TABLES FOR TESTING THE SIGNIFICANCE OF LINEAR REGRESSION IN THE CASE OF TIME-SERIES AND OTHER SINGLE-VALUED SAMPLES

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INTRODUCTION.

The problem of testing the significance of secular trend in a time series is one of frequent occurrence in statistical studies. A time-series is only a particular case of a class of samples in which only one single value of the dependent variate, say y, is available for each value of the independent variate x. Prof. Mahalanobis has suggested that such samples may be called single-valued samples for convenience of reference. Our general problem then is to test the significance of linear regression in the case of single-valued samples.

The linear regrssion equation is given by

$$y = a + b.(x - \bar{x}) \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where $a = \bar{y} = S(y)/N$

$$b = \frac{S(y-\bar{y})(x-\bar{x})}{S(x-\bar{x})^2}$$

In the case of large samples it is possible to test the significance of the coefficient of correlation 'r', or of the regression coefficient $b = r(s_y/s_x)$ with the help of the respective standard errors.

Difficulties arise however in dealing with small samples, especially when N is less than say 30. In such cases it is convenient to use R. A. Fisher's method of the analysis of variance.

In this method the analysis rests on the following algebraic identity:—

$$\begin{aligned} S_N(y-\bar{y})^2 &= S_p[S_{n_p}(y-\bar{y}_p)^2] + S_p[n_p(\bar{y}_p-\bar{y})^2] \\ &= S_p[S_{n_p}(y-\bar{y}_p)^2] + S_p[n_p(y'_p-\bar{y})^2] + S_p[n_p(y'_p-\bar{y}_p)^2] \quad \ldots \quad (2) \end{aligned}$$

where $\bar{y}$ = observed general mean value of 'y',

$\bar{y}_p$ = observed mean value 'y' in the pth array,

y'_p = estimate of $\bar{y}_p$ obtained by the linear regression equation

n_p = no. of observations in pth array

S_N = summation for all values of y from 1 to N

S_p = summation for all arrays from 1 to p

S_{n_p} = summation for all values of y within the pth array from 1 to p.

It is convenient to show the analysis in the form of a table with the appropriate degrees of freedom.

Factor of Variation	Degrees of Freedom	Sum of Squares	Estimated Variance
Deviations from Linear Regression	$p-2$	$S_p[n_p(y'_p-\bar{y}_p)^2]$	s_2^2
Linear Regresions	1	$S_p[n_p(y'_p-\bar{y})^2]$	s_1^2
Between p-arrays	$p-1$	$S_p[n_p(\bar{y}_p-\bar{y})^2]$	s_b^2
Within p-arrays	$N-p$	$S_p[S_{n_p}(y-\bar{y}_p)^2]$	s_w^2
TOTAL ..	$N-1$	$S_N(y-\bar{y})^2$	s_0^2

When s_1^2 is significantly greater than s_w^2, the linear regression may be considered significant and not otherwise. Further when s_2^2 does not differ appreciably from s_w^2, the linear regression is also adequate, but when s_2^2 is significantly greater than s_w^2 the regression is non-linear.

The analysis takes a very simple form in the case of single-valued samples when only one value of 'y' is available for each value of 'x'. Here $n_p=1$, $N=p$, and the analysis of variance takes the following form.

Factor of Variation	Degrees of Freedom	Sum of Squares	Estimated Variance
Deviations from Linear Regression	$N-2$	$S(y-y')^2$	s_2^2
Linear Regression	1	$S(y'-\bar{y})^2$	s_1^2
TOTAL ...	$N-1$	$S(y-\bar{y})^2$	s_y^2

Although s_w^2 cannot be estimated from such data, it is possible to calculate the standard error of 'b' if we can assume that for each value of 'x' the values of 'y' are distributed normally with the same variance, say σ_w^2. Now the best estimate of σ_w^2 is given by $S\,(y-y')^2/(N-2)$ that is by s_2^2. Writing the observed regression coefficient in the samples as $b=S(x-\bar{x})\,(y-\bar{y})/S(x-\bar{x})^2$ we get the standard deviation of b

$$s_b=\frac{s_w}{\sqrt{S(x-\bar{x})^2}}=\left(\frac{S(y-y')^2}{(N-2)\,S(x-\bar{x})^2}\right)^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots \qquad (3)$$

