ON DEVIATIONS OF THE SAMPLE MEAN

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1. Introduction. Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables. Let a be a constant, $-\infty < a < \infty$, and for each $n = 1, 2, \cdots$ let

$$p_n = P\left(\frac{X_1 + \cdots + X_n}{n} \geqq a\right). \tag{1}$$

It is assumed throughout the paper that the distribution of X_1 and the given constant a satisfy the conditions stated in the following paragraph. These conditions imply that $p_n > 0$ for each n, and that $p_n \to 0$ as $n \to \infty$. The object of the paper is to obtain an estimate of p_n, say q_n, which is precise in the sense that

$$q_n/p_n = 1 + o(1) \quad \text{as} \quad n \to \infty. \tag{2}$$

Let t be a real variable, and let $\varphi(t)$ denote the moment generating function (m.g.f.) of X_1, i.e., $\varphi(t) = E(e^{tX_1})$, $0 < \varphi \leqq \infty$. Define

$$\psi(t) = e^{-at}\varphi(t). \tag{3}$$

Let T denote the set of all values t for which $\varphi(t) < \infty$. *We suppose that* $P(X_1 = a) \neq 1$, *that* T *is a non-degenerate interval, and that there exists a positive* τ *in the interior of* T *such that* $\psi(\tau) = \inf_t |\psi(t)| = \rho$ (say). These conditions are satisfied if, for example, $\varphi(t) < \infty$ for all t, $E(X_1) = 0$, $a > 0$, and $P(X_1 > a) > 0$. In any case, τ and ρ are uniquely determined by

$$\frac{\varphi'(\tau)}{\varphi(\tau)} = a \quad \text{and} \quad \rho = \psi(\tau), \tag{4}$$

where $\varphi' = d\varphi/dt$, and we have $0 < \rho < 1$.

There are three separate cases to be considered.

Case 1: The distribution function (d.f.) of X_1 is absolutely continuous, or, more generally, this d.f. satisfies Cramér's condition (C) [1, p. 81].

Case 2: X_1 is a lattice variable, i.e., there exist constants x_0 and $d > 0$ such that X_1 is confined to the set $\{x_0 + rd: r = 0, \pm 1, \pm 2, \cdots\}$ with probability one.

Case 3: Neither Case 1 nor Case 2 obtains.

We can now state

Theorem 1. *There exists a sequence* $b_1, b_2, \cdots$ *of positive numbers* b_n *such that*

$$p_n = \frac{\rho^n}{(2\pi n)^{\frac{1}{2}}} b_n [1 + o(1)], \qquad \log b_n = O(1) \tag{5}$$

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as $n \to \infty$. *In Cases* 1 *and* 3, b_n *is independent of* n. *This last also holds in Case* 2 *if* $P(X_1 = a) > 0$.

The proof of Theorem 1, and of Theorem 2 below, is given in Sections 2-5. The present determination of b_n is given by (4), (9) and (33) in Cases 1 and 3, and by (4), (8), (37), (38) and (46) in Case 2. The following refinements of Theorem 1 are available in Cases 1 and 2:

THEOREM 2. (Cases 1 and 2). *For each* $j = 1, 2, \cdots$ *there exists a bounded (possibly constant) sequence* $c_{j,1}, c_{j,2}, \cdots$ *such that, for any given positive integer* k,

$$(6) \qquad p_n = \frac{\rho^n}{(2\pi n)^{\frac{1}{2}}} b_n \left[1 + \frac{c_{1,n}}{n} + \frac{c_{2,n}}{n^2} + \cdots + \frac{c_{k,n}}{n^k}\right]\left[1 + O\left(\frac{1}{n^{k+1}}\right)\right]$$

as $n \to \infty$.

The sequences $\{c_{j,n}\}$ are given explicitly for Cases 1 and 2 in Sections 3 and 4 respectively. It would be interesting to know whether (6) holds in Case 3 as well, perhaps with the $\{c_{j,n}\}$ determined according to the formula for Case 1.

Estimates in the form (5) or (6) were first obtained by Cramér [2, pp. 20-21] in the case when X_1 has an absolutely continuous component (so that Case 1 obtains). Cramér showed that in the latter case (6) holds for every k (with b_n and each $c_{j,n}$ independent of n), and determined b_n. Our method of proof in the general case (cf. Sections 2-5) is essentially a variant or extension of Cramér's method. Case 2 was treated recently by Blackwell and Hodges [3] by a different method. It is shown in [3] that (6) holds for $k = 1$ in Case 2, under the restriction on n and a that $P(X_1 + \cdots + X_n = na) > 0$ for every admissible n, and the requisite b_n and $c_{1,n}$ (which are then independent of n) are determined explicitly. Some other references bearing on the problem under consideration are [4], [5] and [6].

