Local Equivalence of Rank-Two Quantum Mixed States

Sergio Albeverio[a] [1], Shao-Ming Fei[b,c], Debashish Goswami[d] [2]

[a] Institut für Angewandte Mathematik, Universität Bonn, D-53115
[b] Department of Mathematics, Capital Normal University, Beijing 100037
[c] Max Planck Institute for Mathematics in the Sciences, D-04103 Leipzig
[d] Statistics and Mathematics Unit, Indian Statistical Institute, Kolkata 700108

Abstract

We investigate the equivalence of quantum mixed states under local unitary transformations. For a class of rank-two mixed states, a sufficient and necessary condition of local equivalence is obtained by giving a complete set of invariants under local unitary transformations, such that two states in this class are locally equivalent if and only if all these invariants have equal values for them.

Quantum entanglement has been extensively investigated as a key physical resource to realize quantum information tasks such as quantum cryptography, quantum teleportation and quantum computation [1]. Due the fact that the properties of entanglement for multipartite quantum systems remain invariant under local unitary transformations on the subsystems, the entanglement can be characterized in principle by all the invariants under local unitary transformations. For instance, the trace norms of realigned or partial transposed density matrices in entanglement measure and separability criteria are some of these invariants [2]. Therefore a complete set of invariants gives rise to the classification of the quantum states under local unitary transformations. Two quantum states are locally equivalent if and only if all these invariants have equal values for these states.

There have been many results on calculation of invariants [3, 4] related to the equivalence of quantum states under local unitary transformations, e.g. for general two-qubit systems [5], three-qubit states [6, 7], some generic mixed states [8, 9, 10], some classes of tripartite pure and mixed states [11]. However till now we still have no operational criteria to judge

the equivalence for two general bipartite mixed states under local unitary transformations. In this letter we investigate the local equivalence under local unitary transformations for a class of rank-two bipartite mixed quantum states in arbitrary dimensions, and present an operational criterion.

Let H_1 and H_2 be m and n-dimensional complex Hilbert spaces, with $|e_\alpha\rangle$, $\alpha = 1, ..., m$, and $|f_\beta\rangle$, $\beta = 1, ..., n$, $m \leq n$, as orthonormal bases. Let ρ_1 and ρ_2 be two bipartite density matrices defined on $H_1 \otimes H_2$ with rank $r(\rho_1) = r(\rho_2) = 2$. ρ_1 and ρ_2 are said to be equivalent under local unitary transformations if there exist unitary operators U_1 on H_1 and U_2 on H_2 such that

$$\rho_2 = (U_1 \otimes U_2)\rho_1(U_1 \otimes U_2)^\dagger, \tag{1}$$

where $\dagger$ stands for transpose and conjugation.

As ρ_1 and ρ_2 are rank-two density matrices, they have the following decompositions according to their eigenvalues and eigenvectors:

$$\rho_i = \sum_{\alpha=1}^{2} \lambda_\alpha^i |\nu_\alpha^i\rangle\langle\nu_\alpha^i|, \quad i = 1, 2,$$

where λ_α^i and $|\nu_\alpha^i\rangle$, $\alpha = 1, 2$, are the nonzero eigenvalues and eigenvectors of the density matrix ρ^i respectively, $\sum_{\alpha=1}^{2} \lambda_\alpha^i = 1$. $|\nu_\alpha^i\rangle$ has generally the form

$$|\nu_1^i\rangle = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{\alpha\beta}^i |e_\alpha\rangle \otimes |f_\beta\rangle, \quad |\nu_2^i\rangle = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} b_{\alpha\beta}^i |e_\alpha\rangle \otimes |f_\beta\rangle,$$

where $a_{\alpha\beta}^i$, $b_{\alpha\beta}^i \in \mathbb{C}$, $\sum_{\alpha\beta} a_{\alpha\beta}^i a_{\alpha\beta}^{i*} = \sum_{\alpha\beta} b_{\alpha\beta}^i b_{\alpha\beta}^{i*} = 1$, $i = 1, 2$, $*$ denotes complex conjugation.

