* = s^2+s+1 = b^*$, $r = s+1 = k$, $\lambda = 1$, s a prime power, and $p+x \leqslant s-1$.

(ii) $v^* = 4s-1 = b$, $r = 2s-1 = k$, $\lambda = s-1$, $s > 1$, $p = 1$, $x = 0$.

Let us now come to the EQ design d_2 with v treatments in b blocks of size k each obtained from a GD design d_1 with parameters $v^* = v+p$, $b^* = b-x$, r, k, λ_1, $\lambda_2 = \lambda_1+1$, m, n.

Theorem 3.2. *Let d_1 be a GD $(v^*, b^*, r, k, \lambda_1, \lambda_2 = \lambda_1+1, m, n)$ and d_2 an EQ design with parameters $v = v^*-p$, $b = b^*+x$, k obtained from d_1. Then d_2 is E-optimal in $D_0(v, b, k)$ provided the parameters satisfy the following conditions*

(*a*) $v-pr-xk \geqslant 2$

(*b*) $v \leqslant (v-pr-xk)\, n-p\lambda_2$.

Proof. Since $z^N_{d_11} = (r(k-1)+\lambda_1)/k$ we have from Theorem 2.3 $(r(k-1)+\lambda_1)/k \leqslant z^N_{d_21}$. Let d be any design in $D_0(v, b, k)$. Then from Theorem 2.4

$z^N_{d1} \leqslant (\bar{r}(k-1)+\bar{\lambda})/k$ provided $v \leqslant (v-s)\,(v-t)$ where $bk = v\bar{r}+s$, $0 \leqslant s < v$, $\bar{r}(k-1) = (v-1)\bar{\lambda}+t, 0 \leqslant t < v-1$. Therefore $(r(k-1)+\lambda_1)/k \leqslant z^N_{d_21} \leqslant (\bar{r}(k-1)+\bar{\lambda})/k$

and $z^N_{d_21} = (\bar{r}(k-1)+\bar{\lambda})/k$ (and thus d_2 is E-optimal in $D_0(v, b, k)$) provided

$$r = \bar{r},\ \lambda_1 = \bar{\lambda} \quad \ldots \quad (3.2a)$$

and

$$v \leqslant (v-s)\,(v-t) \quad \ldots \quad (3.2b)$$

Now, on lines similar to Theorem 3.1, from (3.2a) we get the following two conditions : (i) $v-pr-xk \geqslant 1$ and (ii) $v-p\lambda_1 \geqslant 2$. Again from (3.2b), since $s = xk+pr$ and $v-t = v^*-p-r(k-1)+\lambda_1(v^*-p-1) = n-p(\lambda_1+1)$ on simplification, we get condition (b). Finally note that if $v-pr-xk = 1$ condition (b) reduces to $n(m-1)+p\lambda_1 \leqslant 0$ which is never true. Hence for d_2 to be E-optimal, we have a modified condition (i)$'$ $v-pr-xk \geqslant 2$ (i.e. condition (a)). Furthermore, since $r > \lambda_1$ condition (i)$' \Longrightarrow$ (ii) and the result follows.

We have two corollaries for situations when $x = 0$ or $p = 0$. The case $p = 0$ give rise to E-optimal designs of Constantine (1981) and Jacroux (1982).

Corollary 3.3. *Let d_1 be a $GD(v^*, b, r, k, \lambda_1, \lambda_2 = \lambda_1+1, m, n)$ and d_2 a quotient design with parameters $v = v^*-p$, b, k obtained from d_1. Then d_2 is E-optimal in $D_0(v, b, k)$ provided the parameters satisfy*

(*a*) $v-pr \geqslant 2$

(*b*) $v \leqslant (v-pr)\,(n-p\lambda_2)$.

Theorem 3.7 of Bagchi (1988) is a particular case of the above corollary when $p = 1$.