If β is the population value of "b" then

$$(b-\beta)/s_b \qquad \text{or} \qquad \frac{(b-\beta)\sqrt{[(N-2).S(x-\bar{x})^2]}}{\sqrt{S(y-y')^2}} \quad \ldots \qquad (4)$$

has been shown by R. A. Fisher, to be distributed in the same way as 'Students' z ["Application of 'Student's' Distribution," *Metron*, Vol. V. No. 3, (1925), pp. 90—104]. Fisher has noted that this method is valid whatever may be the distribution of x, provided that y is normally distributed with the same variance in each array, and the regression of 'y' on 'x' is linear in the population sampled. He has also given numerical examples (*Statistical Methods for Research Workers*, 4th edition, 1932, Example 22, pp. 127-28) to show how the t-table can be used for testing the significance of the regression coefficient.

CONSTRUCTION OF THE TABLES.

It will be noticed that in the present method the test of the significance of 'b' ultimately reduces to Fisher's z-test, the value of z being given by

$$z = \tfrac{1}{2}\log_e(s_1^2/s_2^2), \text{ or } e^{2z} = s_1^2/s_2^2 \quad \dots \quad \dots \quad \dots \quad (5{\cdot}0)$$

Now

$$s_1^2 = S(y' - \bar{y})^2 = b^2.S(x - \bar{x})^2 \quad \dots \quad \dots \quad \dots \quad (5{\cdot}1)$$

and

$$s_2^2 = S(y - y')^2/(N-2)$$

$$= [S(y-\bar{y})^2 - b^2.S(x-\bar{x})^2]/(N-2) \quad \dots \quad \dots \quad (5{\cdot}2)$$

Therefore

$$e^{2z} = s_1^2/s_2^2 = b^2.s_x^2.(N-2)/(s_y^2 - b^2.s_x^2) \quad \dots \quad \dots \quad (6)$$

Hence we get

$$b^2 = \frac{s_y^2}{s_x^2}\cdot\frac{e^{2z}}{(e^{2z}+N-2)} \quad (7)$$

We notice therefore that if we take the five per cent. (or one per cent.) value of z from Fisher's z-table (Table VI, Statistical Methods for Research Workers, pp. 224-227) with $n_1 = 1$, and $n_2 = N-2$ we can immediately calculate the corresponding five per cent. (or one per cent.) value of 'b' with the help of the above equation.

The calculation of these critical values of 'b' is further simplified by the fact that values of e^{2z} have been given by P. C. Mahalanobis in his 'Auxiliary Tables for Fisher's z-test in the Analysis of Variance' (*Ind. Jour. Agri. Sc.*, Vol. II, Part VI, Dec., 1932, pp. 679-693).

Calling Mahalanobis's x as u (in order to prevent confusion with the independent variate), we have $u = e^{2z}$, and hence

$$b = \frac{s_y}{s_x}\cdot\left(\frac{u}{(u+N-2)}\right)^{\frac{1}{2}} \quad \dots \quad \dots \quad (8)$$

Table 1 (five per cent.) and Table 2 (one per cent.) give the critical values of 'b' calculated from the above formula for various values of the ratio (s_y/s_x) ranging from zero to one.

USE OF THE TABLES.

Example 1. Let us consider an example given by G. C. Whipple (*Vital Statistics*, 1st edition, p. 374) showing the median age of the population in U.S.A. from one Census to another.

