A NOTE ON RANDOM COIN TOSSING

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Abstract

Harris and Keane [**?**] studied absolute continuity/singulariry of two probabilites on the coin-tossing space, one representing independent tosses of a fair coin, while in the other a biased coin is tossed at renewal times of an independent renewal process and a fair coin is tossed at all other times. We extend their results by allowing possibly different biases at the different renewal times. We also invesigate the contiguity and asymptotic separation properties in this kind of set-up and obtain some sufficient conditions.

Key Words. Renewal process, absolute continuity, singularity, contiguity, asymptotic separation, martingale convergence theorem.

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1 Introduction

Kakutani's dichotomy results on the absolute continuity and singularity of two infinite products of probability measures (Kakutani (1948)) are very well known. Harris and Keane [?] examined such dichotomy results in the following setup.
Suppose we have two coins, one of which is fair (unbiased) and the other one possibly biased. Consider independent tosses of the coins, using the biased coin at the renewal times of an independent renewal process, while using the fair coin at all other times. This gives rise to a probability measure on the infinite coin-tossing space. On the infinite coin-tossing space, consider the probability measure. Harris and eane [?] examined absolute continuity/singularity of this probability with respect to the probability given by independent tosses of the fair coin. They obtained sufficient conditions for the first measure to be either absolutely continuous or mutually singular with respect to the second measure.

Further results in this direction have been obtained by Levin et al [?]. They showed that there is a critical value of the bias at which a phase transition takes place.

In this note we consider the situation where instead of using a coin with a fixed bias each time a biased coin is to be tossed, we allow using coins with possibly different biases. We obtain some sufficient conditions for the absolute continuity and singularity in Section 2.

In [?], Thelen showed that Kakutani type dichotomy holds in case of contiguity and asymptotic separation which generalize the concept of absolute continuity and mutual singularity. In section 3, we obtain sufficient conditions for contiguity and asymptotic separation to hold in our coin-tossing set-up.

2 Coin tossing with varying bias

Let $\mathbf{N}$ be the set of nonnegative integers. We use the space $\mathbf{X} = \{-1, 1\}^{\mathbf{N}}$ to represent infinite sequences of coin tosses. Let $\{X_n\}$ be the coordinate random variables on X i.e. $X_n(x) = x_n$ where $x = (x_0, x_1, \ldots) \in X$. The σ-algebra F on X is the usual product σ-algebra. We will be concerned with various probabilities on the space $(\mathbf{X}, F)$. We now describe how these probabilities arise.

Let Ω be the space $\{0,1\}^{\mathbf{N}}$, equipped with the usual product σ-algebra and a probability P on it. Also, let $\{\Delta_n\}$ denote the coordinate random variables on Ω. We will denote $u_n = P(\Delta_n = 1)$.

Harris and Keane [**?**] considered the special case when the $\{\Delta_n\}$ are the indicators of the successive renewal times of some underlying renewal process, that is, $\Delta_n = 1$ or 0 according as a renewal takes places at time n or not. In that case, the u_n are just the probabilities of a renewal at times n. Thus, $u_0 = 1$ and for any $0 < n_1 < \cdots < n_k$, $P(\{\Delta_0 = \Delta_{n_1} = \ldots = \Delta_{n_k} = 1\}) = u_{n_1} u_{n_2-n_1} \ldots u_{n_k-n_{k-1}}$.

For $\theta \in (0,1]$, a coin with bias θ means a coin that yields values ± 1 with probabilities $(1 \pm \theta)/2$ respectively. Let $\theta = (\theta_{\mathbf{0}}, \theta_{\mathbf{1}}, \ldots \theta_{\mathbf{n}}, \ldots)$ be a sequence in (0,1]. The idea is to consider the probability μ_θ on $\mathbf{X}$ that represents independent tosses using, at time n, a fair coin if $\Delta_n = 0$ and a coin with bias θ_n if $\Delta_n = 1$.

Here is the precise definition of the probability measure μ_θ on F. Conditional on the sequence $\mathbf{\Delta} = \{\Delta_n\}$, we have the probability measure $\mu_{\theta,\mathbf{\Delta}}$ on F given by

$$\mu_{\theta,\mathbf{\Delta}}((x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n} [\Delta_i \frac{1 + x_i\theta_i}{2} + (1 - \Delta_i)\frac{1}{2}] = \prod_{i=0}^{n} \frac{1}{2}(1 + \theta_i x_i \Delta_i).$$

By averaging these conditional measures over $\mathbf{\Delta}$, we define

$$\mu_\theta((x_0, x_1, \ldots, x_n)) = \int_\Omega \mu_{\theta,\mathbf{\Delta}}((x_0, x_1, \ldots, x_n))dP = \int_\Omega \prod_{i=0}^{n} \frac{1}{2}(1 + \theta_i x_i \Delta_i)dP.$$

Here $(x_0, x_1, \ldots, x_n)$ denotes the set where $X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n$.

