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NEYMAN FACTORIZATION AND MINIMALITY OF PAIRWISE SUFFICIENT SUBFIELDS

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Assume that every probability measure P in $\mathscr{P}$ of a statistical structure $(X, \mathscr{A}, \mathscr{P})$ has a density $p(x, P)$ w.r.t. a (not necessarily σ-finite) measure m. Let $\mathscr{B}$ be any subfield and suppose that the densities are factored as $p(x, P) = g(x, P)h(x)$ where g is $\mathscr{B}$-measurable. Then $\mathscr{B}$ is pairwise sufficient and contains supports of P's. Assume further that m is locally localizable and $\mathscr{B}$ is pairwise sufficient and contains supports of P's. Then the densities are factored as above.

Two partial orders are introduced for pairwise sufficient subfields. Assuming that every P has a support, a subfield is constructed which is the smallest with supports under the first partial order, and is the smallest under the second. This is used to give a simple proof of existence of the minimal sufficient subfield for the coherent case. In the (uncountable) discrete case it is proved that under the first partial order there are infinitely many minimal pairwise sufficient subfields and hence there is none that is smallest.

1. Introduction. This paper generalizes the Neyman factorization theorem to undominated statistical structures and discusses some related problems on minimality of pairwise sufficient subfields.

A statistical structure means a triplet $(X, \mathscr{A}, \mathscr{P})$ consisting of a space X, a σ-field $\mathscr{A}$ and a family of probability measures $\mathscr{P} = \{P\}$. For any measure m on $\mathscr{A}$, the family of all the m-null sets are written $\mathscr{N}(m)$. Similarly $\mathscr{N}(\mathscr{P})$ denotes the family of all $\mathscr{P}$-null sets, the sets A in $\mathscr{A}$ with $P(A) = 0$ for all P in $\mathscr{P}$. When $\mathscr{N}(m) = \mathscr{N}(n)$ (resp $\mathscr{N}(m) = \mathscr{N}(\mathscr{P})$), we write $m \sim n$ (resp $m \sim \mathscr{P}$). Sub-σ-fields of $\mathscr{A}$ are simply called subfields. For any subfield $\mathscr{B}$ of $\mathscr{A}$, we write $\mathscr{B}^* = \mathscr{B}\mathscr{N}(\mathscr{P})$, $\dot{\mathscr{B}} = \cap\, \{\mathscr{B} \vee \mathscr{N}(P_1 + P_2);\, P_1, P_2 \in \mathscr{P}\}$ and $\hat{\mathscr{B}} = \cap\, \{\mathscr{B} \vee \mathscr{N}(P);\, P \in \mathscr{P}\}$. Here $\mathscr{B} \vee \mathscr{C}$ means the subfield generated by $\mathscr{B}$ and $\mathscr{C}$. For two families of sets $\mathscr{B}$ and $\mathscr{C}$, $\mathscr{B} \subset \mathscr{C}[\mathscr{P}]$ means $\mathscr{B} \subset \mathscr{C}^*$. If $\mathscr{B}^* \supset \mathscr{C}$ also holds, we write $\mathscr{B} = \mathscr{C}[\mathscr{P}]$. This relation "$\subset [\mathscr{P}]$" defines a partial order on the family of all the subfields of $\mathscr{A}$, which we call the partial order (I). Another partial order (II) is introduced by the relation $\mathscr{B} < \mathscr{C}[\mathscr{P}]$, which means that $\mathscr{B} \subset \hat{\mathscr{C}}$. When m is a measure on $\mathscr{A}$, $\mathscr{A}(m)$ denotes the family of all sets A in $\mathscr{A}$ such that $m(A) < \infty$. While $\mathscr{A}l(m)$ denotes the family of all subsets A of X such that $A \cap E \in \mathscr{A}$ for all $E \in \mathscr{A}(m)$, or equivalently, for all E in $\mathscr{A}$ which are σ-finite w.r.t. m. Such sets are called locally-$\mathscr{A}$-measurable w.r.t. m.

All the functions appearing in this paper are extended-real-valued. $I(x; A)$ stands for the indicator function of a set A. A function which is $\mathscr{A}l(m)$-measurable is called locally-$\mathscr{A}$-measurable w.r.t. m.

When $\pi(x)$ denotes any propositional function of x, $[\pi(x)]$ denotes the set of all x in X which satisfy $\pi(x)$. The statement $\pi(x)[m]$ means that $X - [\pi(x)] \subset N$ for some N in $\mathscr{N}(m)$.

Varying conditions, depending on the contexts, will be imposed on statistical structures. One such condition, which will be assumed in Section 2, is the following.

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CONDITION A. There exists a measure m on $\mathscr{A}$ which satisfies:
(a) Every P in $\mathscr{P}$ has a density $p(x, P)$ w.r.t. m,

and

(b) $\mathscr{P} \sim m$.

As a matter of fact, if a measure satisfies (a), there exists another measure satisfying (a) and (b) (see Remark 1.1(a) below).

The measure m is called a dominating measure of $(X, \mathscr{A}, \mathscr{P})$. The densities may be assumed to be nonnegative. Under Condition A, m has the finite subset property, that is, every A in $\mathscr{A}$ with $m(A) = \infty$ has a subset B with $0 < m(B) < \infty$ (see Mussmann (1972, Lemma 2.9)). A statistical structure is called dominated if the dominating measure m is σ-finite. It is discrete if X is an uncountable set, $\mathscr{A}$ is the power set of X, all P are discrete and the only $\mathscr{P}$-null set is the empty set (see Basu and Ghosh (1967)). It is weakly dominated if m is localizable and locally weakly dominated if m is locally localizable (see Definition 2.1). The latter includes the former, which includes the first two cases since a σ-finite measure and the counting measure on the power set are localizable. It is known that weak domination is equivalent to "coherence" due to Hasegawa and Perlman (1974), and "compactness" due to Pitcher (1965) (see Diepenbrock (1971, Theorem 9.1), Mussmann (1972), Theorem 2.13 and Morimoto (1973, Appendix)).

Another condition is concerned with supports of $\mathscr{P}$ which are defined in the following

DEFINITION 1.1. Suppose that $\mathscr{P}$ is a family of probability measures on $(X, \mathscr{A})$ and that $P \in \mathscr{P}$. If there exists a set S in $\mathscr{A}$ which satisfies (a) and (b) below, then S is called a support of P.
(a) $P(S) = 1$, and
(b) $A \in \mathscr{A}$, $A \subset S$ and $P(A) = 0$ imply that $A \in \mathscr{N}(\mathscr{P})$.

In case there exists a support of P for every P in $\mathscr{P}$, we simply say that supports of $\mathscr{P}$ exist, or $\mathscr{P}$ has supports. A support of $\mathscr{P}$ is often denoted by $S(P)$.

CONDITION B. $\mathscr{P}$ has supports.

REMARK 1.1. (a) Conditions A and B are equivalent. In fact, under Condition A, $[p(x, P) > 0]$ is a support of P. That Condition B implies A is Lemma 9.3 of Diepenbrock (1971). Hence, existence of a measure satisfying Condition A(a) implies existence of a measure satisfying Condition A(a) and (b) via Condition B.

(b) If there are two supports S and T of P, then $(S \Delta T) \in \mathscr{N}(\mathscr{P})$.

(c) Under Condition A, if a subfield $\mathscr{B}$ contains a support of P, then for any dominating measure m there exists a density $q(x, P)$ of P w.r.t. m such that $[q(x, P) > 0] \in \mathscr{B}$. If $\mathscr{B}$ contains a support of P for every P in $\mathscr{P}$, then we say that $\mathscr{B}$ contains supports of $\mathscr{P}$.

In Section 2 we give generalizations of the Neyman factorization theorem. A subfield $\mathscr{B}$ is said to have a *Neyman factorization* when each $p(x, P)$ is factored as

$$p(x, P) = g(x, P)h(x) \quad \text{a.e.} \tag{1.1}$$

where $g(x, P)$ is a nonnegative $\mathscr{B}$-measurable function for each P in $\mathscr{P}$ and $h(x)$ is a nonnegative function. The measure to which "a.e." refers and the measurability requirement on $h(x)$ vary from context to context.

Neyman factorization provides a criterion—a necessary and sufficient condition—for sufficiency of a subfield in the dominated case, and for sufficiency of an inducible subfield for the discrete case (Basu and Ghosh (1967), Theorem 2). Mussmann (1971, Theorem 4.5(a)) proved that in the weakly dominated case, sufficiency of a subfield implies Neyman factorization. And, hence, if a subfield includes a sufficient subfield, then it has a Neyman factorization. However, the converse of the last statement is not true unless some additional assumption, e.g., localizability of m on the subfield (see Mussmann (1971, Theorem 4.5(b)), is imposed (for a counter-example see Example 4.1).