MEDIAN AGE OF POPULATION IN THE UNITED STATES

Year	Median Age
1800	16·0
1810	16·0
1820	16·5
1830	17·2
1840	17·9
1850	19·1
1860	19·7
1870	20·4
1880	21·3
1890	21·9
1900	23·4
1910	24·4

$N=12$

$S(x-\bar{x})^2 = N(N^2-1)/12 = 143$

$S(x-\bar{x})(y-\bar{y}) = 112{\cdot}7$

$b = S(x-\bar{x})(y-\bar{y})/S(x-\bar{x})^2 = 0{\cdot}788$

$S(y-\bar{y})^2 = 90{\cdot}58$

$s_y^2/s_x^2 = S(y-\bar{y})^2/S(x-\bar{x})^2 = 0{\cdot}63$

The regression coefficient is found to be 0·788, *i.e.* the median age increases at the rate of 0·787 years in 10 years, *i.e.* at a rate of about 1 month per year. Can this rate be considered significant?

The test given by Fisher (*Statistical Methods for Research Workers*, pp. 123-131) is applied as follows.

$$S(y-\bar{y})^2 = 90{\cdot}58$$
$$b^2S(x-\bar{x})^2 = 88{\cdot}50$$
$$S(y-\bar{y})^2 - b^2S(x-\bar{x})^2 = 2{\cdot}08.$$

Standard error of $b = \sqrt{[(2{\cdot}08/10)\ (1/143)]} = 0{\cdot}0381$. And $t = 0{\cdot}788/0388 = 20{\cdot}66$ which is significant.

With the help of the present tables the test is considerably simplified. We simply calculate the quantity $s_y^2/s_x^2 = 0{\cdot}63$. Looking up Tables 1 and 2 we find.

s_y^2/s_x^2	Values of b	
	5 p.c.	1 p.c.
0·60	·4462	·5483
0·65	·4344	·5707

The observed value 0·787 being greater than the one per cent. value is definitely significant. Thus mere inspection of the present Table is sufficient, once s_y^2/s_x^2 is calculated.

RANGE OF THE TABLE.

It will be noticed that '*b*' varies directly as (s_y/s_x) for a given value of N. Thus if $(s_y/s_x)^2$ is increased 100 times, b will be increased only 10 times. For example for $N=10$, we have

s_y^2/s_x^2	Values of b	
	5 p.c.	1 p.c.
·05	0·1413	0·1710
5·00	1·413	1·710
500·00	14·13	17·10

Similarly if $(s_y/s_x)^2$ is reduced to 1/100 of its value, b is reduced to 1/10.

Thus, for $N=15$

s_y^2/s_x^2	Values of y	
	5 p.c.	1 p.c.
·95	·5009	·6249
·0095	·0501	·0625
·000095	·0050	·0062

It will be seen, therefore, that the Tables given here may be used with slight modifications for all ranges of $(s_y/s_x)^2$.

INTERPOLATION.

When N is given, b is a straight line function of (s_y/s_x). Hence intermediate values of '*b*' can be obtained by linear interpolation, for any value of (s_y/s_x).

In example 1, for example, we can use the following scheme for interpolation.

s_y^2/s_x^2	s_y/s_x	Values of b	
		5 p.c.	1 p.c.
0·60	·7746	·4462	·5483
0·65	·8062	·4644	·5707
0·63	·7937	·4572	·5618

For facility of calculation, values of (s_y/s_x) have been given in the Tables next to $(s_y/s_x)^2$, but interpolations would be seldom necessary in actual practice.

A few more examples will now be given to illustrate the use of the Tables for various ranges.

Example 2. Acreage under Rice in India from 1919-1920 to 1928-1929. (*Statistical Abstracts for British India*, 1932, p. 400).

Year	Acreage $\times 10^{-6}$
1920-21	781
-22	797
-23	806
-24	772
-25	793
-26	802
-27	785
-28	766
-29	811

$N = 9$

$S(x-\bar{x})^2 = 60$

$S(x-\bar{x})(y-\bar{y}) = 15$

$b = 0{\cdot}25$

$S(y-\bar{y})^2 = 1904$

$s_y^2/s_x^2 = 32$

From Table 1 we get 5% values of 'b' for $s_y^2/s_x^2 = 0{\cdot}30$ and $0{\cdot}35$. From which we easily find the values of 'b' for $s_y^2/s_x^2 = 30$ and 35.

s_y^2/s_x^2	Value of b 5 p.c.
·30	·3650
·35	·3942
30	3·650
35	3·942

The observed regression (0·25) is clearly insignificant.