Let A_i and B_i denote the $m \times n$ matrices with entries $a_{\alpha\beta}^{(i)}$ and $b_{\alpha\beta}^{(i)}$ respectively. We consider the necessary and sufficient conditions of equivalence under local unitary transformations for a class of rank-two states satisfying the following conditions:

$$A_i^\dagger A_i = B_i^\dagger B_i, \quad A_i A_i^\dagger = B_i B_i^\dagger \quad \text{for } i = 1, 2. \tag{2}$$

[Theorem] The density matrices ρ_1 and ρ_2 are equivalent under local unitary transformations if and only if the following hold:

(i) $Tr(\rho_1^2) = Tr(\rho_2^2)$;

(ii) $Tr((A_1 B_1^\dagger)^\alpha) = Tr((A_2 B_2^\dagger)^\alpha)$, $\forall \alpha = 1, ..., m$;

(iii) $r(A_1) = r(A_2)$, $r(B_1) = r(B_2)$, $r((B_1^\dagger A_1)^\alpha) = r((B_2^\dagger A_2)^\alpha)$, $\forall \alpha = 1, ..., m$.

[Proof] It is straightforward to see that (i)-(iii) above hold if ρ_1 and ρ_2 are equivalent under local unitary transformations, in the sense of eq.(1).

We prove the converse. Two pairs of $(m \times n)$ matrices, (A, B) and (C, D), are called contragrediently equivalent if $A = SCT^{-1}$, $B = TDS^{-1}$ for some invertible matrices S

and T. It is shown in [12] that the pairs (A, B) and (C, D) are contragrediently equivalent if and only if AB is similar to CD and $r(A) = r(C)$, $r(B) = r(D)$, $r(BA)^\alpha = r(DC)^\alpha$, $r(AB)^\alpha = r(CD)^\alpha$ for all $\alpha = 1, ..., m$.

Therefore from the conditions (ii) and (iii) we have that the pairs $(A_1, B_1^\dagger)$ and $(A_2, B_2^\dagger)$ are contragrediently equivalent and there are invertible (but not necessarily unitary) matrices S and T such that

$$SA_2 = A_1T, \quad TB_2^{-1} = B_1^{-1}S. \tag{3}$$

Eq. (3) can be rewritten as,

$$\begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix} \begin{pmatrix} 0 & A_2 \\ B_2^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_1^\dagger \\ A_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix}.$$

By assumption (2), the matrices $W_1 := \begin{pmatrix} 0 & A_2 \\ B_2^\dagger & 0 \end{pmatrix}$ and $W_2 := \begin{pmatrix} 0 & B_1^\dagger \\ A_1 & 0 \end{pmatrix}$ are normal.

If two normal matrices M, N and an invertible matrix X satisfy $XMX^{-1} = N$, then one has $U_X M U_X^\dagger = N$, where $X = U_X |X|$ is the polar decomposition of X and U_X is unitary [13]. Therefore from the observation that the unitary part of the polar decomposition in $\begin{pmatrix} 0 & S \\ T & 0 \end{pmatrix}$ is nothing but $\begin{pmatrix} 0 & U_S \\ U_T & 0 \end{pmatrix}$, we have

$$\begin{pmatrix} 0 & U_T \\ U_S & 0 \end{pmatrix} \begin{pmatrix} 0 & A_2 \\ B_2^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_1^\dagger \\ A_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & U_T \\ U_S & 0 \end{pmatrix},$$

which is equivalent to

$$A_2 = U_S^\dagger A_1 U_T, \quad B_2 = U_S^\dagger B_1 U_T.$$

Here U_S and U_T are unitary (as S, T are invertible). The condition (i) and $Tr(\rho_1) = Tr(\rho_2) = 1$ together imply that the density matrices ρ_1 and ρ_2 have the same eigenvalues. Therefore $\rho_2 = (U_1 \otimes U_2)\rho_1(U_1 \otimes U_2)^\dagger$, where $U_1 = U_S^\dagger$, $U_2 = (U_T)^t$ (t denoting transpose). $\square$

The Theorem gives a sufficient and necessary condition for local equivalence of two rank-two mixed states satisfying (2). The class of quantum states satisfying (2) is not trivial. As a simple example, we consider the two-qubit systems. In this case A and B are 2×2 matrices. It is easily verified that the following matrices satisfy the required conditions,

$$A(\theta) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad B(\gamma) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\gamma & \sin\gamma \\ \sin\gamma & -\cos\gamma \end{pmatrix}.$$