In the following Section 2 it is shown that p_n can be expressed as $\rho^n I_n$, where I_n is a certain integral; $0 < I_n < 1$, and $I_n = O(n^{-\frac{1}{2}})$ as $n \to \infty$. I_n can be estimated by application of certain refinements [1], [7] of the central limit theorem. This estimation of I_n is carried out in Sections 3, 4 and 5 for Cases 1, 2 and 3 respectively. It may be added here that, as was pointed out in [2], direct application of the central limit theorem (or refinements thereof) to p_n defined by (1) does not, in general, yield approximations q_n which satisfy (2).

In Section 6 we describe certain numerical approximations to p_n which are suggested by Theorems 1 and 2 and their proofs.

2. Lemmas. Let $Y_1 = X_1 - a$, and let F be the (left-continuous) distribution function (d.f.) of Y_1, $F(y) = P(Y_1 < y)$. Let G be defined by $G(z) = \int_{-\infty<y<z} \rho^{-1} e^{\tau y}\, dF(y)$. Since $E(e^{\tau Y_1}) = \psi(\tau) = \rho$, it is clear that G is a probability d.f.. Let Z_1 be a random variable distributed according to G.

LEMMA 1. *The m.g.f. of* Z_1 *exists in a neighborhood of the origin. We have*

$$(7) \qquad E(Z_1) = 0, \qquad 0 < \operatorname{Var}(Z_1) < \infty.$$

PROOF. Let $\xi(t)$ denote the m.g.f. of Z_1. Then $\xi(t) = \psi(\tau + t)/\rho$ for all t, by (3) and the definition of Z_1. Since $\psi(t) < \infty$ in a neighborhood of $t = \tau$, it follows that $\xi(t) < \infty$ in a neighborhood of $t = 0$. Consequently, $E|Z_1|^r < \infty$ for $r = 1, 2, 3, \cdots$ and $E(Z_1^r) = \{d^r\xi/dt^r\}_{t=0}$. In particular, $E(Z_1) = \{d\xi/dt\}_{t=0} = \psi'(\tau)/\rho = 0$, since $\psi(t)$ is minimum at $t = \tau$, and τ is in the interior of T. It remains to show that $\operatorname{Var}(Z_1) > 0$. Suppose to the contrary that $\operatorname{Var}(Z_1) = 0$; then $P(Z_1 = 0) = 1$; hence $P(Y_1 = 0) = 1$, i.e., $P(X_1 = a) = 1$, which is contrary to our assumptions. This completes the proof.

Let $\operatorname{Var}(Z_1)$ be denoted by σ^2. It follows from the preceding paragraph and (4) that

$$\sigma^2 = \frac{\varphi''(\tau)}{\varphi(\tau)} - a^2. \tag{8}$$

Define

$$\alpha = \sigma\tau, \qquad (0 < \alpha < \infty). \tag{9}$$

Let $Z_1, Z_2, \cdots$ be a sequence of independent and identically distributed random variables. For each n, let

$$U_n = \frac{Z_1 + \cdots + Z_n}{n^{\frac{1}{2}}\sigma} \tag{10}$$

and

$$H_n(x) = P(U_n < x), \qquad (-\infty < x < \infty). \tag{11}$$

LEMMA 2. $p_n = \rho^n I_n$, *where*

$$I_n = n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x}[H_n(x) - H_n(0)]\,dx. \tag{12}$$

PROOF. Let $Y_j = X_j - a$ for $j = 1, 2, \cdots, n$. Then

$$\begin{aligned} p_n &= P(Y_1 + \cdots + Y_n \geq 0) && \text{by (1)} \\ &= \int_{y_1+\cdots+y_n\geq 0} \cdots \int dF(y_1)\cdots dF(y_n) \\ &= \rho^n \int_{z_1+\cdots+z_n\geq 0} \cdots \int e^{-\tau(z_1+\cdots+z_n)}\, dG(z_1)\cdots dG(z_n) \\ &= \rho^n \int_{0\leq x<\infty} e^{-n^{\frac{1}{2}}\alpha x}\, dH_n(x) && \text{by (9), (10), (11)} \\ &= \rho^n I_n^* \quad \text{say.} \end{aligned} \tag{13}$$

It follows by integration by parts that I_n^* defined in (13) is equal to I_n, and this completes the proof.

A theorem of Chernoff [4] states that $p_n \leq \rho^n$ for every n, and that for any given positive $\rho_0 < \rho$, we have $p_n \geq \rho_0^n$ for all sufficiently large n. A simple proof of Chernoff's theorem can be given as follows. Since $0 \leq H_n(x) - H_n(0) \leq 1$ for every n and $x \geq 0$, we have $I_n \leq 1$ and hence $p_n \leq \rho^n$ for every n, by Lemma 2. To establish the second part of the theorem, we note first that $\lim_{n\to\infty} H_n(x) = \Phi(x)$ for every x, where

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2}\, dt \qquad (-\infty < x < \infty), \tag{14}$$

by (7), (10), (11) and the central limit theorem. Let ϵ be a positive constant. Then

$$\begin{aligned} I_n &\geq n^{\frac{1}{2}} \alpha \int_{\epsilon}^{\infty} e^{-n^{\frac{1}{2}}\alpha x} [H_n(x) - H_n(0)]\, dx \\ &\geq [H_n(\epsilon) - H_n(0)]\, n^{\frac{1}{2}}\alpha \int_{\epsilon}^{\infty} e^{-n^{\frac{1}{2}}\alpha x}\, dx \\ &= [H_n(\epsilon) - H_n(0)]\, e^{-n^{\frac{1}{2}}\alpha\epsilon}. \end{aligned}$$