Example 3. Number of female scholars in recognised schools in India. (*Statistical Abstract for British India*, 1932, p. 363).

Year	No. of Female Scholars
1919-20	1249
21	1388
22	1529
23	1771
24	1914
25	2110
26	2205
27	2278
1927-28	2451

$N = 9$

$S(x-\bar{x})^2 = 60$

$S(x-\bar{x})(y-\bar{y}) = 9173$

$b = 153$

$S(y-\bar{y})^2 = 1419364$

$s_y^2/s_x^2 = 23656$

From the Table, we get for $s_y^2/s_x^2 = 0{\cdot}05$, the 5% value of 'b' to be 0·1490. Hence for $s_y^2/s_x^2 = 50000 = {\cdot}05 \times 10^6$, we have the 5 per cent. value of $b = 0{\cdot}1490 \times 10^3 = 149$. For $s_y^2/s_x^2 = 23656$, the 5 per cent. value of b will be less than 149. The observed value of $b = 153$ may therefore be considered significant. Interpolation is not necessary. But if we like we can find the value of b corresponding to the value of $(s_y/s_x)^2 = 23656$ by linear interpolation. The 5 per cent. value of $b = 102{\cdot}6$ and the 1 per cent. $b = 122{\cdot}8$. The observed regression ($b = 153$) is thus definitely significant.

Unequal Intervals

These Tables may be used even when x does not increase by equal intervals, for as already noted above the test is independent of the distribution of 'x', the only assumption being that y's are normally distributed with equal variance in each array of x.

Example 4. Relation between Height and Weight of Dry Fibre in Jute Variety D89, Bengal, 1933. (Unpublished Data from Dacca Farm, Bengal).

Height in inches	Weight of Dry Fibre
97	41·5
99	35·5
96	17·0
96	23·5
94	38·5
97	38·0
111	42·25
111	38·75
135	41·5
135	42·25
102	47·0
117	51·5
111	47·0
98	42·5
99	45·0

$N = 15$

$S(x-\bar{x})^2 = 2558$

$S(x-\bar{x})(y-\bar{y}) = 645{\cdot}65$

$b = 0{\cdot}25$

$S(y-\bar{y})^2 = 1101$

$s_y^2/s_x^2 = 0{\cdot}43$

From Table 1, for $N = 15$ and $s_y^2/s_x^2 = 0{\cdot}40$, we find the 5 per cent. value of $b = 0{\cdot}3250$. Hence the 5 per cent. value corresponding to $s_y^2/s_x^2 = 0{\cdot}43$ will be higher, so that the observed regression coefficient $= 0{\cdot}25$ cannot be considered significant.

These Tables can of course also be used to test the significance of $(b-\beta)$, where b is the observed value of the regression coefficient, and β is the corresponding population (or expected) value.

With the greatest pleasure I acknowledge my indebtedness to Prof. P. C. Mahalanobis for his valuable suggestions and to Mr. Sudhirkumar Banerjee, Chief Computer, Statistical Laboratory, for general assistance.

Note by the Editor

One word of caution is necessary. In testing the significance of the regression coefficient we are substituting an estimate of the variance of y based on the sample itself for the population value of the variance. This introduces an uncertainty which increases rapidly as the size of the sample is decreased. It is desirable to remember that the use of the exact expressions developed in recent years for small samples cannot get over this particular difficulty. Great caution is therefore needed in interpreting the significance of regression coefficients based on small samples.

The situation of course is entirely different when the population values of the standard deviations and the correlation coefficient may be considered as known. In this case the exact distribution of the regression coefficient 'b' may be used with safety to test whether the sample values are in reasonable agreement with the population value.