Hence the rank-two density matrix $\rho = \lambda|\psi\rangle\langle\psi| + (1-\lambda)|\phi\rangle\langle\phi|$, where $|\psi\rangle = \sum_{\alpha,\beta=1}^2 a_{\alpha\beta}(\theta)|e_\alpha\rangle \otimes |f_\beta\rangle$, $|\phi\rangle = \sum_{\alpha,\beta=1}^2 b_{\alpha\beta}(\gamma)|e_\alpha\rangle \otimes |f_\beta\rangle$, belongs to the class we are concerning. From the theorem we have that ρ and $\rho' = \lambda|\psi'\rangle\langle\psi'| + (1-\lambda)|\phi'\rangle\langle\phi'|$ with $|\psi'\rangle = \sum_{\alpha,\beta=1}^2 a_{\alpha\beta}(\theta')|e_\alpha\rangle \otimes |f_\beta\rangle$, $|\phi'\rangle = \sum_{\alpha,\beta=1}^2 b_{\alpha\beta}(\gamma')|e_\alpha\rangle \otimes |f_\beta\rangle$, are equivalent under local unitary transformations.

Here the concurrence $C(|\psi\rangle) = C(|\phi\rangle) = 1$. Both pure states $|\psi\rangle$ and $|\phi\rangle$ are maximally entangled. In the special case $\theta = 0$ (resp. $\gamma = 0$), $|\psi\rangle$ (resp. $|\phi\rangle$) is reduced to one of the Bell bases $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ (resp. $|\phi\rangle = (|00\rangle - |11\rangle)/\sqrt{2}$). These states are equivalent under local unitary transformations. Nevertheless, generally ρ and ρ' are not equivalent under local unitary transformations even if $|\psi\rangle$ (resp. $|\phi\rangle$) is equivalent to $|\psi'\rangle$ (resp. $|\phi'\rangle$) under local unitary transformations, unless the same local unitary transformations transform $|\psi\rangle$ to $|\psi'\rangle$ and $|\phi\rangle$ to $|\phi'\rangle$ simultaneously.

Generally a rank-two state can be written as $\rho = \lambda|\nu_1\rangle\langle\nu_1| + (1-\lambda)|\nu_2\rangle\langle\nu_2|$, $0 < \lambda < 1$. The normalized vectors $|\nu_1\rangle$ and $|\nu_2\rangle$ are given by the $m \times n$ matrices $(A)_{\alpha\beta} = a_{\alpha\beta}$ and $(B)_{\alpha\beta} = b_{\alpha\beta}$ respectively, $|\nu_1\rangle = \sum_{\alpha\beta} a_{\alpha\beta}|e_\alpha\rangle \otimes |f_\beta\rangle$, $|\nu_2\rangle = \sum_{\alpha\beta} b_{\alpha\beta}|e_\alpha\rangle \otimes |f_\beta\rangle$, with $Tr(AA^\dagger) = Tr(BB^\dagger) = 1$ due to normalization. Let us consider the general forms of a pair of matrices A and B such that the conditions $A^\dagger A = BB^\dagger$ and $AA^\dagger = BB^\dagger$ are satisfied.

Since $A^\dagger A = B^\dagger B$, we can write down singular value decomposition of A and B as follows:
$$A = U\Delta V^\dagger, \quad B = U'\Delta V'^\dagger,$$
where U, U' and V, V' are unitary matrices and Δ is a diagonal matrix with nonnegative entries. Furthermore, the condition $A^\dagger A = B^\dagger B$ implies $V'^\dagger V \Delta^2 = \Delta^2 V'^\dagger V$. Thus, $V'^\dagger V$ commutes with Δ. Similarly from $AA^\dagger = BB^\dagger$ we conclude that $U'^\dagger U$ also commutes with Δ. Hence we have
$$B = U'\Delta V'^\dagger = UU^\dagger U'\Delta V'^\dagger VV^\dagger = U\Gamma\Delta V^\dagger, \tag{4}$$
where $\Gamma = (U'^\dagger U)^\dagger V'^\dagger V$ is unitary and commutes with Δ. Therefore the pair (A, B) can be transformed into the pair $(\Delta, \Gamma\Delta)$. We call $(\Delta, \Gamma\Delta)$ the canonical form of the pair (A, B). If the diagonal matrix Δ is of the form $diag(d_1, ..., d_1, d_2, ..., d_2, ..., d_k, ..., d_k)$, where d_i is repeated with multiplicity m_i, then Γ must have the block diagonal form $diag(\Gamma_1, ..., \Gamma_k)$, where Γ_i, $i = 1, ..., k$, are $m_i \times m_i$ unitary matrices.