In view of this fact we look for some weaker condition than sufficiency which could be equivalent to Neyman factorization. One possible such condition emerges from Theorem 7 of Morimoto (1972), which proves, for the discrete case, that a subfield has a Neyman factorization if and only if it is pairwise sufficient and contains supports of $\mathscr{P}$.

It turns out, vide Theorem 1, that the said condition—"pairwise sufficiency with supports"—is in fact necessary for Neyman factorization under Condition A. Theorems 2, 3 and 4 then show that the converse of it holds in the locally weakly dominated case. It now appears that Neyman factorization is more a criterion for "pairwise sufficiency with supports" than sufficiency itself, which happens to be the same as the former in the dominated case. An immediate consequence of these theorems is the existence of the smallest pairwise sufficient subfields with supports of $\mathscr{P}$ in the locally weakly dominated case.

Problems concerning existence of minimal sufficient subfields have been considered by various authors (e.g., Bahadur (1954), Pitcher (1957, 1965), Burkholder (1961) and Hasegawa and Perlman (1974, 1975)). Our Section 3 deals with similar problems regarding the smallest and minimal subfields in terms of two partial orders introduced above. The partial order (I) is frequently used for sufficient subfields and Burkholder (1961 Corollary 3) showed that for sufficient subfield being minimal is equivalent to being smallest under the partial order (I). What is usually referred to as "minimality" of sufficient subfields is these two equivalent concepts. On the other hand, (II) seems more natural for pairwise sufficient subfields. Moreover, if we restrict our attention to pairwise sufficient subfields, it is shown that $\mathscr{B} < \mathscr{C}[\mathscr{P}]$ iff $\mathscr{B} \subset \mathscr{C}$ and thus (II) coincides with another partial order introduced by Bahadur (1954, page 429). A smallest (resp minimal) subfield in terms of (I) will be called a *smallest* (resp *minimal*) *pairwise sufficient subfield*, while that in terms of (II) will be called a *pairwise smallest* (resp *minimal*) *sufficient subfield*. The relative position of "pairwise" indicates whether it refers to sufficiency alone or to the property of being smallest (resp minimal) as well. It easily follows from the definitions that pairwise minimal sufficiency coincides with pairwise smallest sufficiency, by an argument similar to that of Burkholder (1961, Corollary 3). We construct, in our Theorem 5, under the assumption that $\mathscr{P}$ has supports (Condition B), a subfield which is pairwise smallest sufficient and smallest pairwise sufficient with supports. In fact, it is also shown that the latter property implies the former. Thus the result mentioned in the last part of Section 2 is proved here under a more general condition.

These results hold in the coherent case of Hasegawa and Perlman (1974), as coherence is stronger than Condition A which is equivalent to Condition B. This observation is used in Theorem 6 to fix the gap in Pitcher's (1965) proof for the existence of the smallest sufficient subfield pointed out by Hasegawa and Perlman.

These results are applied to the discrete case in Section 4. It is proved that a subfield is pairwise sufficient if and only if it separates the smallest sufficient statistic. Theorem 7 gives a concrete form of the subfield constructed in Theorem 5. It also shows the existence of infinitely many minimal pairwise sufficient subfields and, hence, nonexistence of a smallest pairwise sufficient subfield. Thus the equivalence of being minimal and being smallest in terms of (I) is disproved for pairwise sufficiency. The relevance of a recent result on minimal separating subfields by Namba (1977) to the problem or characterization of minimal pairwise sufficient subfields is pointed out.

The results presented here were earlier submitted to the Annals in the form of two independent papers, one by Ghosh and another by Yamada and Morimoto, which were subsequently combined to form the present paper. Recently our attention was called by the referee to two related papers (Siebert (1979) and Luschgy (1978)) which appeared in the meantime. The connection between their results and ours is mentioned in Remark 3.1(b) and after Example 4.1.

2. Generalizations of Neyman factorization theorem.

THEOREM 1. *Let $(X, \mathscr{A}, \mathscr{P})$ be a statistical structure and every P in $\mathscr{P}$ have a density $p(x, P)$ w.r.t. a measure m on $\mathscr{A}$. Assume that $\mathscr{P} \sim m$.*

Suppose that a subfield $\mathscr{B}$ has a Neyman factorization:

$$p(x, P) = g(x, P)h(x) \qquad [m] \tag{2.1}$$

where $g(x, P)$ is a nonnegative $\mathscr{B}$-measurable function for each P and $h(x)$ is a nonnegative function such that $h(x) > 0[m]$.

Then $\mathscr{B}$ is pairwise sufficient and contains supports of $\mathscr{P}$.

PROOF. For any two measures P_1 and P_2 in $\mathscr{P}$ define a $\mathscr{B}$-measurable function k as follows:

$$\begin{aligned} k(x) &= g(x, P_1)/(g(x, P_1) + g(x, P_2)) && \text{where the denominator is positive.} \\ &= 0 && \text{otherwise.} \end{aligned}$$

Take any set A in $\mathscr{A}$ and let B be a measurable subset of A such that

$$(P_1 + P_2)(A - B) = 0$$

and for $x \in B$,

$$p(x, P_1) + p(x, P_2) > 0$$
$$g(x, P_1) + g(x, P_2) > 0$$
$$p(x, P_i) = g(x, P_i)h(x), \qquad i = 1, 2.$$

Then $h(x)$ is positive on B. Hence

$$k(x) = p(x, P_1)/(p(x, P_1) + p(x, P_2))$$

on B, so that

$$\begin{aligned} \int_A k(x)d(P_1 + P_2) &= \int_B p(x, P_1)/(p(x, P_1) + p(x, P_2))\, d(P_1 + P_2), \\ &= P_1(B) = P_1(A). \end{aligned}$$

Hence $\mathscr{B}$ is sufficient for $(X, \mathscr{A}, \{P_1, P_2\})$.

Now take any P in $\mathscr{P}$, and define

$$\begin{aligned} q(x, P) &= p(x, P) && \text{if } p(x, P) > 0, \quad g(x, P) > 0 \\ &= 1 && \text{if } p(x, P) = 0, \quad g(x, P) > 0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Note that $[q(x, P) > 0] = [g(x, P) > 0]$ belongs to $\mathscr{B}$. We are to show that q is another version of the density p. As we have assumed that any density is nonnegative, we have

$$[p(x, P) \neq q(x, P)] \subset [p = 0, g > 0] \cup [p > 0, g = 0] \subset [p \neq g \cdot h] \cup [h = 0],$$

so that $m[p(x, P) \neq q(x, P)] = 0$.

For the weakly dominated and the locally weakly dominated cases we need some preliminaries. Let $(X, \mathscr{A}, m)$ be a measure space.

Define a measure $\bar{m}$ on $(X, \mathscr{A}l(m))$ by

$$\bar{m}(A) = \sup(m(A \cap E); E \in \mathscr{A}(m)).$$

That $\bar{m}$ is in fact a measure is noted by Diepenbrock (1971, Chapter 1, Secion 1). Further, if m has the finite subset property, $\bar{m}$ is an extension of m to $\mathscr{A}l(m)$ (Zaanen (1967, page 257)).

DEFINITION 2.1. A measure m on $\mathscr{A}$ is called *localizable* (resp *locally localizable*) if it satisfies the following condition:

Suppose that $\mathscr{F}$ is any subfamily of $\mathscr{A}(m)$. Then there exists an "ess-sup $\mathscr{F}$" w.r.t. m in $\mathscr{A}$ (resp $\mathscr{A}l(m)$), such that:

(a) $m(F - \text{ess}\cdot\sup \mathscr{F}) = 0$ (resp $\bar{m}(F - \text{ess}\cdot\sup \mathscr{F}) = 0$) for all F in $\mathscr{F}$ and

(b) $m(\text{ess}\cdot\sup \mathscr{F} - A) = 0$ (resp $\bar{m}(\text{ess}\cdot\sup \mathscr{F} - A) = 0$) for all A in $\mathscr{A}$ (resp $\mathscr{A}l(m)$) such that $m(F - A) = 0$ for all F in F.