If ϱ is the correlation, and σ_y and σ_x the standard deviations in the parent population, then the population value of the regression is $\beta = \varrho \cdot \sigma_y / \sigma_x$, the variance of '$b$' the regression in samples of size N is $\sigma_b^2 = \sigma_y^2(1-\varrho^2/\sigma_x^2\ (N-3))$, and the distribution of 'b' is given by

$$z = \frac{z_n}{\left\{\dfrac{\sigma_y^2(1-\rho^2)}{\sigma_x^2} + (b-\beta)^2\right\}^{\frac{1}{2}N}}$$

The use of the above formula has been fully explained by Prof. Karl Pearson in Biometric Tables, Part II, Table XXV, pp. cxxvi-cxxxi. It is useful to refer to the following papers in this connexion.

Karl Pearson: "Further Contributions to the Theory of Small Samples (*Biometrika*, Vol. 17, 1925, pp. 176-199).

U. Romanowsky: "On the moments of standard derivations and correlation coefficients in samples from normal population" (*Metron*, Vol. 5, 1925, pp. 3-46).

R. A. Fisher: "The Goodness of Ft of Regression Formulæ and the Distribution of Regression Co-efficients" (*Jour. Roy. Stat. Soc.* Vol. LXXXV, July, 1922, pp. 597-612).

For large samples the relevant formulæ are given by Karl Pearson. "On the Probable Error of Frequency Constants, Part II" (*Biometrika*, Vol. 9, 1913, pp. 1-10).

It is worth pointing out that in Pearson's formula quoted in this note, σ_x^2, σ_y^2 and ϱ represent population values of the two variances and the coefficient of correlation, while in Fisher's formula used in the text of the paper, s_x^2, s_y^2 and r represent the observed values in the sample. In fact Pearson's value $\sigma_y^2(1-\varrho^2)/\sigma_x^2(N-3)$ is equal to the mathematical expectation or population value of Fisher's expression $s_y^2(1-r^2)/s_x^2(N-2)$.

P. C. M.

(March, 1934).

TABLE 1. FIVE PER CENT. VALUES OF THE REGRESSION COEFFICIENT $(b = \rho .\ \sigma_y/\sigma_x)$. N = 3 to N = 17

σ_y^2/σ_x^2	σ_y/σ_x	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
·05	·2236	·2229	·2124	·1964	·1814	·1687	·1580	·1490	·1418	·1346	·1288	·1236	·1190	·1149	·1112	·1078
·10	·3162	·3152	·3067	·2777	·2566	·2386	·2235	·2107	·1998	·1904	·1821	·1748	·1683	·1657	·1572	·1524
·15	·3873	·3861	·3679	·3402	·3143	·2922	·2737	·2581	·2447	·2332	·2231	·2141	·2062	·1990	·1926	·1867
·20	·4472	·4458	·4248	·3928	·3629	·3374	·3160	·2980	·2826	·2693	·2576	·2473	·2381	·2298	·2224	·2156
·25	·5000	·4985	·4750	·4392	·4057	·3778	·3534	·3332	·3160	·3011	·2880	·2765	·2662	·2570	·2487	·2411
·30	·5477	·5460	·5203	·4810	·4444	·4132	·3871	·3650	·3461	·3298	·3155	·3028	·2916	·2875	·2724	·2640
·35	·5916	·5898	·5620	·5196	·4800	·4464	·4181	·3942	·3738	·3562	·3408	·3271	·3150	·3040	·2942	·2852
·40	·6325	·6305	·6000	·5555	·5132	·4772	·4470	·4215	·3997	·3808	·3643	·3497	·3367	·3250	·3145	·2049
·45	·6708	·6687	·6378	·5892	·5443	·5061	·4741	·4470	·4239	·4039	·3864	·3709	·3571	·3447	·3336	·3233
·50	·7071	·7019	·6717	·6210	·5737	·5335	·4997	·4712	·4468	·4257	·4073	·3910	·3765	·3634	·3516	·3409
·55	·7416	·7393	·7045	·6513	·6017	·5595	·5241	·4942	·4686	·4465	·4272	·4100	·3949	·3818	·3689	·3575
·60	·7746	·7722	·7350	·6803	·6285	·5844	·5474	·5162	·4895	·4664	·4462	·4283	·4124	·3981	·3852	·3734
·65	·8062	·8029	·7659	·7081	·6542	·6083	·5697	·5373	·5094	·4854	·4644	·4457	·4292	·4143	·4009	·3887
·70	·8367	·8341	·7949	·7349	·6789	·6313	·5912	·5576	·5287	·5038	·4819	·4626	·4455	·4300	·4161	·4034
·75	·8660	·8633	·8227	·7606	·7026	·6534	·6120	·5771	·5472	·5214	·4988	·4788	·4611	·4450	·4307	·4175
·80	·8944	·8916	·8497	·7856	·7257	·6748	·6321	·5960	·5652	·5385	·5152	·4945	·4762	·4596	·4448	·4312
·85	·9220	·9191	·8759	·8098	·7481	·6956	·6516	·6144	·5826	·5551	·5311	·5098	·4909	·4738	·4585	·4445
·90	·9487	·9458	·9013	·8332	·7698	·7158	·6704	·6322	·5995	·5712	·5465	·5245	·5051	·4875	·4718	·4574
·95	·9747	·9712	·9260	·8561	·7909	·7354	·6888	·6493	·6159	·5869	·5614	·5389	·5189	·5009	·4847	·4699
1·00	1·0000	·9969	·9500	·8783	·8114	·7545	·7067	·6664	·6319	·6021	·5760	·5529	·5324	·5139	·4973	·4821

