A d-person Differential Game with State Space Constraints

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Abstract We consider a network of d companies (insurance companies, for example) operating under a treaty to diversify risk. Internal and external borrowing are allowed to avert ruin of any member of the network. The amount borrowed to prevent ruin is viewed upon as control. Repayment of these loans entails a control cost in addition to the usual costs. Each company tries to minimize its repayment liability. This leads to a d-person differential game with state space constraints. If the companies are also in possible competition a Nash equilibrium is sought. Otherwise a utopian equilibrium is more appropriate. The corresponding systems of HJB equations and boundary conditions are derived. In the case of Nash equilibrium, the Hamiltonian can be discontinuous; there are d interlinked control problems with state constraints; each value function is a constrained viscosity solution to the appropriate discontinuous HJB equation. Uniqueness does not hold in general in this case. In the case of utopian equilibrium, each value function turns out to be the unique constrained viscosity solution to the appropriate HJB equation. Connection with Skorokhod problem is briefly discussed.

Keywords d-person differential game · State space constraints · Nash equilibrium · Utopian equilibrium · Dynamic programming principle · System of HJB equations · Constrained viscosity solution · Semicontinuous envelope · Deterministic Skorokhod problem · Drift · Reflection

1 Introduction

Consider d insurance companies. Suppose the surplus (or reserve) of Company i, in the absence of any control, is given by

$$S_i(t) = z_i + \int_0^t b_i(r, S_i(r))dr, \quad t \geq 0,$$

where $z_i \geq 0$ is the initial reserve and b_i is the "drift" component. The term b_i incorporates premium rate (>0) of Company i, interest rate (>0) of riskless bonds in which the company may have invested part of its surplus, mean rate (<0) at which claim payments are made, etc. We say Company i is ruined if $S_i(t) < 0$ for some $t \geq 0$. Now suppose the d companies agree on a treaty, to diversify risk, along the following lines. Accordingly, if Company i estimates at some instant of time that it needs an amount $u_i(r)dr$ to avoid ruin, then for $j \neq i$, Company j gives $|R_{ji}|u_i(r)dr$, where $u_i(\cdot) \geq 0$, $R_{ji} \leq 0$, $j \neq i$ and $\sum_{j\neq i} |R_{ji}| \leq 1$. Of course, the shortfall $(1 - \sum_{j\neq i} |R_{ji}|)u_i(r)dr$ has to be procured by Company i from "external" sources. The amount $\sum_{j\neq i} |R_{ji}|u_i(r)dr$ that Company i gets from the other companies of the network is considered a loan on soft interest terms, whereas the amount obtained from external sources carry interest at market rates. As there is mutual obligation among the companies, this is a reasonable way of diversifying the risk.

The function $u(\cdot) = (u_1(\cdot), \ldots, u_d(\cdot))$ is viewed upon as control, with u_i denoting the control for Company i. With the treaty in force, we get the following system of equations to constitute state equations

$$y_i(t) = \int_0^t u_i(r)dr, \tag{1.1}$$

$$z_i(t) = z_i + \int_0^t b_i(r, z_i(r))dr + y_i(t) + \sum_{j\neq i} R_{ij} y_j(t), \tag{1.2}$$

for $t \geq 0$, with the stipulation that

$$z_i(t) \geq 0, \quad t \geq 0 \tag{1.3}$$

for each $i = 1, 2, \ldots, d$. Here $z_i(t) =$ current surplus with Company i at time t, $y_i(t) =$ cumulative amount obtained by Company i from internal and external sources specifically for the purpose of preventing ruin over the period $[0, t]$. As the objective of control is to keep the surplus nonnegative the state constraint (1.3) is clear.

We consider a finite time horizon $T > 0$ to indicate that the treaty may be reviewed at time T. Since repayment of $y_i(\cdot)$ with interest is involved, a cost called *control cost* of the form $\int_0^T M_i(r)u_i(r)dr$ is imposed on Company i. This cost is operative only when the control u_i is exercised. A typical control cost could be

$$\int_0^T e^{a_1(T-r)}\Big(\sum_{j\neq i} |R_{ji}|\Big)u_i(r)dr + \int_0^T e^{a_2(T-r)}\Big(1 - \sum_{j\neq i} |R_{ji}|\Big)u_i(r)dr,$$

where $0 \le a_1 < a_2$ denote respectively interest rates for "internal" and "external" loans. In addition there can also be the usual running cost and terminal cost. Each company tries to minimise its cost, subject to the constraint (1.3). The companies can possibly be in competition. Thus we are lead naturally to a d-person differential game in the d-dimensional orthant with state space constraints, and we seek a Nash equilibrium.

In [20] the above set up has been introduced in a greater generality which included an r.c.l.l input function $w(\cdot)$ (that is, $w(\cdot)$ being right continuous and having left limit at every t). In that set up, $y(\cdot)$, which was treated as control, need not be absolutely continuous or even continuous. Under certain natural monotonicity conditions, it was shown that a Nash equilibrium is given by the solution to the so called deterministic Skorokhod problem. This means, in addition to (1.3) and the analogues of (1.1), (1.2), we stipulate that $y_i(\cdot)$ can increase only when $z_i(\cdot) = 0,\ 1 \le i \le d$. Of course the game in [20] is a d-person dynamic game with state space constraints. Conditions were also given for Nash equilibrium to be the solution of the Skorokhod problem. It is argued in [20] that the above set up constitutes a reinsurance scheme. (See [22] for surplus process and ruin problems in the context of single insurance company.)