Of course the probability on F representing independent tosses of just a fair coin is given by

$$\mu_0((x_0, x_1, \ldots, x_n)) = 2^{-n}.$$

To state our first result, let $\mathbf{\Delta}' = \{\Delta'_n\}$ be an independent copy of $\mathbf{\Delta}$. In other words, we consider the product space $\Omega \times \Omega$ equipped with the product probability $P \otimes P$. $\mathbf{\Delta}$ and $\mathbf{\Delta}'$ are then both defined on this product space as just functions on the first and the second coordinate spaces respectively. Then we have the following Theorem:

Theorem 1 *1.* $\mu_\theta << \mu_0$ *if* $\sum_{i=0}^{\infty} \theta_i^2 < \infty$.
2. $\mu_\theta \perp \mu_0$ *if* $\sum_{i:\Delta_i\Delta_i'=1} \theta_i^2 = \infty$ *a.s.*

Remark 1. Note that for the condition in part 2 to hold it is necessary (and, of course, not sufficient) that, almost surely, $\Delta_i\Delta_i' = 1$ for infinitely many i. The proof of the theorem uses fairly standard martingale techniques.

Remark 2.The condition $\sum \theta_i^2 < \infty$ is actually sufficient for mutual absolute continuity of the measures μ_θ and μ_0.

Let $\rho_n(x)$ be the Radon-Nikodym derivative of μ_θ with respect to μ_0 when both are restricted to the σ-algebra $\sigma\{X_0, X_1, \dots X_n\}$. Clearly

$$\rho_n(x) = \int_\Omega \prod_{i=0}^{n} (1 + \theta_i x_i \Delta_i) dP.$$

Since $\{\rho_n(x)\}_{n\in\mathbf{N}}$ is a non-negative martingale under μ_0, the limit $\rho(x) = \lim_n \rho_n(x)$ exists almost surely.

Proof of part 1. Clearly $\mu_\theta << \mu_0$ on F if and only if the convergence $\rho_n \to \rho$ holds in $L_1(\mu_0)$. For the latter, it suffices to show that $\{\rho_n\}$ is bounded in $L_2(\mu_0)$.
$\int_{\mathbf{X}} \rho_n^2(x) d\mu_0(x) = \int_{\mathbf{X}} \left[\int_\Omega \prod_{i=0}^n (1 + \theta_i x_i \Delta_i) dP\right]^2 d\mu_0$

$$= \int_{\mathbf{X}} d\mu_0(x) \int_{\Omega\times\Omega} \prod_{i=0}^{n} (1 + \theta_i x_i \Delta_i + \theta_i x_i \Delta_i' + \theta_i^2 \Delta_i \Delta_i') d(P \otimes P)$$

(using Fubini's theorem and the fact that $x_i^2 = 1,\ \forall i$)

$$= \int_{\mathbf{X}} d\mu_0(x) \int_{\Omega\times\Omega} \prod_{i=0}^{n} (1 + \theta_i^2 \Delta_i \Delta_i' + \theta_i x_i (\Delta_i + \Delta_i')) d(P \otimes P)$$

$$= \int_{\Omega\times\Omega} d(P \otimes P) \int_{\mathbf{X}} \prod_{i=0}^{n} (1 + \theta_i^2 \Delta_i \Delta_i' + \theta_i x_i (\Delta_i + \Delta_i')) d\mu_0(x),$$

$$= \int_{\Omega\times\Omega} d(P \otimes P) \prod_{i=0}^{n} \int_{\mathbf{X}} (1 + \theta_i^2 \Delta_i \Delta_i' + \theta_i x_i (\Delta_i + \Delta_i')) d\mu_0(x).$$

Note that we were able to take the product $\prod_{i=0}^n$ outside the integral sign above since conditional on $(\mathbf{\Delta}, \mathbf{\Delta}')$, the random variables X_i, $i \geq 0$ are independent.