REMARK 2.1. (a) The extension of m to $\bar{m}$ is not unique. Let X be $[0, 1]$, $\mathscr{A}$ the σ-field generated by all the singletons except $\{0\}$ and m be the counting measure on $\mathscr{A}$. Then $\mathscr{A}l(m) = 2^X$ and $\bar{m}$ gives measure 0 to $\{0\}$ and 1 to all other singletons. On the other hand, the counting measure on 2^X is also an extension of m.

(b) Take the same X and $\mathscr{A}$ as in (a) and let m give measure ∞ to all the sets containing the point 0 and 0 to all other sets. It has no finite subset property. The measure $\bar{m}$, then, is equal to 0 for all the subsets of X and is not an extension of m.

(c) The notion of localizability appears in Segal (1951) and Diepenbrock (1971). Segal defines it in a slightly different setup from the present paper and in our framework it could be interpreted either as localizability or as local localizability, whereas Diepenbrock's definition is equivalent to the latter. Here we find it convenient to distinguish the two notions as well as weak domination and local weak domination. Our nomenclature reflects the difference between the two types of concepts as indicated by their definitions and also by some of their properties (see Lemma 2.2).

DEFINITION 2.2. A family $\{f(x, A); A \in \mathscr{A}(m)\}$ is called an *m-cross-section* if it has the following property:

For each A in $\mathscr{A}(m)$, $f(x, A)$ is an $\mathscr{A}$-measurable function such that

(i) $f(x, A) = 0$ outside A, and

(ii) For any A and B in $\mathscr{A}(m)$, it holds that $f(x, A)I(x, A \cap B) = f(x, B)I(x, A \cap B)[m]$.

LEMMA 2.1. (a) (Diepenbrock (1971, Lemma 3.1)). *Suppose that m and n are two measures on $\mathscr{A}$, with the finite subset property. If $m \sim n$, then a σ-finite set* w.r.t. *m is σ-finite* w.r.t. *n and vice versa.*

(b) (Diepenbrock (1971, Theorem 3.2)). *Suppose that m and n are two measures with the finite subset property such that $m \sim n$. Then m is localizable* (resp *locally localizable*) *if and only if n is localizable* (resp *locally localizable*).

LEMMA 2.2. (Zaanen (1967, page 264, Theorem 2)). *Suppose that m is a measure with the finite subset property. Then m is localizable* (resp *locally localizable*) *if and only if to each m-cross-section $\{f(x, A); A \in \mathscr{A}(m)\}$ there exists an $\mathscr{A}$-measurable* (resp $\mathscr{A}l(m)$-*measurable*) *function $f(x)$ such that*

$$f(x)I(x, A) = f(x, A) \qquad [m]$$

for all A in $\mathscr{A}(m)$.

LEMMA 2.3. *Suppose that $(X, \mathscr{A}, \mathscr{P})$ is weakly dominated. Then*

(a) (Mussmann (1971, Theorem 2.4 and Theorem 4.5), Yamada (1976, Theorem 3.1)). *There exists a localizable dominating measure n such that $\mathscr{P} \sim n$ and for each sufficient subfield $\mathscr{B}$, each P has a $\mathscr{B}$-measurable density* w.r.t. n.

(b) (Kusama and Yamada (1972, Corollary 2.1)). *If a subfield $\mathscr{B}$ is pairwise sufficient, then $\dot{\mathscr{B}}$ is sufficient.*

DEFINITION 2.3. We will tentatively call n in (a) a *pivotal measure* for $(X, \mathscr{A}, \mathscr{P})$. The concept, however, will be redefined under a broader context in Definition 2.4.

Now we state a converse of Theorem 1 for the weakly dominated case.

THEOREM 2. *Let $(X, \mathscr{A}, \mathscr{P})$ be weakly dominated by a pivotal measure n and let $\mathscr{B}$ be a subfield.*

Then $\mathscr{B}$ is pairwise sufficient and contains supports of $\mathscr{P}$ if and only if all P in $\mathscr{P}$ have $\mathscr{B}$-measurable densities w.r.t. n.

PROOF. The "if" part is a special case of the preceding theorem. To prove the "only if" part, we note that each P has a density $q(x, P)$ w.r.t. n such that $[q(x, P) > 0]$ belongs to $\mathscr{B}$. By Lemma 2.3(b) the subfield $\dot{\mathscr{B}}$ is sufficient and each P has a $\dot{\mathscr{B}}$-measurable density $r(x, P)$ w.r.t. n. Hence there exists a $\mathscr{B}$-measurable function $s(x, P)$ which satisfies

$$s(x, P) = r(x, P) \qquad [P].$$

This implies that

$$s(x, P) = q(x, P) \qquad [P].$$

Define a $\mathscr{B}$-measurable function $p(x, P)$ by

$$\begin{aligned} p(x, P) &= s(x, P) \qquad \text{if} \quad q(x\ P) > 0, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

It is easy to check that

$$p(x, P) = q(x, P) \qquad [n].$$

REMARK 2.2. There are pairwise sufficient subfields with supports, which do not include any sufficient subfields. So the "only if" part of Theorem 2 is really more general than Mussmann's result cited in Section 1.

Mere pairwise sufficiency does not guarantee Neyman factorization. Therefore we cannot drop the assumption about support.

See Example 4.1 for these points.

Before proceeding to the locally weakly dominated case, we try to relate it to the weakly dominated case.

Suppose that we are given a statistical structure $(X, \mathscr{A}, \mathscr{P})$ which satisfies Condition A and let $\bar{m}$ be the extension of m on $\mathscr{A}l(m)$ defined before. Then by defining $\bar{P}$ on $\mathscr{A}l(m)$ by

$$\bar{P}(A) = \int_A p(x, P)\, d\bar{m}(x)$$

for $A \in \mathscr{A}l(m)$ and putting $\bar{\mathscr{P}} = \{\bar{P};\ P \in \mathscr{P}\}$ we get an extended statistical structure $(X, \mathscr{A}l(m), \bar{\mathscr{P}})$. As we assume $\mathscr{P} \sim m$ in the original structure, it follows that $\bar{\mathscr{P}} \sim \bar{m}$ and thus the new structure satisfies Condition A. Moreover, the next lemma shows that the latter is weakly dominated provided the former is locally weakly dominated.

LEMMA 2.4. *Suppose that $(X, \mathscr{A}, m)$ is a measure space where m is locally localizable and has the finite subset property. Then $\bar{m}$ is a localizable measure on $\mathscr{A}l(m)$.*

PROOF. Take an $\bar{m}$-crosssection $\{f(x, A); A \in \bar{\mathscr{A}}(\bar{m})\}$, where $\bar{\mathscr{A}}(\bar{m})$ denotes all those sets A in $\mathscr{A}l(m)$ such that $\bar{m}(A)$ is finite. Take a set A in $\mathscr{A}(m)$. Then $f(x, A)$ vanishes outside A and hence is $\mathscr{A}$-measurable.

The subfamily $\{f(x, A); A \in \mathscr{A}(m)\}$ is clearly an m-cross-section. Hence, from the local localizability of m, there exists an $\mathscr{A}l(m)$-measurable function $f(x)$, which satisfies

$$f(x)I(x, A) = f(x, A) \qquad [m] \tag{2.2}$$

for all A in $\mathscr{A}(m)$. We shall show that (2.2) holds for all A in $\bar{\mathscr{A}}(\bar{m})$.

From the definition of $\bar{m}$ there exists a set F in $\mathscr{A}(m)$ which satisfies:

(a) F is a subset of A, and

(b) $\bar{m}(A - F) = 0$.

From (a) we have

$$f(x, A)I(x, F) = f(x, F)I(x, F) \qquad [\bar{m}]$$

as $f(x, A)$ and $f(x, F)$ belong to the same $\bar{m}$-cross-section. It now follows from (b) and

$$f(x)I(x, F) = f(x, F) \qquad [\bar{m}]$$

that (2.2) holds for A.

LEMMA 2.5. *Let $(X, \mathscr{A}, \mathscr{P})$ be locally weakly dominated by two locally localizable measures m and n. Then $\mathscr{A}l(m) = \mathscr{A}l(n)$.*

PROOF. This follows immediately from Lemma 2.1 (a).

LEMMA 2.6. *Let $(X, \mathscr{A}, \mathscr{P})$ be locally weakly dominated by m and define $\bar{\mathscr{P}}$ and $\bar{m}$ as before.*

Let $\bar{n}$ be any localizable dominating measure of the weakly dominated statistical structure $(X, \mathscr{A}l(m), \bar{\mathscr{P}})$. Then each $\bar{P}$ in $\bar{\mathscr{P}}$ has an $\mathscr{A}$-measurable density w.r.t. $\bar{n}$.