Corollary 3.4. *Let d_1 be a $GD(v, b,^* r, k, \lambda_1, \lambda_2 = \lambda_1+1, m, n)$ and d_2 an extended design with parameters v, $b = b^*+x$, k obtained from d_1. Then d_2 is E-optimal in $D_0(v, b, k)$ provided $x \leqslant (v-m)/k$.*

Example 3.2. Consider the following EQ design $d \in D_0$ (11, 17, 3) obtained from GD design SR 26 with parameters $v^* = 12$, $b^* = 16$, $k = 3$ (given in Clatworthy (1973)) by collapsing treatment number 12 with 9 and adding the block (1, 4, 7).

$$d = \begin{array}{ccccccccccccccccc} 1 & 9 & 9 & 11 & 1 & 8 & 9 & 2 & 1 & 5 & 6 & 4 & 1 & 7 & 10 & 4 & 1 \\ 2 & 8 & 10 & 4 & 5 & 3 & 11 & 9 & 9 & 7 & 10 & 2 & 8 & 6 & 11 & 9 & 4 \\ 3 & 7 & 5 & 6 & 6 & 4 & 7 & 10 & 11 & 3 & 8 & 9 & 9 & 2 & 3 & 5 & 7 \end{array}$$

For this EQ design with $p = 1$ and $x = 1$ the conditions in Theorem 3.2 hold and thus d is E-optimal in D_0 (11, 17, 3).

Remark 3.2. As remarked earlier EQ designs obtained from BIB designs are necessarily non-binary unless $p = 0$. This is not the case for EQ designs obtained from GD designs. In fact it is possible to have binary EQ designs of GD designs with $\lambda_1 = 0$ whenever $p \leqslant m(n-1)$. This enables us to obtain E-optimal binary EQ designs, e.g., the design in Example 3.2 is an E-optimal and binary EQ design.

4. E-OPTIMAL ROW-COLUMN DESIGNS

This section shows how E-optimal designs can be obtained in classes $D(v, b, k)$ where bk/v is not an integer.

Let $d \in D(v, b, k)$ where $v > k$ and $b = \rho v$ for some positive integer ρ. Then d is called a Youden design (and denoted by $YD(v, b, r, k, \lambda)$) if d^N is a BIB (v, b, r, k, λ) and $\boldsymbol{M}_d = \rho\ \mathbf{1}\ \mathbf{1}'$. Similarly d is called a Group Divisible Youden design (and denoted by GDYD $(v, b, r, k, \lambda_1, \lambda_2, m, n)$) if d^N is a $GD(v, b, r, k, \lambda_1, \lambda_2, m, n)$ and $\boldsymbol{M}_d = \rho\ \mathbf{1}\ \mathbf{1}'$. Note that corresponding to every BIB design with $b = \rho v$, there exist a Youden design. Similarly, corresponding to every GD design with $b = \rho v$, there exist a group divisible Youden design.

Youden designs and group divisible Youden designs with $\lambda_2 = \lambda_1 + 1$ were proven E-optimal in classes $D(v, b, k)$ by Kiefer (1975) and Cheng (1978) respectively. We now show how these designs can be used to obtain additional E-optimal row-column designs. In what follows, we denote the minimum non-zero eigenvalue of $\boldsymbol{C}_d^N$ by z_{d1}^N.

Theorem 4.1. *Let d_1 be a $YD(v^*, b^*, r, k, \lambda)$ with $b^* = \rho v^*$ and d_2 an EQ design with parameters $v = v^* - p$, $b = b^* + x$, k obtained from d_1. Then d_2 is E-optimal in $D(v, b, k)$ provided the parameters satisfy the following conditions*

(*a*) $v - pr - xk \geqslant 1$

(*b*) $v - p\lambda \geqslant 2$

(*c*) $v \leqslant (v - pr - xk)(v - p\lambda)$.

Proof. We begin by showing that $z_{d_2 1} = z_{d_2 1}^N$. From (1.4), $z_{d_2 1} \leqslant z_{d_2 1}^N$. It is now shown that $z_{d_2 1} \geqslant z_{d_2 1}^N$. First, observe that $\boldsymbol{C}_{d_1}^{(RC)} = \boldsymbol{C}_{d_1}^N$ and $z_{d_1 1} = z_{d_1 1}^N$ since $\boldsymbol{M}_{d_1} = \rho\ \mathbf{1}\ \mathbf{1}'$. Also, by Theorem 3.1, $z_{d_1 1}^N = (r(k-1)+\lambda)/k = z_{d_2 1}^N$ and d_2^N is E-optimal in $D_0(v, b, k)$ under conditions (*a*), (*b*) and (*c*). Finally, note that from Theorem 2.3 we have $z_{d_2 1} \geqslant z_{d_1 1}$. The result $z_{d_2 1} \geqslant z_{d_2 1}^N$ follows.