Since μ_0 is the measure of a fair coin toss process, $\int_{\mathbf{X}} x_i d\mu_0(x) = 0, \ \forall i$. Therefore,

$$\int_{\mathbf{X}} \rho_n^2(x) d\mu_0(x) = \int_{\Omega\times\Omega'} \prod_{i=0}^{n} (1+\theta_i^2 \Delta_i \Delta_i') d(P\otimes P)$$

(since each factor ≥ 1)

$$\leq \int_{\Omega\times\Omega'} \prod_{i=0}^{\infty} (1+\theta_i^2 \Delta_i \Delta_i') d(P\otimes P)$$
$$\leq \prod (1+\theta_i^2) \ \leq \exp\left(\textstyle\sum_{i=0}^{\infty} \theta_i^2\right),$$

which is finite if $\sum \theta_i^2 < \infty$. ☐

Proof of part 2. We will show that under given condition $\rho(x) = 0$ a.s $[\mu_0]$, which will clearly imply that $\mu_\theta \perp \mu_0$. Similar calculation as in the proof of part 1 yields,

$$\int_{\mathbf{X}} \rho_n(x)\rho_n(-x) d\mu_0(x) = \int_{\Omega\times\Omega'} \prod_{i=0}^{n} (1-\theta_i^2 \Delta_i \Delta_i') d(P\otimes P).$$

Now by Fatou's Lemma,

$\int_{\mathbf{X}} \rho(x)\rho(-x) d\mu_0 = \int_{\mathbf{X}} \liminf \rho_n(x)\rho_n(-x) d\mu_0$

$\leq \liminf \int_{\mathbf{X}} \rho_n(x)\rho_n(-x) d\mu_0(x) = \liminf \int_{\Omega\times\Omega} \prod_{i=1}^{n} (1-\theta_i^2 \Delta_i \Delta_i') d(P\otimes P)$.

Since the integrand is bounded above by 1 for all n, we can use DCT to conclude that

$$\int_X \rho(x)\rho(-x) d\mu_0 \leq \int_{\Omega\times\Omega'} \prod_{i=0}^{\infty} (1-\theta_i^2 \Delta_i \Delta_i') d(P\otimes P) ....(*)$$

Clearly, the integrand in $(*)$ equals 0 almost surely $(P\otimes P)$ if $\sum_{i:\Delta_i\Delta_i'=1} \theta_i^2 = \infty, a.s.(P\otimes P)$. This completes the proof. ☐

We now consider the special case when $\mathbf{\Delta}$ arises from an underlying independent renewal process, that is, $\Delta_n = 1$ or 0 according as a renewal takes place at time n or not. Note that in this case the sequence $\{\Delta_n \Delta_n'\}$ is also the sequence of indicators of renewal times of a new renewal process with renewal probabilities $\{u_n^2\}$. From the theory of renewal processes, one knows that

$$(P \otimes P)(\Delta_i \Delta_i' = 0 \; \forall \, i) = (\sum_0^\infty u_i^2)^{-1} > 0 \;\; \text{if} \; \sum_0^\infty u_i^2 < \infty,$$

while

$$(P \otimes P)(\Delta_i \Delta_i' = 1 \text{ for infinitely many } i) = 1 \;\; \text{if} \; \sum_0^\infty u_i^2 = \infty.$$

Remark 3. From the above it is clear that in case $\mathbf{\Delta}$ arises from a renewal process, the condition of part 2 of Theorem 1 will hold only if $\sum_{n=0}^\infty u_n^2 = \infty$.

Going back to the renewal sequence $\{\Delta_n \Delta_n'\}$, let T denote the time till the first renewal takes place, that is, $T = \inf\{n > 0 : \Delta_n \Delta_n' = 1\}$. One then has

Remark 4. If $E(T) < \infty$ and if $\{\theta_k\}_{k=1}^\infty$ is monotone with $\sum_{k=1}^\infty \theta_k^2 = \infty$, then $\sum\limits_{i:\Delta_i \Delta_i' = 1} \theta_i^2 = \infty$ a.s., so that $\mu_\theta \perp \mu_0$.

To see this, let us consider the case of monotone decreasing $\{\theta_k\}$. Denoting $\{S_k, \; k \geq 1\}$ to be the successive renewal times of the underlying renewal process generating $\{\Delta_n \Delta_n'\}$, one has, by the strong law, $\frac{S_k}{k} \to E(T)$ *a.s.* In particular, if M is a positive integer with $M > E(T)$, there will exist, for a.e. ω, a positive integer $k_0(\omega)$, such that $S_k \leq Mk$ for all $k > k_0(\omega)$. But then,

$$\sum_{i:\Delta_i {\Delta'}_i = 1} \theta_i^2 = \sum_{k \geq 1} \theta_{S_k}^2 \geq \sum_{k \geq k_0} \theta_{S_k}^2 \geq \sum_{k \geq k_0} \theta_{Mk}^2 \geq \frac{1}{M} \sum_{k = Mk_0}^\infty \theta_k^2 = \infty.$$

The case when the sequence $\{\theta_k\}$ is monotone increasing is trivial.