Hence $\liminf_{n\to\infty} \{n^{-\frac{1}{2}} \log I_n\} \geq -\alpha\epsilon$. Since $I_n \leq 1$ for every n, and since ϵ is arbitrary, it follows that $n^{-\frac{1}{2}} \log I_n = o(1)$. Hence $n^{-1} \log p_n = \log \rho + o(1)$, by Lemma 2, and this is equivalent to the conclusion desired.

The preceding argument depends only on the central limit theorem. In the following sections we estimate I_n more accurately by substituting the expansions of $H_n(x)$ due to Cramér [1] and Esseen [7] in the right side of (12). The remainder of this section is concerned with preparations for this application of the Cramér-Esseen expansions. Almost all the considerations of the following paragraphs are well known, and we include them here only for the sake of completeness.

Let $\eta(w)$ denote the m.g.f. of Z_1/σ. According to Lemma 1, $\eta < \infty$ in a neighborhood of $w = 0$. For $j = 2, 3, \cdots$ let λ_j be defined by

$$\lambda_2 = \tfrac{1}{2}; \qquad \lambda_j = (j!\sigma^j)^{-1}(d^j/dt^j)[\log \varphi(t)]_{t=\tau} \quad (j = 3, 4, \cdots). \tag{15}$$

It should be noted that $j!\lambda_j$ is the jth cumulant of the distribution of Z_1/σ. The m.g.f. of U_n, with U_n defined by (10), is

$$[\eta(w/n^{\frac{1}{2}})]^n = \exp [n \sum_{j=2}^{\infty} \lambda_j (w/n^{\frac{1}{2}})^j].$$

Clearly, $[\eta(w/n^{\frac{1}{2}})]^n \exp(-w^2/2)$ is analytic in a domain independent of n, and can be expanded there as a power series in w. By regrouping the terms of this series according to powers of n we shall have

$$[\eta(w/n^{\frac{1}{2}})]^n e^{-w^2/2} = \sum_{j=0}^{\infty} n^{-\frac{1}{2}j} P_j(w) \tag{16}$$

where the P_j are polynomials. P_j is of degree $3j$, and P_j is even or odd according as j is even or odd. The first few polynomials are

$$
\begin{aligned}
P_0(w) &= w^0 \equiv 1,\\
P_1(w) &= \lambda_3 w^3,\\
P_2(w) &= \lambda_4 w^4 + \tfrac{1}{2}\lambda_3^2 w^6, \qquad (17)\\
P_3(w) &= \lambda_5 w^5 + \lambda_3\lambda_4 w^7 + \tfrac{1}{6}\lambda_3^3 w^9,\\
P_4(w) &= \lambda_6 w^6 + (\tfrac{1}{2}\lambda_4^2 + \lambda_3\lambda_5) w^8 + \lambda_3^2\lambda_4 w^{10} + \tfrac{1}{24}\lambda_3^4 w^{12}.
\end{aligned}
$$

Write $\Phi^{(0)}(x) = \Phi(x)$ and $\Phi^{(r)}(x) = (d^r/dx^r)\Phi(x)$ for $r = 1, 2, \cdots$, where Φ is given by (14). Let $P_j(-\Phi)$ denote the function of x obtained by replacing w^r with $(-1)^r\Phi^{(r)}(x)$ in the polynomial $P_j(w)$. It is clear that each $P_j(-\Phi)$ is absolutely continuous and of bounded variation in $(-\infty, \infty)$. It should also be noted that $P_j'(-\Phi)$ is square integrable with respect to Lebesgue measure.

In the following, for any function $K(x)$ of bounded variation in $(-\infty, \infty)$, we denote the c.f. of K by $\chi(t \mid K)$, i.e.,

$$\chi(t \mid K) = \int_{-\infty}^{\infty} e^{itx}\, dK(x) \qquad (18)$$

for every real t. If K is absolutely continuous, χ is, of course, $(2\pi)^{\frac{1}{2}}$ times the Fourier transform of K'. The reader may refer to [8, Chapters I–III] for such elements of Fourier transform theory as are used in this paper.

LEMMA 3. *For every j, t, and x*

$$\chi(t \mid P_j(-\Phi)) = P_j(it)e^{-\frac{1}{2}t^2} \qquad (19)$$

and

$$P_j'(-\Phi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-itx} P_j(it)\, d\Phi(t). \qquad (20)$$

PROOF. As is pointed out in [1, p. 49], we have

$$\chi(t \mid \Phi^{(r)}) = (-it)^r e^{-\frac{1}{2}t^2} \qquad (21)$$

for $r = 0, 1, \cdots$. Suppose, for given j, that $P_j(w) = \sum_{r=0}^{N} a_r w^r$, where the a_r and N are constants (depending on j). Then $P_j(-\Phi) = \sum_{r=0}^{N} a_r (-1)^r \Phi^{(r)}(x)$; hence the left side of (19) equals $\sum_0^N a_r (-1)^r \chi(t \mid \Phi^{(r)})$; (19) now follows from (21). The relation (20) follows from (19) by the inversion formula for the Fourier transform, since $dP_j(-\Phi) = P_j'(-\Phi)\, dx$, and $d\Phi(t) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}t^2}\, dt$.