TABLE 1. FIVE PER CENT. VALUES OF THE REGRESSION COEFFICIENT ($b = \rho . \sigma_y/\sigma_x$). N = 18 to N = 30—(*Continued*)

σ_z^2/σ_x^2	σ_y/σ_x	18	19	20	21	22	23	24	25	26	27	28	29	30
·05	·2236	·1047	·1018	·0992	·0968	·0945	·0924	·0904	·0886	·0868	·0852	·0836	·0821	·0807
·10	·3162	·1481	·1440	·1403	·1369	·1337	·1307	·1279	·1252	·1228	·1204	·1182	·1161	·1141
·15	·3873	·1814	·1764	·1719	·1677	·1637	·1600	·1566	1534	·1503	·1475	·1448	·1423	·1398
·20	·4472	·2094	·2037	·1985	·1936	·1890	·1848	·1808	·1771	·1736	·1703	·1672	·1643	·1614
·25	·5000	·2312	·2278	·2219	·2165	·2114	·2067	·2022	·1981	·1942	·1905	·1870	·1837	·1803
·30	·5477	·2565	·2493	·2431	·2371	·2315	·2264	·2215	·2169	·2127	·2086	·2048	·2012	·1977
·35	·5916	·2770	·2693	·2626	·2561	·2501	·2443	·2392	·2343	·2297	·2253	·2212	·2173	·2136
·40	·6325	·2962	·2881	·2807	·2738	·2673	·2614	·2559	·2505	·2456	·2409	·2365	·2323	·2283
·45	·6708	·3141	·3055	·2977	·2904	·2835	·2772	·2713	·2657	·2605	·2555	·2508	·2464	·2422
·50	·7071	·3311	·3221	·3138	·3061	·2989	·2922	·2860	·2801	·2746	·2693	·2644	·2597	·2553
·55	·7416	·3473	·3378	·3201	·3210	·3135	·3063	·2990	·2937	·2880	·2825	·2773	·2724	·2677
·60	·7746	·3627	·3528	·3438	·3353	·3274	·3201	·3132	·3068	·3008	·2950	·2896	·2845	·2796
·65	·8062	·3775	·3672	·3578	·3490	·3408	·3332	·3260	·3193	·3130	·3071	·3014	·2961	·2910
·70	·8367	·3918	·3811	·3713	·3622	·3537	·3458	·3384	·3314	·3249	·3187	·3128	·3073	·3020
·75	·8660	·4055	·3945	·3843	·3749	·3661	·3570	·3502	·3430	·3363	·3299	·3238	·3181	·3126
·80	·8944	·4188	·4074	·3969	·3872	·3781	·3697	·3617	·3543	·3473	·3407	·3344	·3285	·3229
·85	·9220	·4318	·4200	·4092	·3991	·3897	·3811	·3729	·3652	·3580	·3512	·3447	·3387	·3328
·90	·9487	·4422	·4321	·4210	·4107	·4010	·3921	·3837	·3758	·3684	·3614	·3547	·3485	·3425
·95	·9747	·4565	·4440	·4326	·4219	·4120	·4028	·3942	·3861	·3785	·3713	·3644	·3580	·3519
1·00	1·0000	·4683	·4555	·4438	·4329	·4227	·4133	·4044	·3961	·3883	·3809	·3739	·3678	·3610