PROOF. Let $S(P) = [p(x, P) > 0]$, where $p(x, P)$ is any density of P w.r.t. m. $S(P)$ is a σ-finite set w.r.t. $\bar{m}$, and hence w.r.t. $\bar{n}$.

Let $h(x, P)$ be a Radon-Nikodym density of $\bar{m}$ w.r.t. $\bar{n}$ on $S(P)$ and zero outside $S(P)$. Since $h(x, P)$ is $Al(m)$-measurable and vanishes outside a σ-finite set $S(P)$ in $\mathscr{A}$, it is an $\mathscr{A}$-measurable function.

If we take any set A in $\mathscr{A}l(m)$, it follows that

$$\begin{aligned}\bar{P}(A) &= \int_{A \cap S(P)} p(x, P)\, d\bar{m} = \int_{A \cap S(P)} p(x, P)h(x, P)\, d\bar{n} \\ &= \int_A p(x, P)h(x, P)\, d\bar{n},\end{aligned}$$

as required.

LEMMA 2.7. *Assume that $(X, \mathscr{A}, \mathscr{P})$ is locally weakly dominated by m and define $\bar{\mathscr{P}}$ and $\bar{m}$ as before.*

Let $\mathscr{B}$ be a subfield of $\mathscr{A}$, which is pairwise sufficient for $(X, \mathscr{A}, \mathscr{P})$ and contains supports of $\mathscr{P}$. Then it has the same property for $(X, \mathscr{A}l(m), \bar{\mathscr{P}})$.

PROOF. For each P in $\mathscr{P}$ let $p(x, P)$ be a density of P w.r.t. m with $S(P) = [p(x, P)$

$> 0] \in \mathscr{B}$. (see Remark 1.1 (c)). Take $\bar{P}_1$ and $\bar{P}_2$ in $\bar{\mathscr{P}}$ and a $\mathscr{B}$-measurable density $f(x)$ of P_1 w.r.t. $P_1 + P_2$. $S(P_1)$ and $S(P_2)$ are σ-finite w.r.t. m, and so $A \cap (S(P_1) \cup S(P_2))$ belongs to $\mathscr{A}$ for every A in $\mathscr{A}l(m)$. It follows that f is $d\bar{P}_1/d(\bar{P}_1 + \bar{P}_2)$. So $\mathscr{B}$ is pairwise sufficient for $(X, \mathscr{A}l(m), \bar{\mathscr{P}})$.

Now, from the definition of $\bar{P}$, any density of P w.r.t. m serves as a density of $\bar{P}$ w.r.t. $\bar{m}$. So that $p(x, P)$ is a density of $\bar{P}$ w.r.t. $\bar{m}$, whose support belongs to $\mathscr{B}$.

THEOREM 3. *Let $(X, \mathscr{A}, \mathscr{P})$ be locally weakly dominated. Then there exists a locally localizable measure n which satisfies:*

(i) $\mathscr{P} \sim n$,

(ii) *Each P has a density* w.r.t., *n, and*

(iii) *A subfield $\mathscr{B}$ is pairwise sufficient and contains supports of $\mathscr{P}$ if and only if each P has a $\mathscr{B}$-measurable density* w.r.t. *n.*

PROOF. Take a dominating measure m and denote a density of P w.r.t. m by $p(x, P)$. Then, take a pivotal measure $\bar{n}$ of the weakly dominated statistical structure $(X, \mathscr{A}l(m), \bar{\mathscr{P}})$, vide Lemma 2.3 (a) and Lemma 2.4.

By Lemma 2.6 there exists an $\mathscr{A}$-measurable density of $\bar{P}$ w.r.t. $\bar{n}$. It follows that the restriction of $\bar{n}$ to $\mathscr{A}$, which will be denoted by n, is locally localizable by Lemma 2.1 (b). Further, n satisfies (i) and (ii), and the "if" part of (iii) is a special case of Theorem 1.

For the "only if" part, notice that $\mathscr{B}$ is pairwise sufficient for $(X, \mathscr{A}l(m), \bar{\mathscr{P}})$ and contains supports of $\bar{\mathscr{P}}$, by Lemma 2.7. Then by Theorem 2, we have a $\mathscr{B}$-measurable density of $\bar{P}$ w.r.t. $\bar{n}$, which, at the same time, is a density of P w.r.t. n.

COROLLARY 2.1. *Suppose that $(X, \mathscr{A}, \mathscr{P})$ is locally weakly dominated. Then there exists a smallest pairwise sufficient subfield with supports of $\mathscr{P}$.*

The subfield is obviously the one generated by (a version of) all densities of $\mathscr{P}$ with respect to the pivotal measure.

The same result is shown in a more general context in Theorem 5.

We now give a definition of a pivotal measure of a locally weakly dominated statistical structure.

DEFINITION 2.4. A locally localizable measure n is called a pivotal measure of a locally weakly dominated statistical structure $(X, \mathscr{A}, \mathscr{P})$ if it has the following property.

(i) It is a dominating measure of $(X, \mathscr{A}, \mathscr{P})$.

(ii) A subfield $\mathscr{B}$ is pairwise sufficient for $(X, \mathscr{A}, \mathscr{P})$ and contains supports of $\mathscr{P}$ if and only if every P in $\mathscr{P}$ has a $\mathscr{B}$-measurable density w.r.t. n.

This definition agrees with that in Definition 2.3.

For the dominating measures other than the pivotal measures we give

THEOREM 4. *Assume that $(X, \mathscr{A}, \mathscr{P})$ is locally weakly dominated by m and let $\mathscr{B}$ be a subfield of $\mathscr{A}$.*

Then $\mathscr{B}$ is pairwise sufficient for $(X, \mathscr{A}, \mathscr{P})$ and contains the supports of $\mathscr{P}$ if and only if it has a Neyman factorization:

$$p(x, P) = g(x, P)h(x) \qquad [\bar{m}],$$

where $g(x, P)$ is a nonnegative $\mathscr{B}$-measurable function for each P in $\mathscr{P}$ and $h(x)$ is a nonnegative $\mathscr{A}l(m)$-measurable function.

PROOF. The "if" part is proved by applying Theorem 1 to $(X, \mathscr{A}(m), \bar{\mathscr{P}})$. For, $\mathscr{A}(m)$-measurability of h and $\bar{m} \sim \bar{\mathscr{P}}$ imply $h > 0$ $[\bar{m}]$ and thus the assumptions of Theorem 1 are fulfilled by $(X, \mathscr{A}(m), \bar{\mathscr{P}})$. Hence $\mathscr{B}$ is pairwise sufficient for $(X, \mathscr{A}(m), \bar{\mathscr{P}})$, and contains supports of $\bar{\mathscr{P}}$. It follows that $\mathscr{B}$ is pairwise sufficient for $(X, \mathscr{A}, \mathscr{P})$, because each P in $\mathscr{P}$ is the restriction of some $\bar{P}$ in $\bar{\mathscr{P}}$ on $\mathscr{A}$. Further, $\mathscr{B}$ contains supports of $\mathscr{P}$, as the same function $p(x, P)$ serves as a density of P and $\bar{P}$.

For the "only if" part define measures n and $\bar{n}$ as in Theorem 3. Then by Lemma 2.7, $\mathscr{B}$ is pairwise sufficient for $(X, \mathscr{A}(m), \bar{\mathscr{P}})$ and contains supports of $\bar{\mathscr{P}}$. Hence there exists $g(x, P)$, a $\mathscr{B}$-measurable density of $\bar{P}$ w.r.t. $\bar{n}$ as well as P w.r.t. n.

The remainder of the proof is a construction of $h(x)$. It goes the same way as Mussmann (1971, Theorem 4.5). As the reference is unpublished, we sketch an outline of the proof here.

Let us take any set A in $\mathscr{A}(n)$. Since $n \sim m$ and both the measures have the finite subset property, A, being σ-finite w.r.t. n, is also σ-finite w.r.t. m (see Lemma 2.1 (a)). Hence the Radon-Nikodym theorem gives us a density of n w.r.t. m, when both are restricted on A. Write it $k(x, A)$ and define

$$h(x, A) = k(x, A) \quad \text{if} \quad x \in A,$$
$$= 0 \quad \text{otherwise.}$$

It is easy to see that the family of functions

$$\{h(x, A); \quad A \in \mathscr{A}(n)\}$$

is an n-cross-section.