A probability d.f. $K(x)$ is said to satisfy condition (C) if

$$\limsup_{|t|\to\infty} |\chi(t \mid K)| < 1.$$

In the following lemma the F_j are arbitrary probability d.fs.

LEMMA 4. *If F_1 satisfies* (C), *and if F_1 is absolutely continuous with respect to F_2, then F_2 also satisfies* (C).

PROOF. In this proof, for any probability d.f. K let K^* denote the symmetrized d.f. defined by $K^*(x) = \int_{-\infty}^{\infty} K(x + y)\, dK(y)$. We then have

$$\chi(t \mid K^*) = \int_{-\infty}^{\infty} \cos(tx)\, dK^* = |\chi(t \mid K)|^2$$

for all t.

Suppose, contrary to the lemma, that there exists a sequence $\{t_j : j = 1, 2, \cdots\}$ such that $|t_j| \to \infty$ and $|\chi(t_j \mid F_2)| \to 1$ as $j \to \infty$. It then follows from the above paragraph with $K = F_2$ that $\int_{-\infty}^{\infty} \cos(t_j x)\, dF_2^* \to 1$. Hence $\cos(t_j x) \to 1$ in F_2^*-measure. Since F_2-measure dominates F_1-measure, it is easily seen that F_2^*-measure dominates F_1^*-measure. Consequently, $\cos(t_j x) \to 1$ in F_1^*-measure. It now follows from the above paragraph with $K = F_1$ that $|\chi(t_j \mid F_1)|^2 \to 1$ as $j \to \infty$, which is impossible. This completes the proof.

We conclude this section with a description of the functions $S_1(x)$, $S_2(x)$ which occur in the Euler-Maclaurin sum formulae, and which are required in the analysis of Case 2. It is convenient to define S_1 as follows:

$$S_1(x) = \tfrac{1}{2} - x \quad \text{for} \quad 0 < x \leq 1; \qquad S_1(x+1) = S_1(x). \tag{22}$$

For $j \geq 2$, S_j may be defined as

$$S_j(x) = \begin{cases} \dfrac{1}{2^{j-1}} \displaystyle\sum_{r=1}^{\infty} \dfrac{\cos(2\pi r x)}{(\pi r)^j} & (j \text{ even}) \\[2ex] \dfrac{1}{2^{j-1}} \displaystyle\sum_{r=1}^{\infty} \dfrac{\sin(2\pi r x)}{(\pi r)^j} & (j \text{ odd}). \end{cases} \tag{23}$$

Each S_j is a bounded and periodic function; S_j is absolutely continuous for $j \geq 2$; and at each non-integral x we have

$$S_1'(x) = -1, \; S_{j+1}'(x) = (-1)^j S_j(x) \qquad (j = 1, 2, \cdots). \tag{24}$$

3. I_n in Case 1. Suppose that the d.f. of X_1 satisfies (C). Since $Y_1 = X_1 - a$, it is plain that F, the d.f. of Y_1, also satisfies (C). It is easily seen that F and G (the d.f. of Z_1) are absolutely continuous with respect to each other. It therefore follows from Lemma 4 with $F_1 = F$ and $F_2 = G$ that G also satisfies (C).

Let k be an arbitrary but fixed positive integer. It follows from the conclusion of the preceding paragraph by Cramér's theorem [1, p. 81] that $H_n(x) = K_n(x) + R_n(x)$, where

$$K_n(x) = \sum_{j=0}^{k} n^{-\frac{1}{2}j} P_j(-\Phi) \tag{25}$$

and $R_n(x)$ is of the order $n^{-(k+1)/2}$ uniformly in x. It follows hence from (12) that

$$I_n = n^{\frac{1}{2}}\alpha \int_0^{\infty} e^{-n^{\frac{1}{2}}\alpha x} [K_n(x) - K_n(0)]\, dx + O(n^{-\frac{1}{2}k-\frac{1}{2}}). \tag{26}$$