The following theorem gives a nice sufficient condition for absolute continuity of μ_θ with respect to μ_0 in the special case when the $\{\Delta_n\}$ arise from a renewal process. It extends Theorem 1, part 2 of Harris and Keane [**?**]. For the proof, one can essentially repeat the argument given in [**?**] and therefore, we omit it here.

Theorem 2 *If* $\{\Delta_n\}$ *are the indicators of the successive renewal times of a renewal process with renewal probabilities* $\{u_n\}$*, then*

$$\mu_\theta << \mu_0 \quad \textit{if} \quad \sum_{n=1}^{\infty} u_n^2 < 1 + (\sup_i \theta_i)^{-2}.$$

3 Contiguity and asymptotic separation

Contiguity and asymptotic separation are useful generalizations of absolute continuity and singularity and they have important applications in asymptotic theory of statistics. For a discussion of contiguity and asymptotic separation see Thelen [?]; Greenwood and Shiryayev [?]; Lipcer, Pukelsheim and Shiryayev [?] and Oosterhoff and van Zwet [?].

Definition Let $\{\Omega_n, F_n, (\mu_n, \tilde{\mu}_n)\}$ be a sequence of experiments. The sequence $\{\tilde{\mu}_n\}$ is said to be contiguous to the sequence $\{\mu_n\}$ (write $\tilde{\mu}_n \nabla \mu_n$) if for each sequence $\{B_n\}$ where $B_n \in F_n$, $\forall\ n$ and $\mu_n(B_n) \to 0$, as $n \to \infty$, we have $\tilde{\mu}_n(B_n) \to 0$ as $n \to \infty$. The sequence $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ are mutually contiguous if $\{\mu_n\}$ is contiguous to $\{\tilde{\mu}_n\}$ and vice versa.

The sequence $\{\mu_n\}$ is asymptotically separated from $\{\tilde{\mu}_n\}$ (write $\tilde{\mu}_n \Delta \mu_n$) if there exists a subsequence $\{n'\}$ and a corresponding subsequence of subsets $\{B_{n'}\}$ where $B_{n'} \in F_{n'}\ \forall\ n'$ and $\mu_{n'}(B_{n'}) \to 1$ as $n' \to \infty$ but $\tilde{\mu}_{n'}(B_{n'}) \to 0$. Note that it is possible to have two subsequences, along one of which μ_n and $\tilde{\mu}_n$ are contiguous, while along the other μ_n and $\tilde{\mu}_n$ are asymptotically separated which, of course is equivalent to asymptotic separation on the entire sequence.

It is easy to see that in the special case of $(\Omega_n, F_n) \equiv (\Omega, F)$, $\mu_n \equiv \mu$ and $\tilde{\mu}_n \equiv \tilde{\mu}$, $\forall\ n$, contiguity is equivalent to absolute continuity and asymptotic separation is equivalent to singularity.

We will now consider two sequences of probability measures $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ on the coin-tossing space $(\mathbf{X}, F)$, each constructed in a manner similar to those in the previous section. We then give some sufficient conditions for contiguity and asymptotic separation of these sequences.

For each $n \geq 1$, we have two sequences $\theta_n = \{\theta_{n,i}\}_{i \geq 0}$ and $\phi_n = \{\phi_{n,i}\}_{i \geq 0}$ of numbers in $(0, 1)$. Let μ_n, for each n, be the probability measure on $(\mathbf{X}, F)$ corresponding to a sequence of independent tosses of coins, where the ith toss uses a coin of bias $\theta_{n,i}$. Thus

$$\mu_n((x_0, x_1, \cdots, x_k)) = \prod_{i=0}^{k} \frac{1 + \theta_{n,i} x_i}{2}.$$

On the other hand, $\tilde{\mu}_n$, for each n, is the measure associated with a sequence of independent tosses where the ith toss uses a coin with bias $\phi_{n,i}$ if $\Delta_i = 1$ and bias $\theta_{n,i}$ if $\Delta_i = 0$. Thus

$$\tilde{\mu}_n((x_0, x_1, \cdots, x_k)) = \int_\Omega \prod_{i=0}^{k} \left[\Delta_i \frac{1 + \phi_{n,i} x_i}{2} + (1 - \Delta_i) \frac{1 + \theta_{n,i} x_i}{2} \right] dP.$$

Clearly, on the σ-algebra $\sigma\{X_0, X_1, \ldots X_k\}$, the measure $\tilde{\mu}_n$ is absolutely continuous with respect to μ_n, for each n, with the density given by

$$\rho_n^{(k)}(x_0, x_1, \ldots, x_k) = \int_\Omega \prod_{i=0}^{k} \left[1 + \frac{\Delta_i x_i (\phi_{n,i} - \theta_{n,i})}{1 + \theta_{n,i} x_i} \right] dP$$

and also, by the martingale convergence theorem, the limit $\rho_n = \lim_{k \to \infty} \rho_n^{(k)}$ exists $[\mu_n]$-almost surely.