Lemma 2.2 now gives us a nonnegative function $h(x)$ which is $\mathscr{A}(n)$-measurable and hence (by Lemma 2.5) $\mathscr{A}(m)$-measurable and satisfies

$$h(x)I(x, A) = h(x, A) \qquad [n]$$

for all A in $\mathscr{A}(n)$. Thus we have

$$\int I(x, A)h(x)\, d\bar{m} = \int_A h(x)\, d\bar{m} = \int h(x, A)\, dm$$
$$= n(A) < \infty.$$

By a standard technique we have, for any nonnegative $\mathscr{A}$-measurable n-integrable function $f(x)$,

$$\int f(x)h(x)\, d\bar{m} = \int f(x)\, dn. \tag{2.3}$$

We will now prove that

$$\int_A g(x, P)\, h(x)\, d\bar{m} = \int_A p(x, P)\, d\bar{m} \tag{2.4}$$

for all A in $\mathscr{A}(m)$ which would give a desired factorization. Note that (2.3) implies (2.4) for A in $\mathscr{A}$. Since $[g(x, P) > 0]$ can be written as $\sum E_i$ where $E_i \in \mathscr{A}$, (2.4) clearly holds for all A in $\mathscr{A}(m)$.

REMARK 2.3. (a) In case $(X, \mathscr{A}, \mathscr{P})$ is weakly dominated, m can be assumed to be localizable. Then $h(x)$ can be taken to be $\mathscr{A}$-measurable.

(b) However, as Example 4.4 shows, $\mathscr{A}$-measurability of $h(x)$ does not follow in general.

EXAMPLE 2.1. Let X be a two-dimensional Euclidean space whose generic point is (y, z), and $\mathscr{A}$ be the family of sets A such that all of its y-sections, denoted by A_y, are Borel sets. Define a localizable measure m on $\mathscr{A}$ by

$$m(A) = \sum \{l(A_y); -\infty < y < \infty\},$$

where l is the linear Lebesgue measure.

Now, let $\theta = (\xi, \eta)$ be a two-dimensional parameter and to each θ such that $-\infty < \xi, \eta < \infty$, define P_θ by

$$\begin{aligned} p((y, z), P_\theta) &= \exp[-(z-\eta)^2/2] \quad &&\text{if } y = \xi, \\ &= 0 &&\text{otherwise.} \end{aligned}$$

Let $\mathscr{B}$ be the family of all those sets A in $\mathscr{A}$ such that $A_y = \varnothing$ except for a countable number of y and their complements. Then $\mathscr{B}$ has a Neyman factorization, is pairwise sufficient and contains supports of $\mathscr{P} = \{P_\theta; \theta \in \Theta\}$. $\mathscr{A}$ itself is the smallest sufficient subfield.

The following two examples, both having nonlocally localizable dominating measures, suggest possibility of extending our results further to some more general cases.

EXAMPLE 2.2. Consider the concept of uniform distributions from a wider scope. Let $(X, \mathscr{A}, m)$ be a measure space and $\mathscr{F} = \{F(\theta); \theta \in \Theta\}$ be any subfamily of $\mathscr{A}(m)$ and let $m(F(\theta)) > 0$ for all θ in Θ. Define a probability measure P_θ for each θ in Θ by

$$P_\theta(A) = m(A \cap F(\theta))/m(F(\theta))$$

for all A in $\mathscr{A}$. Let us call $\mathscr{P} = \{P_\theta; \theta \in \Theta\}$ the family of uniform distributions on $\mathscr{F}$, w.r.t. m.

Here the subfield generated by $\mathscr{F}$, denoted by $\mathscr{D}$, is the smallest pairwise sufficient subfield with supports. The smallest sufficient (or pairwise sufficient) subfield does not necessarily exist, as Example 2 of Basu and Ghosh (1967) and our Example 4.4 illustrate. In this case Neyman factorization is necessary and sufficient for pairwise sufficiency with supports without any condition on the dominating measure m.

EXAMPLE 2.3. A measure given by Halmos (1950, page 131) serves as an example of a nonlocalizable dominating measure. Suppose that Y and Z are uncountable sets and that $|Y| < |Z|$. In the product space $Y \times Z$, a set of the form $L(b) = \{(y, b); y \in Y\}$, $(b \in Z)$, is called a horizontal line, and $M(a) = \{(a, z); z \in Z\}$, $(a \in Y)$, is called a vertical line. A subset A of $Y \times Z$ is called full on $L(b)$ (or $M(a)$) if $L(b) - A$ (resp $M(a) - A$) is countable. Following Halmos we define a σ-field of a subsets of $Y \times Z$ by

$$\mathscr{A} = \{A; A \text{ is full or countable on each horizontal or vertical line}\},$$

and a measure n on $\mathscr{A}$ by

$$n(A) = \text{The number of horizontal and vertical lines on which } A \text{ is full.}$$

It is easily seen that n is not σ-finite, that it has the finite subset property and that local $\mathscr{A}$-measurability w.r.t. it coincides with $\mathscr{A}$-measurability. We further prove that n is not localizable and, hence, is not locally localizable.

Take a set A in $\mathscr{A}(n)$. Then the number of horizontal lines on which A is full is finite. Let their union be denoted by L and define $f(x, A) = I(x, A \cap L)$. Naturally $\{f(x, A); A \in \mathscr{A}(n)\}$ is an n-cross-section.

If we assume that n is localizable, then there exists an $\mathscr{A}$-measurable function $f(x)$ which satisfies

$$f(x)I(x, A) = f(x, A) \qquad [n]$$

for each A in $\mathscr{A}(n)$. Denote by E the set on which $f(x)$ is nonzero. By taking $L(b)$ as A we see that E is full on any horizontal line. Hence $|E| \geq |Z| (|Y| - \aleph_0) \geq |Z|$. On the other hand, by taking $M(a)$ as A we see E is countable on any vertical line. Hence, $|E| \leq |Y|\aleph_0 \leq |Y|$, which is a contradiction.

We can construct a "kth product" of the measure n on $X = \prod_{i=1}^{k} (Y_i \times Z_i)$, as follows. Denote by $x = (y_1, z_1; \cdots; y_k, z_k)$ a generic point of X. Define $\mathscr{A}$ and a measure m on $\mathscr{A}$ in an analogous way:

$\mathscr{A} = \{A$; A is full or countable on each $(2k - 1)$-dimensional hyperplane which is orthogonal to a coordinate axis$\}$,

$$m(A) = \text{The number of such hyperplanes on which } A \text{ is full.}$$

The proof that m is not locally localizable goes the same way as above, except that notations are more cumbersome.

A family $\mathscr{P} = \{P_\theta; \theta \in \Theta\}$ of probability measures having m as a dominating measure is defined as follows. First we define a function $q(y, z; \theta)$ on $Y \times Z$ by

$$q(y, z; \theta) = I(y, \alpha_1)\xi_1 + I(y, \alpha_2)\xi_2 + I(z, \beta_1)\eta_1 + I(z, \beta_2)\eta_2,$$

where $\theta = (\alpha_1, \alpha_2, \beta_1, \beta_2, \xi_1, \xi_2, \eta_1, \eta_2)$, α_1 and α_2 are two distinct points in Y, β_1 and β_2 are two distinct points in Z, ξ_1, ξ_2, η_1 and η_2 are nonnegative, $\xi_1 + \xi_2 + \eta_1 + \eta_2 = 1$, and $I(y, \alpha_1) = 1$ if $y = \alpha_1$ and $= 0$ otherwise, etc. Now define P_θ by $\dfrac{dP_\theta}{dm} = \prod_{i=1}^{k} q(y_i, z_i; \theta)$. The statistical structure thus constructed, with a sufficiently large k, has a sufficient statistic which is calculated as follows:

Look at y_i's in x and see if there are two different y-values each assumed by more than two y_i's. In that event, record the two values and the number of y_i's assuming each of them. Otherwise, record all the different y-values in x and the number of y_i's assuming each of them. The statistic thus calculated is denoted by $u(y_1, \cdots, y_k)$. Then look at z_i's and define $v(z_1, \cdots, z_k)$ similarly. The statistic $t(x) = (u(y_1, \cdots, y_k), v(z_1, \cdots, z_k))$ is a sufficient statistic and has a Neyman factorization.

3. Minimality of pairwise sufficient subfields. Take any two measures P_1 and P_2 in $\mathscr{P}$. Throughout this section let $r(x; P_1, P_2)$ be any fixed density of P_1 w.r.t. $(P_1 + P_2)$.