We have

$$\chi(t \mid K_n) = \sum_{j=0}^{k} n^{-\frac{1}{2}j} P_j(it)\, e^{-\frac{1}{2}t^2} \tag{27}$$

by (19) and (25). Let $f_n(x) = \exp(-n^{\frac{1}{2}}\alpha x)$ for $x \geq 0$ and $f_n(x) = 0$ otherwise. Then $\int_{-\infty}^{\infty} e^{itx} f_n(x)\,dx = 1/(n^{\frac{1}{2}}\alpha - it) = g_n(t)$ say. Consequently, by first using integration by parts and then Parseval's formula, it follows that

$$
\begin{aligned}
n^{\frac{1}{2}}\alpha \int_0^{\infty} e^{-n^{\frac{1}{2}}\alpha x} [K_n(x) - K_n(0)]\,dx &= \int_0^{\infty} e^{-n^{\frac{1}{2}}\alpha x} K_n'(x)\,dx \\
&= \int_{-\infty}^{\infty} f_n(x) K_n'(x)\,dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{g_n(t)}\,\chi(t \mid K_n)\,dt.
\end{aligned} \tag{28}
$$

It follows from (26), (27) and (28) that

$$
\alpha(2\pi n)^{\frac{1}{2}} I_n = \int_{-\infty}^{\infty} \left(1 + \frac{it}{n^{\frac{1}{2}}\alpha}\right)^{-1} \left(\sum_{j=0}^{k} n^{-\frac{1}{2}j} P_j(it)\right) d\Phi(t) + O(n^{-\frac{1}{2}k}). \tag{29}
$$

Define

$$
\mu_{r,s} = \int_{-\infty}^{\infty} (it)^r P_s(it)\,d\Phi(t) \qquad (r, s = 0, 1, 2, \cdots). \tag{30}
$$

Since P_s is an even [odd] polynomial if s is even [odd], and since $\int_{-\infty}^{\infty} t^{2j+1}\,d\Phi(t) = 0$ for $j = 0, 1, 2, \cdots$, it follows that each $\mu_{r,s}$ is a real constant, and that

$$
\mu_{r,s} = 0 \quad \text{if } r + s \text{ is odd.} \tag{31}
$$

Now, $(1 + itn^{-\frac{1}{2}}\alpha^{-1})^{-1} = \sum_{0 \le r < k} (-itn^{-\frac{1}{2}}\alpha^{-1})^r + n^{-\frac{1}{2}k} t^k \omega_n(t)$, where $|\omega|$ is bounded in n and t. Since Φ has finite moments of all orders, it therefore follows from (29), (30) and (31) that

$$
\alpha(2\pi n)^{\frac{1}{2}} I_n = \sum_{0 \le j < \frac{1}{2}k} n^{-j} \left\{ \sum_{r+s=2j} \left(-\frac{1}{\alpha}\right)^r \mu_{r,s} \right\} + O(n^{-\frac{1}{2}k}). \tag{32}
$$

Since $p_n = \rho^n I_n$, and since $\mu_{0,0} = 1$, it follows by replacing k with $2k + 2$ in (32) that (6) holds for any given k, with

$$
b_n = \alpha^{-1} \tag{33}
$$

and

$$
c_{j,n} = \sum_{r+s=2j} \left(-\frac{1}{\alpha}\right)^r \mu_{r,s} \qquad (j = 1, 2, \cdots) \tag{34}
$$

for every n. This establishes Theorem 2, and hence also Theorem 1, in Case 1.

It follows from (17) and (30) that the coefficients $\mu_{r,s}$ required to compute $c_{1,n}$ according to (34) are

$$
\begin{aligned}
\mu_{2,0} &= -1 \\
\mu_{1,1} &= 3\lambda_3 \\
\mu_{0,2} &= 3\lambda_4 - \frac{15}{2}\lambda_3^2,
\end{aligned} \tag{35}
$$

where the λ_j are given by (15). Similarly, $c_{2,n}$ can be computed from

$$
\begin{aligned}
\mu_{4,0} &= 3 \\
\mu_{3,1} &= -15\lambda_3 \\
\mu_{2,2} &= -15\lambda_4 + 105\lambda_3^2 \qquad (36)\\
\mu_{1,3} &= -15\lambda_5 + 105\lambda_3\lambda_4 - \frac{315}{2}\lambda_3^3 \\
\mu_{0,4} &= -15\lambda_6 + 105(\tfrac{1}{2}\lambda_4^2 + \lambda_3\lambda_5) - \frac{945}{2}\lambda_3^2\lambda_4 + \frac{10395}{24}\lambda_3^4.
\end{aligned}
$$

We conclude this section with a remark concerning the role of Cramér's theorem [1, p. 81] in the preceding argument. Suppose that H_n is absolutely continuous, and that H_n' is square integrable over $(-\infty, \infty)$. It then follows, by integrating (12) by parts and using Parseval's formula, that

$$
(29^*) \qquad \alpha(2\pi n)^{\frac{1}{2}} I_n = \int_{-\infty}^{\infty} \left(1 + \frac{it}{n^{\frac{1}{2}}\alpha}\right)^{-1} \left\{\left[\eta\left(\frac{it}{n^{\frac{1}{2}}}\right)\right]^n e^{\frac{1}{2}t^2}\right\} d\Phi(t)
$$

where η is, as before, the m.g.f. of Z_1/σ. (The square integrability condition is imposed here for the validity of Parseval's formula, and can be replaced by others, e.g., that $(1 + t^2)^{-\frac{1}{2}} \mid \eta(it) \mid$ be integrable). According to (16), the function in curly brackets on the right side of (29*) can be expressed as $\sum_{j=0}^{\infty} n^{-\frac{1}{2}j} P_j(it)$. By comparing (29) and (29*) it is seen that, from a technical point of view, the role of Cramér's theorem in the present special case is to guarantee that when $\sum_{j=0}^{\infty} n^{-\frac{1}{2}j} P_j$ is replaced by $\sum_{j=0}^{k-1} n^{-\frac{1}{2}j} P_j$ on the right side of (29*), the error introduced is indeed of the order $n^{-\frac{1}{2}k}$. The same remark, but with (29*) replaced by a rather different formula for I_n, applies to the role of Esseen's theorem in the argument of the following section.