In fact, if we have another pair of matrices A' and B', associated with the eigenvectors $|\nu_1'\rangle$ and $|\nu_2'\rangle$ of another rank-two density matrix in the class considered, with canonical form $(\Delta', \Gamma'\Delta')$, then $|\nu_1'\rangle$, $|\nu_2'\rangle$ and $|\nu_1\rangle$, $|\nu_2\rangle$ are equivalent under local unitary transformations if and only if $\Delta = \Delta'$ and $w_i\Gamma_i w_i^\dagger = \Gamma_i'$ for some unitary matrix w_i, $i = 1, ..., k$.

Therefore under the local unitary transformation $\rho \rightarrow (U \otimes V^*)\rho(U \otimes V^*)^\dagger$, a rank-two mixed state in our class has the standard form: $\rho = \lambda|\mu_1\rangle\langle\mu_1| + (1-\lambda)|\mu_2\rangle\langle\mu_2|$, where $|\mu_1\rangle = \sum_\alpha d_\alpha|e_\alpha\rangle \otimes |f_\alpha\rangle$, $|\mu_2\rangle = \sum_{\alpha\beta}(\Gamma\Delta)_{\alpha\beta}|e_\alpha\rangle \otimes |f_\beta\rangle$. In particular, if all the singular values are distinct, then two such density matrices are equivalent under local unitary transformation if and if they have exactly the same standard form.

We have investigated the equivalence under local unitary transformations for a class of rank-two bipartite mixed quantum states. A complete set of invariants has been presented

such that any two of these states are locally equivalent if and only if all these invariants have equal values for these related density matrices.

Our method can be applied to another classification of quantum states, defined by local operations and classical communication (LOCC). Two states have the same kind of entanglement if they can be obtained from each other by LOCC with nonzero probability [14]. There have been many results for bipartite and multipartite pure states for their equivalence under SLOCC [15, 16, 17]. In [15] Dür et al showed that for pure three-qubit states there are six different classes of entanglement under SLOCC. Verstraete et al considered the entanglement of four-qubit case under SLOCC and concluded that there exist nine families of states corresponding to nine different ways of entanglement [16]. Nevertheless for mixed states few is known yet.

Corresponding to pure states, we say that ρ_1 and ρ_2 are equivalent under SLOCC if there exist *invertible* (but not necessarily unitary) matrices P and Q such that

$$\rho_2 = (P \otimes Q)\rho_1(P \otimes Q)^\dagger. \tag{5}$$

[**Proposition**] The density matrices ρ_1 and ρ_2, with B_1 and B_2 nonsingular, are equivalent under SLOCC if the following hold:

(i) $Tr(\rho_1^2) = Tr(\rho_2^2)$;

(ii) $Tr((A_1 B_1^{-1})^\alpha) = Tr((A_2 B_2^{-1})^\alpha)$, $\forall\ \alpha = 1, ..., m$;

(iii) $r(A_1) = r(A_2)$, $r(B_1) = r(B_2)$, $r((B_1^{-1}A_1)^\alpha) = r((B_2^{-1}A_2)^\alpha)$, $\forall\ \alpha = 1, ..., m$.

[**Proof**] From the conditions (ii) and (iii) we have that the pairs (A_1, B_1^{-1}) and (A_2, B_2^{-1}) are contragrediently equivalent. Hence there are invertible (but not necessarily unitary) matrices S and T such that $SA_2 = A_1 T$, $TB_2^{-1} = B_1^{-1}S$.

That is, we have $A_2 = S^{-1}A_1 T$, $\quad B_2 = S^{-1}B_1 T$. Accounting to the condition (i) which implies that the density matrices ρ_1 and ρ_2 have the same eigenvalues, the above relations give rise to the equivalence of ρ_1 and ρ_2 under SLOCC. $\qquad\square$

The classification of quantum states under local operations is of significance in quantum information processing. We have presented some criteria for the equivalence of some bipartite mixed states in arbitrary dimensions. Our results can be generalized to the case of multipartite states by considering bipartite decompositions. In terms of the method used in [11], our equivalence criteria for bipartite mixed states can be also used to study the equivalence of tripartite pure states.

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