TABLE 2. ONE PER CENT. VALUES OF THE REGRESSION COEFFICIENT ($b = \rho \cdot \sigma_y/\sigma_x$) N = 3 to N = 17

σ_y^2/σ_x^2	σ_y/σ_x	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
·05	·2236	·2236	·2214	·2144	·2051	·1955	·1865	·1784	·1710	·1643	·1583	·1528	·1479	·1433	·1392	·1354
·10	·3162	·3162	·3130	·3031	·2900	·2765	·2638	·2522	·2418	·2323	·2238	·2161	·2091	·2027	·1960	·1915
·15	·3873	·3873	·3834	·3713	·3552	·3387	·3231	·3089	·2961	·2846	·2742	·2647	·2562	·2483	·2411	·2345
·20	·4472	·4472	·4427	·4287	·4102	·3911	·3731	·3567	·3419	·3286	·3166	·3057	·2958	·2867	·2784	·2708
·25	·5000	·5000	·4950	·4793	·4586	·4378	·4172	·3989	·3823	·3674	·3540	·3418	·3307	·3206	·3113	·3028
·30	·5477	·5477	·5422	·5251	·5024	·4790	·4569	·4369	·4188	·4024	·3877	·3744	·3622	·3511	·3410	·3316
·35	·5916	·5916	·5857	·5672	·5426	·5174	·4936	·4719	·4523	·4347	·4188	·4044	·3913	·3793	·3685	·3582
·40	·6325	·6324	·6262	·6064	·5801	·5531	·5277	·5045	·4836	·4648	·4474	·4323	·4183	·4055	·3938	·3830
·45	·6708	·6707	·6641	·6431	·6153	·5866	·5596	·5351	·5129	·4929	·4749	·4585	·4437	·4300	·4176	·4062
·50	·7071	·7070	·7000	·6779	·6486	·6184	·5899	·5641	·5406	·5196	·5006	·4835	·4677	·4533	·4402	·4281
·55	·7416	·7415	·7342	·7110	·6802	·6485	·6187	·5916	·5670	·5440	·5250	·5069	·4903	·4754	·4617	·4490
·60	·7746	·7745	·7669	·7426	·7105	·6774	·6462	·6179	·5923	·5692	·5483	·5294	·5123	·4966	·4823	·4690
·65	·8062	·8061	·7981	·7729	·7394	·7050	·6726	·6431	·6164	·5924	·5707	·5510	·5332	·5169	·5019	·4882
·70	·8367	·8366	·8283	·8021	·7674	·7317	·6981	·6674	·6397	·6148	·5923	·5719	·5534	·5364	·5209	·5066
·75	·8660	·8659	·8573	·8302	·7943	·7573	·7225	·6908	·6621	·6363	·6130	·5919	·5728	·5552	·5392	·5244
·80	·8944	·8943	·8855	·8575	·8203	·7822	·7462	·7135	·6839	·6572	·6331	·6113	·5916	·5734	·5569	·5416
·85	·9220	·9219	·9128	·8839	·8457	·8063	·7692	·7355	·7050	·6775	·6527	·6302	·6098	·5911	·5740	·5583
·90	·9487	·9486	·9392	·9095	·8701	·8296	·7915	·7568	·7254	·6971	·6716	·6484	·6274	·6082	·5907	·5744
·95	·9747	·9746	·9650	·9344	·8940	·8524	·8132	·7775	·7453	·7162	·6900	·6662	·6447	·6249	·6068	·5902
1·00	1·0000	·9999	·9900	·9587	·9172	·8745	·8343	·7977	·7646	·7348	·7079	·6835	·6614	·6411	·6226	·6053