4. I_n in Case 2. Suppose that X_1 is a lattice variable. Let d be the maximum span of X_1, i.e., $d > 0$ is the g.c.d. of the differences between consecutive possible values of X_1. Let x_0 be the number such that $a \leq x_0 < a + d$, and such that the possible values of X_1 are included in the set $\{x_0 + rd : r = 0, \pm 1, \pm 2, \cdots\}$. Let

$$
(37) \qquad \beta = d/\sigma, \qquad \gamma = \tau d, \qquad \kappa = (x_0 - a)/d
$$

It should be noted that $0 \leq \kappa < 1$. For each n, let

$$
(38) \qquad \theta_n = n\kappa - [n\kappa], \qquad 0 \leq \theta_n < 1,
$$

where $[x]$ denotes the greatest integer contained in x.

Let k be an arbitrary but fixed positive integer. It follows from Esseen's theorem for the lattice case [7, p. 61] that $H_n(x) = K_n(x) + L_n(x) + R_n(x)$, where $K_n(x)$ is given by (25), R_n is of the order $n^{-(k+1)/2}$ uniformly in x, and L_n

is defined as follows. For any $j = 1, 2, \cdots$ let $h_j = 1$ if $j \equiv 1$ or $2 \pmod 4$ and $h_j = -1$ if $j \equiv 0$ or $3 \pmod 4$. Then

$$L_n(x) = \sum_{j=1}^{k} n^{-\frac{1}{2}j} h_j \beta^j S_j(n^{\frac{1}{2}}\beta^{-1}x - \theta_n) K_n^{(j)}(x) = \sum_{j=1}^{k} M_{j,n}(x) \text{ say,} \tag{39}$$

where $K_n^{(j)}$ is the jth derivative of K_n. It follows hence from (12) that

$$\begin{aligned} I_n = n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [K_n(x) - K_n(0)] dx \\ + \sum_{j=1}^{k} n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [M_{j,n}(x) - M_{j,n}(0)] dx + O(n^{-\frac{1}{2}k-\frac{1}{2}}). \end{aligned} \tag{40}$$

The first term on the right side of (40) is (cf., (28)) equal to $\int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} K_n^{(1)}(x)\, dx$. We observe next that, for $j \geqq 2$,

$$\begin{aligned} n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [M_{j,n}(x) - M_{j,n}(0)]\, dx &= \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} M'_{j,n}(x)\, dx \\ &= n^{-\frac{1}{2}j} h_j \beta^j \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} [S_j(y_n) K_n^{(j+1)}(x) \\ &\qquad + (-1)^{j-1} n^{\frac{1}{2}}\beta^{-1} S_{j-1}(y_n) K_n^{(j)}(x)]\, dx \\ &= n^{-\frac{1}{2}j} h_j \beta^j \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} S_j(y_n) K_n^{(j+1)}(x)\, dx \\ &\qquad - n^{-\frac{1}{2}(j-1)} h_{j-1} \beta^{j-1} \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} S_{j-1}(y_n) K_n^{(j)}(x)\, dx \\ &= N_{j,n} - N_{j-1,n} \text{ (say).} \end{aligned} \tag{41}$$

In (41), we have put $n^{\frac{1}{2}}\beta^{-1}x - \theta_n = y_n$, and used integration by parts, (24), and the identity $(-1)^j h_j = h_{j-1}$. In order to evaluate the contribution of $M_{1,n}$ to the right side of (40), suppose for the moment that $0 < \theta_n < 1$, and let

$$\zeta_0 = 0, \qquad \zeta_r = (r - 1 + \theta_n)\beta/n^{\frac{1}{2}} \quad (r = 1, 2, \cdots). \tag{42}$$

Let A_r denote the open interval (ζ_r, ζ_{r+1}). Then $S_1(y_n)$ is linear in x over each A_r (cf. (22)), and its derivative there equals $-n^{\frac{1}{2}}\beta^{-1}$. By writing $\int_0^\infty = \sum_{r=0}^{\infty} \int_{A_r}$, and applying integration by parts to $\int_{A_r}$, it follows without difficulty that

$$\begin{aligned} n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} M_{1,n}(x)\, dx = - \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x} K_n^{(1)}(x)\, dx \\ + N_{1,n} + M_{1,n}(0) + \beta n^{-1} \sum_{r=1}^{\infty} e^{-\gamma(r-1+\theta_n)} K_n^{(1)}(\zeta_r), \end{aligned} \tag{43}$$

where $\gamma = \alpha\beta = \tau d$ (cf. (37)). Now, $S_1(x)$ is a left-continuous function of x. It follows hence that, for given n, the left and right sides of (43) are right-continuous in θ_n. Since (43) holds for each θ_n in (0, 1), we conclude that (43) is valid for $\theta_n = 0$ also.