THEOREM 5. *Assume that every P in $\mathscr{P}$ has a support $S(P)$. Then the subfield $\mathscr{D}$ generated by all the functions*

$$I(x; S(P)) \qquad P \in \mathscr{P},$$

and

$$r(x; P_1, P_2)I(x; S(P_1) \cup S(P_2)), \; P_1, P_2 \in \mathscr{P}$$

is a smallest pairwise sufficient subfield with supports of $\mathscr{P}$, and is a pairwise smallest sufficient subfield.

PROOF. First we prove that $\mathscr{D}$ is a smallest pairwise sufficient subfield with supports of $\mathscr{P}$.

Let $\mathscr{B}$ be a pairwise sufficient subfield which contains a support $T(P)$ for each P in $\mathscr{P}$. There exists a $\mathscr{B}$-measurable density $s(x; P_1, P_2) = dP_1/d(P_1 + P_2)$ for any P_1 and P_2 in $\mathscr{P}$. Then $S(P) \,\Delta\, T(P) \in \mathscr{N}(\mathscr{P})$ and it follows that for any real a

$$[r(x; P_1, P_2)I(x; S(P_1) \cup S(P_2)) > a] \,\Delta\, [s(x; P_1, P_2)I(x; T(P_1) \cup T(P_2)) > a] \in \mathscr{N}(\mathscr{P})$$

which shows that $\mathscr{B} \supset \mathscr{D} \, [\mathscr{P}]$.

Now we prove that pairwise smallest sufficiency of any subfield $\mathscr{C}$ follows from its being smallest pairwise sufficient with supports. Let $\mathscr{B}$ be a pairwise sufficient subfield and write $\mathscr{S} = \{S(P);\ P \in \mathscr{P}\}$. We are to show that $\mathscr{B} > \mathscr{B} \vee \mathscr{S}\,[\mathscr{P}]$, and it is enough to show that for any $S(P)$ in $\mathscr{S}$ and any P_1 and P_2 in $\mathscr{P}$ there exists a set B in $\mathscr{B}$ such that $(P_1 + P_1)(B \,\Delta\, S(P)) = 0$.

Noting that $\mathscr{B}$ is pairwise sufficient for the family of all the finite convex combinations of measures in $\mathscr{P}$, we have a function $f(x)$, a $\mathscr{B}$-measurable density of P w.r.t. $(P_1 + P_2 + P)$. Let $B = [f(x) > 0]$. Then we have

$$\int_{B \cap (X - S(P))} f(x)\, d(P_1 + P_2 + P) = P(B \cap (X - S(P))) = 0,$$

and, since $f(x) > 0$ in B, $(P_1 + P_2)(B \cap (X - S(P)) = 0$. On the other hand, $P((X - B) \cap S(P)) \leq P(X - B) = 0$. By the definition of a support this implies $(P_1 + P_2)((X - B) \cap S(P)) = 0$, so that $(P_1 + P_2)(B \,\Delta\, S(P)) = 0$.

It follows that

$$\mathscr{B} > \mathscr{B} \vee \mathscr{S} \supset \mathscr{C} \qquad [\mathscr{P}]$$

because $\mathscr{B} \vee \mathscr{S}$ is pairwise sufficient and contains supports of $\mathscr{P}$.

Remark 3.1. (a) In view of the equivalence of Conditions A and B and the paragraph following Condition A in Section 1, Theorem 5 holds true under each of the conditions mentioned in the said paragraph, including coherence.

(b) A recent paper by Siebert (1979) includes a proof of the same subfield being pairwise smallest sufficient. The same paper and a paper by Luschgy (1978) gave examples of nonexistence of pairwise smallest sufficient subfields.

We quote here the following lemma by Hasegawa and Perlman (1974, Lemma 3.3).

Lemma 3.1. *Assume that $(X, \mathscr{A}, \mathscr{P})$ is coherent. If a subfield $\mathscr{B}$ is sufficient, then $\mathscr{B}^* = \dot{\mathscr{B}}$.*

Remark 3.2. If we apply this lemma to each pair $\{P_1, P_2\}$ in $\mathscr{P}$, then we have $\dot{\mathscr{B}} = \hat{\mathscr{B}}$, provided that $\mathscr{B}$ is pairwise sufficient.

Proposition 3.1. *Assume that $(X, \mathscr{A}, \mathscr{P})$ is coherent and $\mathscr{B}$ is sufficient for $(X, \mathscr{A}, \mathscr{P})$. Then*

$$\mathscr{B}^* \supset \hat{\mathscr{D}}.$$

Proof. Since $\mathscr{D}$ is pairwise smallest sufficient (vide Remark 3.1(a)), $\mathscr{D} \subset \mathscr{B} \vee \mathscr{N}(P_1 + P_2)$ for all P_1 and P_2 in $\mathscr{P}$. So by Remark 3.2, $\mathscr{D} \subset \hat{\mathscr{B}} = \dot{\mathscr{B}}$. Hence $\hat{\mathscr{D}} \subset \hat{\mathscr{B}} = \dot{\mathscr{B}} = \mathscr{B}^*$.

Theorem 6. *Assume that $(X, \mathscr{A}, \mathscr{P})$ is coherent. Then $\hat{\mathscr{D}}$ is smallest sufficient.*

The proof of this is the same as that of Theorem 2.5 of Pitcher (1965), except that we use the foregoing proposition to complete the gap pointed out by Hasegawa and Perlman (1974). Since the proof is short it is reproduced here for the sake of completeness.

Proof. $\hat{\mathscr{D}}$ is sufficient by the argument given in the proof of Theorem 2.5 of Pitcher (1965). The proof is completed by appealing to Proposition 3.1.

4. The discrete case. Let X be an uncountable set, $\mathscr{A}$ the power set of X, all P be discrete and only $\mathscr{P}$-null set be the empty set. This case falls in the case of weak domination if we take the dominating measure m to be the counting measure and $p(x, P)$ as the probability function of P. A *statistic* is defined as a class of mutually disjoint nonempty sets which collectively cover X (for details see Morimoto (1972 or 1973)).

DEFINITION 4.1. A subfield $\mathscr{B}$ *separates* a statistic $\mathscr{T} = \{T\}$ if for any two sets T and S in $\mathscr{T}$ there exists a set B in $\mathscr{B}$ which contains T and does not meet S.

DEFINITION 4.2. A subfield $\mathscr{B}$ is *separating* if for any pair of points in X there exists a set in $\mathscr{B}$ which contains one and only one of them.

Basu and Ghosh (1967) proved the existence of the "smallest sufficient statistic" denoted henceforth by $\mathscr{M} = \{M\}$. They showed that all the unions of its sets constitute the smallest sufficient subfield. It was shown that two points x and y belong to a same set in $\mathscr{M}$ if and only if

$$(4.1)\quad p(x, P) > 0 \Leftrightarrow p(y, P) > 0 \qquad \text{for all } P \text{ in } \mathscr{P}, \text{ and}$$

$$p(x, P)/p(y, P) \quad \text{is independent of } P.$$

LEMMA 4.1. *Let $(X, \mathscr{A}, \mathscr{P})$ be a discrete statistical structure. A subfield $\mathscr{B}$ is pairwise sufficient if and only if it separates $\mathscr{M}$.*

PROOF. The "only if" part is Theorem 3 of Morimoto (1972). Let us take up the "if" part. Take P_1 and P_2 in $\mathscr{P}$ and define

$$q(x) = p(x, P_1)/(p(x, P_1) + p(x, P_2))$$

on the countable set $[p(x, P_1) + p(x, P_2) > 0]$. Because of (4.1), $q(x)$ is constant on each set in $\mathscr{M}$. Therefore $[p(x, P_1) + p(x, P_2) > 0]$ is a countable union of sets in $\mathscr{M}$, say $= \cup_{i=1}^{\infty} M(i)$.

We can get a disjoint sequence $\{D(n); n = 1, 2, \cdots\}$ such that $\cup_{n=1}^{\infty} D(n) = X$, $D(n)$ contains $M(n)$, meets no other $M(i)$, and belongs to $\mathscr{B}$, for each n.

Now extend $q(x)$ to X in such a way that it is constant on each $D(n)$. The function $q(x)$ thus extended is a $\mathscr{B}$-measurable density of P_1 w.r.t. $P_1 + P_2$, so that $\mathscr{B}$ is pairwise sufficient.

The following lemma is due to B. V. Rao.