TABLE 2. ONE PER CENT. VALUES OF THE REGRESSION COEFFICIENT ($b = \rho . \sigma_y/\sigma_x$). N = 18 to N = 30—(*Continued.*)

σ_y^2/σ_x^2	σ_y/σ_x	18	19	20	21	22	23	24	25	26	27	28	29	30
·05	·2236	·1319	·1286	·1235	·1227	·1200	·1175	·1151	·1130	·1109	·1089	·1070	·1052	·1035
·10	·3162	·1864	·1818	·1775	·1733	·1697	·1662	·1629	·1597	·1568	·1540	·1513	·1484	·1464
·15	·3873	·2284	·2227	·2174	·2125	·2079	·2036	·1995	·1957	·1920	·1886	·1853	·1822	·1793
·20	·4472	·2637	·2572	·2511	·2454	·2400	·2350	·2304	·2259	·2217	·2177	·2140	·2104	·2070
·25	·5000	·2949	·2876	·2807	·2743	·2684	·2628	·2575	·2526	·2479	·2435	·2393	·2353	·2315
·30	·5477	·3230	·3150	·3075	·3005	·2940	·2879	·2821	2767	·2715	·2667	·2621	·2577	·2535
·35	·5916	·3489	·3402	·3321	·3246	·3176	·3109	·3047	·2989	·2933	·2881	·2831	·2783	·2739
·40	·6325	·3730	·3638	·3551	·3471	·3395	·3324	·3258	·3193	·3136	·3080	·3027	·2976	·2928
·45	·6708	·3956	·3858	·3766	·3681	·3601	·3526	·3455	·3389	·3326	·3266	·3210	·3156	·3105
·50	·7071	·4170	·4067	·3970	·3880	·3796	·3717	·3642	·3572	·3506	·3443	·3383	·3327	·3273
·55	·7416	·4373	·4265	·4163	·4060	·3981	·3898	·3820	·3747	·3677	·3611	·3549	·3489	·3433
·60	·7746	·4568	·4455	·4349	·4250	·4158	·4071	·3990	·3913	·3840	·3772	·3706	·3644	·3586
·65	·8062	·4754	·4636	·4526	·4424	·4328	·4237	·4153	·4073	·3997	·3925	·3858	·3793	·3722
·70	·8367	·4934	·4812	·4697	·4501	·4491	·4398	·4310	·4227	·4148	·4074	·4004	·3937	·3873
·75	·8660	·5107	·4980	·4862	·4752	·4640	·4552	·4461	·4375	·4294	·4217	·4144	·4075	·4009
·80	·8944	·5274	·5144	·5021	·4908	·4801	·4701	·4607	·4519	·4434	·4355	·4280	·4208	·4140
·85	·9220	·5437	·5302	·5176	·5059	·4949	·4846	·4740	·4658	·4575	·4469	·4412	·4338	·4268
·90	·9487	·5594	·5456	·5326	·5206	·5093	·4986	·4857	·4793	·4704	·4610	·4540	·4464	·4392
·95	·9747	·5743	·5605	·5472	·5348	·5232	·5123	·5021	·4924	·4823	·4740	·4664	·4586	·4512
1·00	1·0000	·5897	·5751	·5614	·5487	·5368	·5256	·5151	·5052	·4958	·4860	·4785	·4705	[illegible]