Since S_k and $K_n^{(k+1)}$ are bounded functions, it is plain from the definition of $N_{j,n}$ (cf. (41)) that $N_{k,n}$ is of the order $n^{-\frac{1}{2}k-\frac{1}{2}}$. It therefore follows from (40), (41) and (43) that

$$I_n = \beta n^{-\frac{1}{2}} \sum_{r=1}^{\infty} e^{-\gamma(r-1+\theta_n)} K_n^{(1)}(\zeta_r) + O(n^{-\frac{1}{2}k-\frac{1}{2}}). \tag{44}$$

Now, according to (20) and (25),

$$K_n^{(1)}(\zeta_r) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-it\zeta_r} \left(\sum_{j=0}^{k} n^{-\frac{1}{2}j} P_j(it)\right) d\Phi(t) \tag{45}$$

for every r. Let us write

$$z = e^{-\gamma}, \qquad b_n = [\beta/(1-z)]z^{\theta_n}. \tag{46}$$

It follows from (44) and (45) that

$$\begin{aligned} b_n^{-1} (2\pi n)^{\frac{1}{2}} I_n = \int_{-\infty}^{\infty} & \frac{(1-z) \exp[-it\beta\theta_n/n^{\frac{1}{2}}]}{(1 - z\exp[-it\beta/n^{\frac{1}{2}}])} \\ & \cdot \left(\sum_{0}^{k} n^{-\frac{1}{2}j} P_j(it)\right) d\Phi(t) + O(n^{-\frac{1}{2}k}). \end{aligned} \tag{47}$$

For any θ and any $j = 0, 1, 2, \cdots$ let

$$\ell_j(\theta) = \frac{1}{j!}\left\{\frac{d^j}{dw^j}\left(\frac{(1-z)e^{-\theta w}}{(1-ze^{-w})}\right)\right\}_{w=0}. \tag{48}$$

It then follows easily from (31) and (47) that

$$b_n^{-1}(2\pi n)^{\frac{1}{2}} I_n = \sum_{0 \le j < k/2} n^{-j} \Big[\sum_{r+s=2j} \beta^r \ell_r(\theta_n) \mu_{r,s}\Big] + O(n^{-\frac{1}{2}k}). \tag{49}$$

By replacing k with $2k + 2$ in (49) we see that (6) holds for any given k, with b_n given by (46), and

$$c_{j,n} = \sum_{r+s=2j} \beta^r \ell_r(\theta_n) \mu_{r,s}. \tag{50}$$

This establishes Theorem 2 in Case 2, and hence also the first part of Theorem 1. To complete the proof of Theorem 1 in Case 2, we see from (37), (38) that $P(X_1 + \cdots + X_n = na) > 0$ implies $\theta_n = 0$. Consequently if $P(X_1 = a) > 0$ then $\theta_n = 0$ for every n, and hence $b_n = \beta/(1-z)$ for every n.

It may be worthwhile to note that in the present case b_n can be expressed as $\alpha^{-1}[\gamma e^{\gamma(1-\theta_n)}/(e^{\gamma} - 1)]$, which shows that, in general, b_n oscillates about the value α^{-1} (cf. (33)) as $n \to \infty$ through the sequence 1, 2, $\cdots$.

An alternative formula for the coefficients ℓ_j required in (50) is

$$\ell_j(\theta) = (-1)^j \sum_{r+s=j} \frac{\theta^r}{r!s!}\left\{(1-z)\left(z\frac{d}{dz}\right)^s (1-z)^{-1}\right\}. \tag{51}$$

From (51) it is easily seen that, with $u = s/(1 - s)$,

$$
\begin{aligned}
&\ell_0 = 1\\
&\ell_1 = -(\theta + u)\\
(52)\quad &\ell_2 = \tfrac{1}{2}[(\theta + u)^2 + u(1 + u)]\\
&\ell_3 = -\tfrac{1}{6}[(\theta + u)^3 + 3u(1 + u)\theta + u(1 + u)(1 + 5u)]\\
&\ell_4 = \tfrac{1}{24}[(\theta + u)^4 + 6u(1 + u)\theta^2 + 4u(1 + u)(1 + 5u)\theta\\
&\qquad\qquad + 23u^4 + 36u^3 + 14u^2 + u].
\end{aligned}
$$

The coefficients $c_{1,n}$ and $c_{2,n}$ can be computed from (35), (36), (50) and (52). The formulae for b_n and $c_{1,n}$ with $\theta = 0$ agree with the results of [3].