Now suppose $d \in D(v, b, k)$ is any design. Then by (1.4) $z_{d1} \leqslant z_{d1}^N$. Since d_2^N is E-optimal in $D_0(v, b, k)$, $z_{d_2 1} = z_{d_2 1}^N \geqslant z_{d_1}^N \geqslant z_{d_1}$ and d_2 is E-optimal in $D(v, b, k)$.

The results of Das and Dey (1989) and Jacroux (1982) follow as corollaries to Theorem 4.1 when $x = 0$ or $p = 0$.

Corollary 4.1. *Let d_1 be a YD (v^*, b, r, k, λ) and d_2 a quotient design with parameters $v = v^* - p$, b, k obtained from d_1. Then d_2 is E-optimal in $D(v, b, k)$ provided the parameters satisfy*

(*a*) $v - pr \geqslant 2$

(*b*) $v \leqslant (v - pr)\ (v - p\lambda)$.

Corollary 4.2. *Let d_1 be a $YD(v, b^*, r, k, \lambda)$ and d_2 an extended design with parameters v, $b = b^*+x$, k obtained from d_1. Then d_2 is E-optimal in $D(v, b, k)$ provided $x \leqslant (v-1)/k$.*

Example 4.1. The design d as given in Example 3.1 when considered as a row-column design is E-optimal in $D(12, 14, 4)$. This EQ row-column design is obtained from $YD(13, 13, 4, 4, 1)$ by collapsing treatment number 13 with 12 and adding a column (1, 2, 3, 4). Also the $YD(13, 13, 4, 4, 1)$ when extended by $x \leqslant 3$ arbitrary columns is E-optimal in $D(13, 13+x, 4)$.

Note that for the series of designs given after Example 3.1, the conditions in Theorem 4.1 are satisfied and hence, these designs can be used to obtain E-optimal row-column designs.

Theorem 4.2. *Let d_1 be a $GDYD$ $(v^*, b^*, r, k, \lambda_1, \lambda_2 = \lambda_1+1, m, n)$ with $b^* = pv^*$ and d_2 an EQ design with parameters $v = v^*-p$, $b = b^*+x$, k obtained from d_1. Then d_2 is E-optimal in $D(v, b, k)$ provided the parameters satisfy the following conditions*

(*a*) $v-pr-xk \geqslant 2$

(*b*) $v \leqslant (v-pr-xk)\ (n-p\lambda_2)$.

Proof. The result follows from (1.4), Theorems 2.3, 3.2 and using arguments analogous to those given in the proof of Theorem 4.1.

Corollary 4.3. *Let d_1 be a $GDYD(v^*, b, r, k, \lambda_1, \lambda_2 = \lambda_1+1, m, n)$ and d_2 a quotient design with parameters $v = v^*-p$, b, k obtained from d_1. Then d_2 is E-optimal in $D(v, b, k)$ provided the parameters satisfy*

(*a*) $v-pr \geqslant 2$

(*b*) $v \leqslant (v-pr)\ (n-p\lambda_2)$.

Corollary 4.4. *Let d_1 be a $GDYD$ $(v, b^*, r, k, \lambda_1, \lambda_2 = \lambda_1+1, m, n)$ and d_2 an extended design with parameters v, $b = b^*+x$, k obtained from d_1. Then d_2 is E-optimal in $D(v, b, k)$ provided $x \leqslant (v-m)/k$.*

Remark 4.1. As against the results of Das and Dey (1989) and Jacroux (1982) where YD and GDYD were extended only by disjoint columns, here we find that any arbitrary columns may be added to obtain E-optimal designs. Moreover Jacroux (1982) proves the E-optimality of extended GDYD for $x \leqslant \min\,[2, (v-m)/k]$ and thereby limited extension of GDYD to a maximum of two disjoint columns. However, Corollary 4.4 has no such limitation (except that $x \leqslant (v-m)/k$). This enables us to obtain several E-optimal extended GDYD with $x \geqslant 3$.