We will first focus on the problem of contiguity. By Proposition 3.2 of Levin, Pemantle and Peres [**?**], we know that for any n, $\tilde{\mu}_n$ is either absolutely continuous or singular with respect to μ_n. On the other hand, from the definition of contiguity it follows easily that if $\tilde{\mu}_n \perp \mu_n$ for infinitely many n, then contiguity of $\{\tilde{\mu}_n\}$ with respect to $\{\mu_n\}$ cannot hold. So, except possibly for a finitely many n, $\tilde{\mu}_n$ must be absolutely continuous with respect to μ_n, for contiguity to hold.

As before, we will derive the conditions for $L_2(\mu_n)$-boundedness of $\{\rho_n^{(k)}\}_{k \geq 0}$ to guarantee absolute continuity of $\tilde{\mu}_n$ with respect to μ_n. Indeed, we have

$$\int_{\mathbf{X}} [\rho_n^{(k)}(x)]^2 d\mu_n(x) = \int_{\mathbf{X}} \left\{ \int_\Omega \prod_{i=0}^{k} \left[1 + \frac{\Delta_i x_i (\phi_{n,i} - \theta_{n,i})}{1 + \theta_{n,i} x_i} \right] dP \right\}^2 d\mu_n(x)$$

$$= \int_{\mathbf{X}} d\mu_n(x) \int\limits_{\Omega \times \Omega} \prod_{i=0}^{k} \left\{ 1 + \frac{\Delta_i x_i (\phi_{n,i} - \theta_{n,i})}{1 + \theta_{n,i} x_i} \right\} \left\{ 1 + \frac{\Delta'_i x_i (\phi_{n,i} - \theta_{n,,i})}{1 + \theta_{n,i} x_i} \right\} d(P \otimes P)$$

$$= \int_{\mathbf{X}} d\mu_n(x) \int\limits_{\Omega \times \Omega} \prod_{i=0}^{k} \left\{ 1 + \Delta_i \Delta'_i \frac{(\phi_{n,i} - \theta_{n,i})^2}{(1 + \theta_{n,i} x_i)^2} \right.$$

$$+\frac{\Delta_i x_i(\phi_{n,i}-\theta_{n,i})}{1+\theta_{n,i}x_i} + \frac{\Delta'_i x_i(\phi_{n,i}-\theta_{n,i})}{1+\theta_{n,i}x_i}\Big\} \, d(P\otimes P)$$

$$= \int_{\Omega\times\Omega} \int_{\mathbf{X}} d\mu_n(x) \prod_{i=0}^k \Big\{1 + \Delta_i\Delta'_i \frac{(\phi_{n,i}-\theta_{n,i})^2}{(1+\theta_{n,i}x_i)^2}$$

$$+\frac{\Delta_i x_i(\phi_{n,i}-\theta_{n,i})}{1+\theta_{n,i}x_i} + \frac{\Delta'_i x_i(\phi_{n,i}-\theta_{n,i})}{1+\theta_{n,i}x_i}\Big\} \, d(P\otimes P).$$

The integration over $\mathbf{X}$ with respect to the measure μ_n is easy to perform. First of all, μ_n is a product probability so that the product $\prod_{i=0}^k$ can be pushed outside the integral. Next, $\mu_n((x_i)) = (1+\theta_{n,i}x_i)/2$, so that $\int_{\mathbf{X}} \frac{1}{(1+\theta_{n,i}x_i)2} \, d\mu_n(x) = \frac{1}{1-\theta_{n,i}^2}$ and $\int_{\mathbf{X}} \frac{x_i}{1+\theta_{n,i}x_i} \, d\mu_n(x) = 0$.