5. I_n in Case 3. If X_1 is not a lattice variable, then neither is Z_1. It follows hence from a theorem of Esseen [7, p. 49] that $H_n(x) = \Phi(x) + n^{-\frac{1}{2}}f(x) + n^{-\frac{1}{2}}r_n(x)$, where $f(x) = (\text{const.})\,(1 - x^2)\exp(-\tfrac{1}{2}x^2)$, and $r_n(x) \to 0$ uniformly in x as $n \to \infty$. The contribution of $n^{-\frac{1}{2}}f$ to I_n is $n^{-\frac{1}{2}}\int_0^\infty e^{-(n)^{\frac{1}{2}}\alpha x}f'(x)\,dx$, which is easily seen to be of the order n^{-1}. It follows that

$$
\begin{aligned}
I_n &= n^{\frac{1}{2}}\alpha \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x}[\Phi(x) - \Phi(0)]\,dx + o(n^{-\frac{1}{2}})\\
&= \int_0^\infty e^{-n^{\frac{1}{2}}\alpha x}\Phi'(x)\,dx + o(n^{-\frac{1}{2}}) \qquad (53)\\
&= e^{\frac{1}{2}n\alpha^2}[1 - \Phi(n^{\frac{1}{2}}\alpha)] + o(n^{-\frac{1}{2}})\\
&= (2\pi n)^{-\frac{1}{2}}\alpha^{-1} + o(n^{-\frac{1}{2}}).
\end{aligned}
$$

In (53), we have used integration by parts, a linear change of variable, and the leading term of the asymptotic formula [9, p. 179]

$$
(54)\quad 1 - \Phi(x) = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}\{x^{-1} - x^{-3} + 3x^{-5} + O(x^{-7})\} \text{ as } x \to \infty.
$$

It follows from (53) that (5) holds, with $b_n = \alpha^{-1}$ for every n. This completes the proof of Theorem 1.

Since $\Phi(x) + n^{-\frac{1}{2}}f(x) = K_n(x)$, where K_n is defined by (25) with $k = 1$, the conclusion of the preceding paragraph is also available from the argument of Section 3. We have used a direct calculation instead because this calculation suggests the form of the numerical approximations described in the following section.

6. Concluding remarks. Suppose, in a given case, and for given n and a, that it is required to compute the numerical value of p_n defined by (1). In this section we consider approximations of the form

$$
(55)\qquad q_n = \rho^n e^{\frac{1}{2}v_n^2}((1 - \Phi(v_n)),
$$

where ρ and Φ are defined by (4) and (14), and v_n is a suitably chosen number.

We shall describe four choices of v_n, called v_n^*, $v_n^{(0)}$, $v_n^{(1)}$, and $v_n^{(2)}$. The resulting values of q_n are denoted by q_n^*, $q_n^{(0)}$, etc.

First consider

$$v_n^* = n^{\frac{1}{2}}\alpha \tag{56}$$

where α is given by (4), (8) and (9). This choice of v_n amounts (cf. (53)) to approximating I_n by replacing H_n with Φ on the right side of (12). It therefore follows from the Esseen-Berry theorem that we always have

$$| p_n - q_n^* | \leq 2C \frac{\rho^n}{n^{\frac{1}{2}}} \frac{E | Z_1 |^3}{\sigma^3} \tag{57}$$

where C is a universal constant. Wallace [10, p. 637] states that $C \leq 2.05$.

Next, consider

$$v_n^{(0)} = n^{\frac{1}{2}}/b_n \tag{58}$$

where b_n is defined by (33) in Cases 1 and 3, and by (46) in Case 2. (Of course, $q_n^{(0)} = q_n^*$ in Cases 1 and 3). Then $q_n^{(0)}$ satisfies (2), and the $o(1)$ term in (2) is known to be of the order n^{-1} in Cases 1 and 2. Finally, let $c_{j,n}$ be defined according to Section 4 in Cases 1 and 3, and according to Section 5 in Case 2. Define

$$v_n^{(1)} = v_n^{(0)} [1 - (b_n^2 + c_{1,n})/n] \tag{59}$$

if the expression within the square brackets is positive and $v_n^{(1)} = 0$ otherwise; and

$$v_n^{(2)} = v_n^{(1)} [1 + (b_n^4 + c_{1,n}^2 - b_n^2 c_{1,n} - c_{2,n})/n^2] \tag{60}$$

if the expression in square brackets is positive and $v_n^{(2)} = 0$ otherwise. Then $q_n^{(j)}$ also satisfies (2), and $o(1) = O(n^{-j-1})$ in Cases 1 and 2 ($j = 1, 2$). The stated theoretical properties of the approximations $q_n^{(j)}$ are easy consequences of (5), (6), (54), and (58).

Although (unlike q_n^*) the approximations $q_n^{(j)}$ are derived from asymptotic expansions corresponding to the case when $n \to \infty$ and a is held fixed, the usefulness of these approximations may be wider than is suggested by the derivation. Some evidence to this effect is provided by the fact that if X_1 is normally distributed then $p_n = q_n^{(0)} = q_n^{(1)} = q_n^{(2)}$ for every admissible a and every n.

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