Example 4.2. Consider the GDYD (24, 24, 5, 5, 0, 1, 6, 4) d whose allocation of treatments to rows and columns is

$$d = \begin{array}{cccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 1 \\ 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 1 & 2 & 3 & 4 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array}$$

The design $d \in D(24, 24, 5)$ is obtained from GD design R 153 having cyclic solution. Now consider additional columns (1, 2, 7, 13, 19), (3, 4, 9, 15, 21) and (5, 6, 11, 17, 23). Since $(v-m)/k = (24-6)/5 = 18/5$, we may add either one or two or all the three additional columns to d to obtain E-optimal designs in $D(24, 24+x, 5)$, $x = 1, 2, 3$. Furthermore the design obtained by collapsing treatment number 24 with 18 in d and then keeping it as such or adding one or two additional columns would be E-optimal in $D(23, 24+x, 5)$, $x = 0, 1, 2$. Finally, we also have by collapsing treatment number 24 with 18 and treatment number 23 with 17 in d, a design which is E-optimal in $D(22, 24, 5)$.

The following series of GD designs satisfy the reuqirements of corollaries 4.3 or 4.4, leading to E-optimal row-column designs.

(*i*) $v^* = s^2 = b^*$, $r = s = k$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s = n$, s a prime power, and either $p = 1$, $x = 0$ or $p = 0$, $x \leqslant s-1$.

(*ii*) $v^* = s^2-1 = b^*$, $r = s = k$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = s+1$, $n = s-1$, s a prime power, and either $p = 1$, $x = 0$ or $p = 0$, $x \leqslant s-1$.

5. Tabulation

In this section we give the parameter sets of E-optimal block and row-column designs satisfying the theorems in Sections 3 and 4. The parameters v, b, k of these designs along with the values of $p \geqslant 1$ and $x \geqslant 0$ are given in two tables. In Table 1 E-optimal designs (in the parametric range $v < b \leqslant 50$, $3 \leqslant k \leqslant 15$), which are derivable from existing BIB designs (or their complements) listed in Hall (1986), have been presented. In Table 2 we give the parameter sets of E-optimal designs (in the parameteric range $b \leqslant 50$, $2 \leqslant k \leqslant 8$), which are derivable from existing GD designs with $\lambda_2 = \lambda_1+1$ listed in Clatworthy (1973). In these tables the parameter set marked with asterisk refer to block as well as row-column design. As such parameter sets not marked with asterisk refer only to the block designs.

When $p = 0$, the parameter sets are not listed since it is clear that corresponding to a BIB (v, b, r, k, λ), we shall get sets $(v, b+x, k)$, $x \leqslant (v-1)/k$ and that corresponding to GD $(v, b, r, k, \lambda_1, \lambda_2 = \lambda_1+1, m, n)$, we shall get the sets $(v, b+x, k)$, $x \leqslant (v-m)/k$. Moreover if $b = \rho v$, the sets would refer to block and row-column designs, otherwise it refers to only block designs.

TABLE 1. PARAMETRIC VALUES OF E-OPTIMAL BLOCK AND ROW-COLUMN DESIGNS ($v < b \leqslant 50$, $3 \leqslant k \leqslant 15$) BASED ON THEOREMS 3.1 AND 4.1