Thus we finally get

$$\int_{\mathbf{X}} [\rho_n^{(k)}(x)]^2 d\mu_n(x) = \int_{\Omega\times\Omega} \prod_{i=0}^k \left\{1 + \Delta_i\Delta'_i \frac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2}\right\} \, d(P\otimes P).$$

A sufficient condition for this to remain bounded over k, for a fixed n, is that $\sum_{i=1}^{\infty} \frac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2} < \infty$, so that this latter condition will imply $\tilde{\mu}_n \ll \mu_n$. But absolute continuity of $\tilde{\mu}_n$ with respect to μ_n even for all but a finitely many n does not guarantee contiguity. In addition, we will need that the densities $\{\rho_n\}$ be tight with respect to $\{\tilde{\mu}_n\}$, i.e. we must have

$$\lim_{k\to\infty} \limsup_n \tilde{\mu}_n(\rho_n > k) = 0.$$

See page 31, Greenwood and Shiryayev [**?**]. If $\{\rho_n\}$ is uniformly bounded then, of course, $\{\rho_n\}$ is tight. So we proceed to obtain conditions for uniform boundedness of $\{\rho_n\}$.

Since $\rho_n(x) \le \sup_k \rho_n^{(k)}(x) = \sup_k \int_\Omega \prod_{i=0}^k \left[1 + \frac{\Delta_i x_i(\phi_{n,i}-\theta_{n,i})}{1+\theta_{n,i}x_i}\right] dP$, a simple upper bound for $\rho_n(x)$ is given by

$$\rho_n(x) \;\le\; \prod_{i=0}^{\infty} \left[1 + \frac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}}\right]. \qquad (**)$$

So a sufficient condition for the ρ_n to be uniformly bounded is

$$\limsup_n \sum_{i=0}^{\infty} \frac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}} < \infty\,.$$

Since $\sum_{i=0}^{\infty} \frac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2} < \sum_{i=0}^{\infty} \frac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}}$, the above condition is, therrfore, sufficient for contiguity of $\{\tilde{\mu}_n\}$ with respect to $\{\mu_n\}$.

We thus have the following Theorem:

Theorem 3 $\tilde{\mu}_n \nabla \mu_n$ *if the following two conditions hold:*

i) $\limsup_n (\sup_i \theta_{n,i}) < 1$ *and*

ii) $\limsup_n \sum_{i=0}^{\infty} (\phi_{n,i} - \theta_{n,i}) < \infty$.

Remark 5. Using arguments similar to the proof of Theorem 2 and the upper bound (**) obtained above, one can show that when $\{\Delta_n\}$ are indicators of renewal times of a renewal process with probability $P(\Delta_n = 1) = u_n$, the following two conditions are sufficient to guarantee $\tilde{\mu}_n \nabla \mu_n$:

(i) $\limsup_n \sum_{i=0}^{\infty} \frac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}} < \infty$.

(ii) $\sum_{n=1}^{\infty} u_n^2 < 1 + \frac{1}{\theta}$ where $\theta = \limsup_n \frac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2}$.

Next we will find sufficient conditions under which asymptotic separation occurs. Since having $\tilde{\mu}_n \perp \mu_n$ for infinitely many n is clearly sufficient for $\tilde{\mu}_n \Delta \mu_n$, a simple set of sufficient conditions would be those that guarantee $\rho_n(x) = 0$ a.s. $[\mu_n]$ for infinitely many n.

As before, we will use the technique of Harris and Keane [**?**] . But for that we need the following two lemmas:

Lemma 1 *Consider the probabilities arising from two sequences of independent coin tossings. The first uses a fair coin throughout, while the second uses a sequence of coins with biases* $\{\theta_i\}$. *Then these two probabilities are mutually absolutely continuous if* $\sum_{i=0}^{\infty} \theta_i^2 < \infty$.

Proof. Since both are product probabilities, we can use Kakutani's [**?**] criterion. Let $\nu = \prod_{i=1}^{\infty} \nu_0$ and $\tilde{\nu} = \prod_{i=0}^{\infty} \nu_{\theta_i}$ denote the two measures. Here, $\nu_0(1) = \nu_0(-1) = \frac{1}{2}$, whereas $\nu_{\theta_i}(1) = \frac{1+\theta_i}{2} = 1 - \nu_{\theta_i}(-1)$, for all i.