LEMMA 4.2. *Let $\mathscr{D}$ be the subfield generated by $\mathscr{M}$ and $\mathscr{D}(M)$ be the subfield generated by all the sets in $\mathscr{M}$ except M. Suppose that $\mathscr{B}$ is a proper subfield of $\mathscr{D}$. Then $\mathscr{B}$ separates $\mathscr{M}$ if and only if $\mathscr{B} = \mathscr{D}(M)$ for some M in $\mathscr{M}$.*

PROOF. Suppose $\mathscr{B}$ separates $\mathscr{M}$. As $\mathscr{B} \neq \mathscr{D}$ there exists M in $\mathscr{M}$ which does not belong to $\mathscr{B}$. We shall show that any set K in $\mathscr{M}$ other than M belongs to $\mathscr{B}$. As $\mathscr{B}$ separates $\mathscr{M}$, there is a set B in $\mathscr{B}$ which contains K and does not meet M. Since $B \in \mathscr{D}$, either B or $X - B$ is a countable union of sets in $\mathscr{M}$. In the latter case, however, from the separating property of $\mathscr{B}$ it would contain M, contrary to the assumption. So B is a countable union of sets in $\mathscr{M}$, say, $= \cup_{i=1}^{\infty} M(i)$. For each $M(i)$ there exists a set $B(i)$ in $\mathscr{B}$ which contains K and does not meet $M(i)$. The set $(\cap_{i=1}^{\infty} B(i)) \cap B$, obviously in $\mathscr{B}$, is equal to K.

The converse is obvious.

THEOREM 7. *Let $(X, \mathscr{A}, \mathscr{P})$ be a discrete statistical structure and let $\mathscr{M}$ be the smallest sufficient statistic. Then*

(a) *The subfield $\mathscr{D}$ generated by $\mathscr{M}$ is the smallest pairwise sufficient subfield with supports, and hence is pairwise smallest sufficient.*

(b) *The subfields generated by all but any one of the sets of $\mathscr{M}$ are minimal pairwise sufficient.*

(c) *No smallest pairwise sufficient subfield exists.*

PROOF. (a) follows immediately from Theorem 5 since the $\mathscr{D}$ of Theorem 5 is identical with the $\mathscr{D}$ of Theorem 7. The remaining parts follow from Lemmas 4.1 and 4.2.

REMARK 4.1. Intersection of two pairwise sufficient subfields may not be pairwise sufficient, unlike the case of sufficiency (cf., Burkholder (1961, Theorem 4)). For example, take two such subfields as described in (b). Their intersection does not separate $\mathscr{M}$.

REMARK 4.2. The subfields described in (b) are pairwise sufficient and do not have Neyman factorization.

REMARK 4.3. Let us see what happens if 2^X is replaced by some subfield $\mathscr{A}$. Assume only that $\mathscr{A}$ is separating.

Let $\{p(x, \theta);\ \theta \in \Theta\}$ be a family of nonnegative functions $p(x, \theta)$ on X which assign positive values to a countable number of points and satisfies $\sum \{p(x, \theta);\ x \in X\} = 1$. Define $\mathscr{P} = \{P_\theta;\ \theta \in \Theta\}$, where $P_\theta(A) = \sum \{p(x, \theta);\ x \in A\}$ for all A in $\mathscr{A}$ and for all θ in Θ. Let $\mathscr{M}$ and $\mathscr{D}$ be as before. The counting measure m on $\mathscr{A}$ is locally localizable, because $\mathscr{A}l(m) = 2^X$.

Here again, a subfield $\mathscr{B}$ is pairwise sufficient if and only if it separates $\mathscr{M}$. In fact, if $\mathscr{B}$ separates $\mathscr{M}$ it is pairwise sufficient for $(X, 2^X, \mathscr{P})$ and, hence, for $(X, \mathscr{A}, \mathscr{P})$. This proves the "if".

The "only if" part is proved as follows. As $\mathscr{A}$ is separating and all the sets in $\mathscr{M}$ are countable sets, $\mathscr{A}$ separates $\mathscr{M}$. By Lemma 4.1, $\mathscr{A}$ is pairwise sufficient for $(X, 2^X, \mathscr{P})$. It follows from pairwise sufficiency of $\mathscr{B}$ for $(X, \mathscr{A}, \mathscr{P})$ that $\mathscr{B}$ is pairwise sufficient for $(X, 2^X, \mathscr{P})$. Hence, again by Lemma 4.1, $\mathscr{B}$ separates $\mathscr{M}$.

If we assume further that $\mathscr{A}$ contains all the singletons, it follows that m is not localizable, $\mathscr{A}$ includes $\mathscr{D}$, $p(x, \theta)$ are $\mathscr{A}$-measurable and $(X, \mathscr{A}, \mathscr{P})$ satisfies the assumptions of Theorem 4. (Incidentally, if $\mathscr{A}$ contains no singletons, then m is localizable, because $\mathscr{A}l(m) = \{\varnothing\}$.) Theorem 7 remains true in this case. For, as we have seen, any subfield of $\mathscr{A}$ which is pairwise sufficient for $(X, \mathscr{A}, \mathscr{P})$ is pairwise sufficient for $(X, 2^X, \mathscr{P})$, so that the subfields which are proved to be smallest or minimal pairwise sufficient under $(X, 2^X, \mathscr{P})$ have the same property under $(X, \mathscr{A}, \mathscr{P})$. The same property is not true for sufficiency and there may not exist a smallest sufficient subfield (see Example 4.4 and Example 2 of Basu and Ghosh (1967)). A subfield has a Neyman factorization if and only if it includes $\mathscr{D}$.

EXAMPLE 4.1. As the simplest case which, however, retains all the essential features of the discrete case, let us take up the family of all the one-point distributions on X. That is, we define $\mathscr{P} = \{P_\theta;\ \theta \in X\}$ where $P_\theta(\{\theta\}) = 1$. Taking the counting measure m as the dominating measure we have

$$p(x, P_\theta) = \frac{dP_\theta}{dm} = 1 \qquad x = \theta$$

$$= 0 \qquad \text{otherwise}$$

The subfield $\mathscr{D}$ given in Theorem 7(a) is now the family of all countable and cocountable sets. For all P_θ in P, $p(x, P_\theta)$ is $\mathscr{D}$-measurable, so that $\mathscr{D}$ has a Neyman factorization. $\mathscr{D}$ contains no sufficient subfield, as the smallest sufficient subfield is $\mathscr{A} = 2^X$.

Let a be a fixed point in X and let $\mathscr{D}(a)$ be the subfield generated by all the singletons except a. It is the specialization of $\mathscr{D}(M)$ in Theorem 7(b) to this case. It does not contain the support of P_a which is $\{a\}$, and it does not have Neyman factorization as $p(x, a)$ cannot be factored as is required. This shows that the assumptions in Theorem 2, 3 and 4 concerning supports cannot be deleted.

The foregoing example was treated also by Luschgy (1978, Example 1).

We now ask whether there exist minimal pairwise sufficient subfields other than those given in Theorem 7(b). Take up the preceding example again and note that a subfield is pairwise sufficient if and only if it separates points. We wish to find out if there exist minimal separating subfields other than those generated by all but one singletons.

Our question, which has now taken a set theoretical appearance, was answered recently by Namba (1977). He obtained, among other things, a fine representation theorem which completely charcterizes the minimal separating subfields, and found an example of such a subfield that does not contain any singleton. The following example will illustrate his results in the present context.

EXAMPLE 4.2. Let $I = \{i\}$ be an uncountable set of indices i and 2^I be the space of all functions on I to $\{0, 1\}$. Points in 2^I are written $x = (x(i); i \in I)$, or simply $x(i)$ and O will be the point such that $O(i) = 0$ for all i in I.

Now we define, for each nonnegative integer n, X_n to be the set of all points x such that $x(i) = 1$ for at most n indices i in I. Put $X = \bigcup_{n=1}^{\infty} X_n$. A subset B of X is called a clopen set if there exists a countable subset $K(B)$ of I which will be called a support of B, such that $x \in B$ and $x(i) = y(i)$ for all i in $K(B)$ imply $y \in B$. Let $\mathscr{B}$ be the family of all such sets. It is a separating subfield of 2^X without any singleton.

We further prove the following property of X equipped with $\mathscr{B}$ which will be called ω_1-compactness: Suppose that to every x in X there corresponds a "neighbourhood", i.e., a set $B(x)$ in $\mathscr{B}$ of the form $B(x) = \{y; y(i) = x(i), i \in K(B(x))\}$. Then we can choose a countable number of points x_k, $k = 0, 1, \cdots$ such that $\bigcup_{k=0}^{\infty} B(x_k) = X$.