v	b	k	p	x
6*	7	3	1	0
8	12	3	1	0
12*	26	3	1	0
12*	27	3	1	1
14	35	3	1	0
14	36	3	1	1
6*	7	4	1	0
9	15	4	1	0
11*	13	4	2	0
12*	13	4	1	0
12*	14	4	1	1
12*	26	4	1	0
14	20	4	2	0
15	20	4	1	0
15	21	4	1	1
15	22	4	1	2
15	40	4	1	0
23*	50	4	2	0
24*	50	4	1	0
10*	11	5	1	0
18*	21	5	3	0
19*	21	5	2	0
19*	22	5	2	1
20*	21	5	1	0
20*	22	5	1	1
20*	23	5	1	2
20*	42	5	1	0
20*	43	5	1	1
22	30	5	3	0
23	30	5	2	0
23	31	5	2	1
24	30	5	1	0
24	31	5	1	1
24	32	5	1	2
24	33	5	1	3
10*	11	6	1	0
14*	16	6	2	0
15*	16	6	1	0
15*	17	6	1	1
15	24	6	1	0
15*	32	6	1	0
20*	42	6	1	0
20*	43	6	1	1
27*	31	6	4	0
28*	31	6	3	0
28*	32	6	3	1
29*	31	6	2	0
29*	32	6	2	1
29*	33	6	2	2
30*	31	6	1	0
30*	32	6	1	1
30*	33	6	1	2
30*	34	6	1	3
14*	15	7	1	0
20	30	7	1	0
20	31	7	1	1
21*	44	7	1	0
26	36	7	2	0
27	36	7	1	0
27	37	7	1	1
27	38	7	1	2
14*	15	8	1	0
12*	13	9	1	0
18*	19	9	1	0
20	35	9	1	0
23*	25	9	2	0
24*	25	9	1	0
24*	26	9	1	1
24*	50	9	1	0
26	39	9	1	0
36	40	9	1	1
31	44	9	2	0
32	44	9	1	0
32	45	9	1	1
32	46	9	1	2
34*	37	9	3	0
35*	37	9	2	0
35*	38	9	2	1
36*	37	9	1	0
36*	38	9	1	1
36*	39	9	1	2
15*	16	10	1	0
18*	19	10	1	0
24	40	10	1	0
29*	31	10	2	0
30*	31	10	1	0
30*	32	10	1	1
22*	23	11	1	0
32	48	11	1	0
32	49	11	1	1
22*	23	12	1	0
42*	45	12	3	0
43*	45	12	2	0
43*	46	12	2	1
44*	45	12	1	0
44*	46	12	1	1
44*	47	12	1	2
26*	27	13	1	0
38*	40	13	2	0
39*	40	13	1	0
39*	41	13	1	1
26*	27	14	1	0
30*	31	15	1	0
34*	36	15	2	0
35*	36	15	1	0
35*	37	15	1	1
35	48	15	1	0

TABLE 2. PARAMETRIC VALUES OF E-OPTIMAL BLOCK AND ROW-COLUMN DESIGNS ($b \leqslant 50$, $2 \leqslant k \leqslant 8$) BASED ON THEOREMS 3.2 AND 4.2.

v	b	k	p	x	v	b	k	p	x
7*	16	2	1	0	23*	26	5	2	1
9	25	2	1	0	23*	26	5	1	2
11*	36	2	1	0	24*	25	5	1	0
11*	37	2	1	1	24*	26	5	1	1
13	49	2	1	0	24*	27	5	1	2
13	50	2	1	1	32	49	5	3	0
8*	9	3	1	0	33	49	5	2	0
11	16	3	1	0	33	50	5	2	1
11	17	3	1	1	34	49	5	1	0
14	25	3	1	0	34	50	5	1	1
14	26	3	1	1	28	25	6	2	0
14*	30	3	1	0	28	26	6	2	1
15*	32	3	1	0	29	25	6	1	0
15*	33	3	1	1	29	26	6	1	1
16*	36	3	2	0	29	27	6	1	2
17*	36	3	1	0	39	49	6	3	0
17*	37	3	1	1	39	50	6	3	1
17*	38	3	1	2	40	49	6	2	0
19	49	3	2	0	40	50	6	2	1
20	49	3	1	0	41	49	6	1	0
20	50	3	1	1	41	50	6	1	1
11	9	4	1	0	45*	48	7	3	0
14*	15	4	1	0	45*	49	7	3	1
14*	30	4	1	0	45*	49	7	4	0
15*	16	4	1	0	46*	48	7	2	0
15*	17	4	1	1	46*	49	7	2	1
18	25	4	2	0	46*	49	7	3	0
19	25	4	1	0	46*	50	7	2	2
19	26	4	1	1	46*	50	7	3	1
19	27	4	1	2	47*	48	7	1	0
23	42	4	1	0	47*	49	7	2	0
23	43	4	1	1	47*	49	7	1	1
26	49	4	2	0	47*	50	7	2	1
26	50	4	2	1	47*	50	7	1	2
27	49	4	1	0	48*	49	7	1	0
27	50	4	1	1	48*	50	7	1	1
18	16	5	2	0	52	49	8	4	0
19	16	5	1	0	53	49	8	3	0
19	17	5	1	1	53	50	8	3	1
22*	24	5	2	0	54	49	8	2	0
23*	24	5	1	0	54	50	8	2	1
23*	25	5	2	0	55	49	8	1	0
23*	25	5	1	1	55	50	8	1	1

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