By Kakutani's criterion applied to these product probabilities, ν and $\tilde{\nu}$ are mutually absolutely continuous if and only if

$$\prod_{i=1}^{\infty} \left(\frac{\sqrt{1+\theta_i}}{2} + \frac{\sqrt{1-\theta_i}}{2}\right) < \infty \,.$$

Expanding both $\sqrt{1+\theta_i}$ and $\sqrt{1-\theta_i}$ binomially, we have,

$\prod_{i=0}^{\infty}\left[\frac{1}{2}(\sqrt{1+\theta_i}+\sqrt{1-\theta_i})\right]=\prod_{i=0}^{\infty}\left[1-\sum_{n=1}^{\infty}\frac{1.3.5....(4n-3)}{2.4.6....4n}\,\theta_i^{2n}\right]$. The infinite product is positive if $\sum_{i=0}^{\infty}\sum_{n=1}^{\infty}\frac{1.3.5....(4n-3)}{2.4.6....4n}\,\theta_i^{2n}\big] < \infty$. Since the coefficients of θ_i^{2n} are all less than 1, the above sum is less than $\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\theta_i^{2n}$ which is finite if $\sum_{i=0}^{\infty}\theta_i^2$ is finite. □

Going back now to our sequences of probabilities $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ on $(\mathbf{X}, F)$, recall that for each n, ρ_n is the $[\mu_n]$-a.s. limit $\lim_k \rho_n^{(k)}$, where $\rho_n^{(k)}$ is the density of $\tilde{\mu}_n$ with respect to μ_n on $\sigma\{X_0, X_1, \ldots, X_k\}$. We define $\rho_n(x)$ to be $\limsup_k \rho_n^{(k)}(x)$ in case $\lim_k \rho_n^{(k)}(x)$ does not exist.

Lemma 2 *For each n, the event $\{x : \rho_n(x) = 0\}$ is a tail event.*

Only a slight modification of the proof of Lemma 1 of Harris and Keane (see page 32 of [**?**]) gives the result. However, for the sake of completeness we are presenting the proof below.

Proof. Let $F'_m = \sigma(X_m, X_{m+1}, \ldots)$ and $F' = \cap_{m=0}^{\infty} F'_m$ be the tail σ-algebra. We will show that $\{\rho_n = 0\} \in F'$. For this it is enough to show that if $x \in \mathbf{X}$ is such that $\rho_n(x) = 0$, then for any $y \in \mathbf{X}$ with $y_i = x_i$, $\forall\, i > m$ for some m, one has $\rho_n(y) = 0$ also. It clearly suffices to do this only for y of the form $y = (x_0, x_1, \ldots, x_{m-1}, -x_m, x_{m+1}, \ldots)$.

By definition, $$\rho_n(x) = \limsup_k \int_\Omega \prod_{i=0}^{k}\left[1+\Delta_i x_i \frac{\phi_{n,i}-\theta_{n,i}}{1+\theta_{n,i}x_i}\right] dP$$

and $$\rho_n(y) = \limsup_k \int_\Omega \prod_{i=0}^{k}\left[1+\Delta_i y_i \frac{\phi_{n,i}-\theta_{n,i}}{1+\theta_{n,i}y_i}\right] dP.$$

From this and using the fact that $y_m = -x_m$ and $y_i = x_i\ \forall\, i \neq m$, one easily obtains

$$\frac{1-\frac{\phi_{n,m}-\theta_{n,m}}{1-\theta_{n,m}}}{1+\frac{\phi_{n,m}-\theta_{n,m}}{1-\theta_{n,m}}}\,\rho_n(x) \;\leq\; \rho_n(y) \;\leq\; \frac{1+\frac{\phi_{n,m}-\theta_{n,m}}{1-\theta_{n,m}}}{1-\frac{\phi_{n,m}-\theta_{n,m}}{1-\theta_{n,m}}}\,\rho_n(x)\,,$$

from which it follows that $\rho_n(x) = 0$ if and only if $\rho_n(y) = 0$. □

Now, we will prove the following theorem:

Theorem 4 *$\tilde{\mu}_n \Delta \mu_n$ if*

i) $\limsup_n \sum_{i=0}^{\infty} \theta_{n,i}^2 < \infty$ *and*

ii) $\sum_{i:\Delta_i\Delta_i'=1} \phi_{n,i}^2 = \infty$ *[P]-a.s., for infinitely many n.*

Remark 6. Of course, in order for condition (ii) to hold it is necessary that $P(\Delta_i\Delta_i' = 1 \text{ for infinitely many } i) = 1$. In the special case when the $\{\Delta_i\}$ are the indicators of renewal times of an underlying renewal process, we have already seen that this last condition is equivalent to $\sum_{n=0}^{\infty} u_n^2 = \infty$.