It is enough to prove the same property for all X_n. We do this by induction. Take X_1 and write $K = K(B(0))$. If a point x does not belong to $B(0)$ then $x(i) = 1$ for some i in K, so that there is a countable number of such points. Let them be $x_1, x_2, \cdots, x_k, \cdots$ and take 0 as x_0. Thus the ω_1-compactness of X_1 is established.

Next, assume that $X_1, X_2, \cdots, X_n$ are ω_1-compact and consider X_{n+1}. Notice again that a point x outside $B(0)$ satisfies $x(i) = 1$ for some i in K. $B(0)$ and $K(0)$ are different from those considered relative to X_1 or X, but we use the same symbols. Take an integer p such that $1 \le p \le n + 1$ and a point d such that $d(i) = 1$ for p indices i in K and define $D = \{x; x(i) = d(i), i \in K\}$. Since $X_{n+1} - B(0)$ is a countable union of such sets, it suffices to show that D is covered by a countable number of neighbourhoods.

For any x in X let x' be the "restriction" of x on $J = I - K(0)$, i.e., $x' = (x(i); i \in J)$ and $Z = \{x'; x \in D\}$. Then Z is the set of all functions on J to $\{0, 1\}$ whose value is 1 for at most $n + 1 - p$ indices in J. As the cardinality of J is the same as that of I, the assumption of induction on X_{n+1-p} applies to Z, giving rise to a countable number of points whose neighbourhoods collectively cover Z. Let them be $x_1', x_2', \cdots$. Then take all those points x in D whose restrictions on J coincide with any of x_q', $q = 1, 2, \cdots$ and let them be denoted by $x_1, x_2, \cdots, x_k, \cdots$. It is clear that $\bigcup_{k=1}^{\infty} B(x_k) = D$.

This amounts to proving that $\mathscr{B}$ is minimal separating, since the theorem of Namba states (in a more extensive set up) that ω_1-compactness of X defined above is a necessary and sufficient condition for $\mathscr{B}$ to be minimal separating. More generally, let Y be any space, $\mathscr{A}$ a separating σ-field and $\mathscr{F} = \{F_i; i \in I\}$ be its generator. There is a natural correspondence between a point x in X and a function $(x(i); i \in I)$ on I to $\{0, 1\}$ defined by $x(i) = 1 \Leftrightarrow x \in F_i$. Through this correspondence Y and $\mathscr{A}$ are conveniently identified with a subset

X of 2^I and the family $\mathscr{B}$ of all the "clopen" sets in it. The minimality of $\mathscr{A}$ is thus checked through ω_1-compactness of X.

R. V. Ramamoorthi and B. V. Rao gave us the following example of a σ-field which is separating but has no minimal separating subfield. In view of Remark 4.3 pairwise sufficiency of a subfield is equivalent to its separating property in this case. So, it serves as an example with no minimal pairwise sufficient subfield.

EXAMPLE 4.3. Take $X = 2^I$ as in the previous example and let $\mathscr{A}$ be the family of all the clopen subsets of X and $\mathscr{P}$ be the family of all one-point probability measures. Let $\mathscr{B}$ be any separating subfield of $\mathscr{A}$ and let us prove that $\mathscr{B}$ has a separating proper subfield. In what follows, $\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots$ will be either 0 or 1. Define two points $x(i; \epsilon_1)$ $\epsilon_1 = 0, 1$ in X by $x(i; 0) = O$ *and* $x(i; 1) = I$, where $I(i) = 1$ for all i in I. By the separating property of $\mathscr{B}$ there exists $B(\epsilon_1) \in \mathscr{B}$, $\epsilon_1 = 0, 1$, such that $x(i; \epsilon_1) \in B(\epsilon_1)$, $\epsilon_1 = 0, 1$, $B(0) \cap B(1) = \varnothing$ and $B(0) \cup B(1) = X$. Let $K(1)$ denote a common support of $B(0)$ and $B(1)$.

Define four points $x(i; \epsilon_1, \epsilon_2)$ by

$$\begin{aligned} x(i; \epsilon_1, \epsilon_2) &= \epsilon_1 \qquad i \in K(1) \\ &= \epsilon_2 \qquad i \in I - K(1). \end{aligned}$$

There exists four mutually disjoint sets $B(\epsilon_1, \epsilon_2) \in B$ such that $x(i; \epsilon_1, \epsilon_2) \in B(\epsilon_1, \epsilon_2)$ for all ϵ_1 and ϵ_2 and $\cup \{B(\epsilon_1, \epsilon_2); \epsilon_2 = 0, 1\} = B(\epsilon_1)$ for $\epsilon_1 = 0, 1$. Let $K(2)$ denote a common support of $B(\epsilon_1, \epsilon_2)$, $\epsilon_1, \epsilon_2 = 0, 1$ and assume, without loss of generality, that $K(1) \subset K(2)$. Then define eight points $x(i; \epsilon_1, \epsilon_2, \epsilon_3)$ by

$$\begin{aligned} x(i; \epsilon_1, \epsilon_2, \epsilon_3) &= \epsilon_1 \qquad i \in K(1) \\ &= \epsilon_2 \qquad i \in K(2) - K(1) \\ &= \epsilon_3 \qquad i \in I - K(2), \end{aligned}$$

and find out eight disjoint sets $B(\epsilon_1, \epsilon_2, \epsilon_3)$ in a similar way as above and proceed iteratively. We get a branching sequence of sets $B(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ and increasing sequence of sets of indices $(K(n))$. As $\bigcup_{n=1}^{\infty} K(n)$ is a countable set, for each sequence $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n, \cdots)$ $\bigcap_{n=1}^{\infty} B(\epsilon_1, \epsilon_2, \cdots \epsilon_n)$ is not empty and belongs to $\mathscr{B}$, with a support $\bigcup_{n=1}^{\infty} K(n)$. Let it be denoted by $B(\epsilon)$.

Define $\mathscr{C}$ to be the family of all those sets C in $\mathscr{B}$ such that either C or $X - C$ is included by a countable union of the sets $B(\epsilon)$. $\mathscr{C}$ is a separating subfield of $\mathscr{B}$. For, if x and y are two points in X, then there exists a set B in $\mathscr{B}$ such that $x \in B$ and $y \bar{\in} B$. Take the set $B(\epsilon)$ which contains x. Then $B \cap B(\epsilon)$ belongs to $\mathscr{C}$, contains x and does not contain y. Moreover, $\mathscr{C}$ is a proper subfield of $\mathscr{B}$, as it does not contain $B(0)$.

T. Kamae drew our attention to the following example.

EXAMPLE 4.4. Using the same notations as in Remark 4.3, let $X = (-1, 1)$, $\mathscr{A}$ = the Borel field of X, $\Theta = (0, 1)$ and for each θ in Θ let $p(x, \theta)$ satisfy the following conditions.

$$\begin{aligned} &p(\theta, \theta) > 0 \quad \text{and} \quad p(-\theta, \theta) > 0 \\ &p(x, \theta) = 0 \quad \text{if} \quad x \neq \theta \quad \text{and} \quad -\theta \\ &p(\theta, \theta) + p(-\theta, \theta) = 1. \end{aligned}$$

Thus each P_θ is a two point measure with probabilities $p(-\theta, \theta)$ on $-\theta$ and $p(\theta, \theta)$ on θ.

Let $\mathscr{D}$ denote the family of all countable and cocountable symmetric sets. Then by Remark 4.3, $\mathscr{D}$ is a smallest pairwise sufficient subfield with supports of $\mathscr{P}$. If we define

$$\begin{aligned} g(x,\theta) &= p(\theta,\theta) && \text{if} \quad x=\theta \quad \text{or} \quad -\theta \\ &= 0 && \text{otherwise, and} \\ h(x) &= 1 && \text{if} \quad x \geq 0 \\ &= p(x,-x)/p(-x,-x) && \text{if} \quad x<0, \end{aligned}$$

then

$$p(x,\theta) = g(x,\theta)h(x) \tag{4.2}$$

where $g(x,\theta)$ is $\mathscr{B}$-measurable, but if $p(\theta,\theta)$ is non-Borel, $h(x)$ is measurable, not w.r.t. $\mathscr{A}$ but only w.r.t. $2^X = \mathscr{A}(m)$. This is the case for any factorization of the form (4.2), which shows that we cannot have $\mathscr{A}$-measurability of $h(x)$ in Theorem 4. Incidentally, the family of all symmetric Borel sets is sufficient if and only if $p(\theta,\theta)$ is a Borel function and in case it is non-Borel, no smallest sufficient subfield exists.

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