Proof. Routine calculation gives us

$$\begin{aligned}
&\int_{\mathbf{X}} \rho_n^{(k)}(x)\rho_n^{(k)}(-x)d\mu_n(x) \\
&= \int_{\Omega\times\Omega} \int_X \prod_{i=0}^{k} \left[1 - \Delta_i\Delta_i' \frac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2} \right. \\
&\qquad \left. + \Delta_i x_i \tfrac{\phi_{n,i}-\theta_{n,i}}{1+\theta_{n,i}x_i} - \Delta_i' x_i \tfrac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}x_i}\right] d\mu_n(x)\, d(P\otimes P) \\
&= \int_{\Omega\times\Omega} \prod_{i=0}^{k} \left[1 - \Delta_i\Delta_i' \tfrac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2} - \Delta_i' \tfrac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}^2} 2\theta_{n,i}\right] d(P\otimes P)\,.
\end{aligned}$$

Using Fatou's Lemma first and then DCT, one obtains,

$$\begin{aligned}
&\int_{\mathbf{X}} \rho_n(x)\rho_n(-x)d\mu_n(x) \\
&\le \liminf_k \int_{\Omega\times\Omega} \prod_{i=0}^{k} \left[1 - \Delta_i\Delta_i' \frac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2} - \Delta_i' \frac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}^2} 2\theta_{n,i}\right] d(P\otimes P) \\
&= \int_{\Omega\times\Omega} \prod_{i=0}^{\infty} \left[1 - \Delta_i\Delta_i' \tfrac{(\phi_{n,i}-\theta_{n,i})^2}{1-\theta_{n,i}^2} - \Delta_i' \tfrac{\phi_{n,i}-\theta_{n,i}}{1-\theta_{n,i}^2}, 2\theta_{n,i}\right] d(P\otimes P), \cdots\cdots (***)
\end{aligned}$$

so that, $\int_X \rho_n(x)\rho_n(-x)d\mu_n(x) = 0$ if the integrand in $(***)$ is 0, i.e., if $\sum_{i:\Delta_i\Delta_i'=1} \frac{(\phi_{n,i}-\theta_{n,i})^2 + 2\theta_{n_i}(\phi_{n,i}-\theta_{n,i})}{1-\theta_{n,i}^2} = \sum_{i:\Delta_i\Delta_i'=1} \frac{\phi_{n,i}^2-\theta_{n,i}^2}{1-\theta_{n,i}^2} = \infty$ a.s. $[P\otimes P]$. This happens by conditions (i) and (ii) of the Theorem for infinitely many n. So, under the two conditions of the Theorem, $\int_X \rho_n(x)\rho_n(-x)d\mu_n(x) = 0$ for infinitely many n. Let us fix one such n. Since by Lemma 2, $\{\rho_n = 0\}$ is a tail event, either of the following two cases must occur:

(a) $\mu_n(\{x : \rho_n(x) = 0\}) = 1$

(b) $\mu_n(\{x : \rho_n(-x) = 0\}) = 1$.

In case (a), we have $\tilde{\mu}_n \perp \mu_n$ and we are done. In case (b), we have, by Lemma 1 and condition (i) of the Theorem, $\nu(\{x : \rho_n(-x) = 0\}) = 1$, where ν is the probability on $(\mathbf{X}, F)$ as defined in the proof of Lemma 1. But then, by symmetry of the measure ν, we would have $\nu(\{x : \rho_n(x) = 0\}) = 1$, which, in turn, implies $\mu_n(\{x : \rho_n(x) = 0\}) = 1$ (again by Lemma 1 and condition (i) of the Theorem). Thus we are back to case (a). □

Remark 7. In case $\{\Delta_i\}$ are the indicators of renewal times of a renewal process, we have a similar result like the one stated in Remark 4. Denoting $T = \inf\{n > 0 : \Delta_n \Delta'_n = 1\}$, one has the following:

If $E(T) < \infty$ and if $\frac{\phi_{n,i}^2 - \theta_{n,i}^2}{1 - \theta_{n,i}^2}$ is monotone in i and $\sum_{i=1}^{\infty} \frac{\phi_{n,i}^2 - \theta_{n,i}^2}{1 - \theta_{n,i}^2} = \infty$ for infinitely many n, then $\tilde{\mu}_n \Delta \mu_n$

Remark 8. We have derived only a set of sufficient conditions. It would be interesting to derive reasonable necessary conditions but the problem does not seem to be easy. Moreover, in Levin et al [**?**] it has been shown that in the renewal setup Kakutani like dichotomy holds, i.e. the probability measures of the two dependent processes are either mutually absolutely continuous or singular. On the other hand Thelen [**?**] gave sufficient conditions for contiguity/asymptotic separation dichotomy in case of two sequences of measures in independent setup. It would be interesting to know whether contiguity/asymptotic separation dichotomy holds in the renewal setup.

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