As another illustration, consider d interdependent sectors of an economy; these can even be different sections of the same company. If one sector faces severe financial strain, other sectors can pitch in previously-agreed-upon fractions of the money needed. Once again we are lead to (1.1), (1.2) and the state constraint (1.3). In this case, however, the different sectors may not be in competition, but each sector will try to minimise its cost. This leads again to a d-person differential game with state space constraints, and we seek to simultaneously minimise cost of each sector. We call an optimal control in this situation to be a *utopian equilibrium*; the name is derived from a comment in [15]. In Sect. 5 of [19] a more general model has been considered without the game theoretic trappings; under fairly strong monotonicity conditions it has been proved that the deterministic Skorokhod problem provides the utopian equilibrium. See [11, 21] for earlier results, and [19] for additional comments.

The purpose of this paper is to study the d-person differential game (in the orthant with state space constraints) using the framework of HJB equations and constrained viscosity solutions. Soner [24] has been the first to consider control problems with state space constraints. Since then it is known that the appropriate way to study such problems is through the so called constrained viscosity solutions to HJB equations.

In the case of utopian equilibrium, there are d control problems each with a d-dimensional control set; should all the d problems attain their minima at the *same* control we have a utopian equilibrium. Under some conditions the value function for the i-th player is shown to be the unique bounded uniformly continuous constrained viscosity solution to the appropriate HJB equation.

In the case of Nash equilibrium, there are *d interlinked* control problems with one dimensional control sets. The Hamiltonian can be discontinuous in the time variable. We show that the value function is a constrained viscosity solution to the discontinuous HJB equation in an appropriate sense, involving the semicontinuous envelopes of the Hamiltonian. It is also shown that uniqueness does not hold in general.

The paper is organized as follows. In Sect. 2 we describe the differential game, and derive the HJB equations as well as the "boundary conditions" dictated by the state space constraint. We also take a preliminary glance at viscosity solutions under somewhat strong regularity assumptions. In Sect. 3 we take a closer look at viscosity solutions as the Hamiltonian will be discontinuous in the context of Nash equilibrium; an appropriate notion of constrained viscosity solution is defined. An Appendix includes a brief discussion on the connection with the deterministic Skorokhod problem of probability theory. An example is given to show that Nash equilibrium need not be unique.

We now indicate some connections with previous works. Besides [24], HJB equations with state constraints have been considered by [9]; see [3, 13] for more information. There have been quite a few papers where Skorokhod problem, deterministic as well as stochastic, has played a major role in control and 2-person zero-sum differential game problems. In many of these, the dynamics of the system is governed by the z-part of the solution to Skorokhod problem; often the so called Skorokhod map is assumed to be Lipschitz continuous on the function space. Moreover the reflection terms are essentially taken to be constants. Existence and uniqueness of the value function as viscosity solution to appropriate PDE are often studied. Costs corresponding to singular controls (which are similar to control costs considered here) and ergodic controls are also investigated. To get a flavour of these one may see [1, 2, 6] and the references therein.

There seems to be quite a few papers on stochastic differential games with N players (and on two player nonzero sum stochastic differential games), with a nondegenerate diffusion term in the dynamics. While [4, 5, 16] use regularity results for systems of nonlinear elliptic/parabolic equations to obtain Nash equilibrium, [7] adopts an approach involving occupation measures. References to earlier works are given in these papers.

In contrast, there do not seem to be many papers dealing with the deterministic set up, that is, on differential games with N players or on two-player nonzero sum differential game; part of the reason could be the absence of a uniformly elliptic term in the Hamiltonian and the consequent non availability of regularity results for the resulting system of PDE's. Olsder [18] illustrates some of the difficulties and curious aspects in the context of two instructive examples concerning two-person nonzero-sum differential games. Cardaliaguet and Plaskacz [10] deal with a class of nonzero sum two person differential games on the line; even with an apparently simple looking dynamics, there are unexpected features like having to discriminate between interesting and uninteresting Nash equilibrium feedbacks; the approach here involves explicitly computing the suitable solution in small intervals. Bressan and Shen [8] consider n-person differential games in one dimension for which the system of HJB equations is strictly hyperbolic, and derive Nash equilibria for such situations. Besides giving references to earlier works, these three papers illustrate some of the difficulties inherent in getting global solutions to d-person differential games.

To the best of our knowledge, there is no previous work dealing with d-person differential games with state space constraints. Our paper gives an example of a situation where a system of first order nonlinear PDE's, with constraints and involving discontinuous Hamiltonian, can be dealt with using the viscosity solution approach. Moreover, the connection with Skorokhod problem also indicates a way of obtaining the constrained viscosity solution.

We now fix some notations. For $1 \le i \le d$, $y \in \mathbb{R}^d$ we denote $y_{-i} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_d)$. Similarly for an $\mathbb{R}^d$-valued function $g(\cdot)$, $1 \le i \le d$, we write

$g_{-i}(\cdot) = (g_1(\cdot), \ldots, g_{i-1}(\cdot), g_{i+1}(\cdot), \ldots g_d(\cdot))$ where $g(\cdot) = (g_1(\cdot), \ldots, g_d(\cdot))$. We shall often identify $g(\cdot) = (g_i(\cdot), g_{-i}(\cdot))$.

For a function $(r, y, z) \mapsto f(r, y, z)$ on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ denote $\partial_0 f(r, y, z) = \frac{\partial f}{\partial r}(r, y, z)$,

$$\nabla_y f(r, y, z) = \left(\frac{\partial f}{\partial y_1}(r, y, z), \ldots, \frac{\partial f}{\partial y_d}(r, y, z) \right),$$
$$\nabla_z f(r, y, z) = \left(\frac{\partial f}{\partial z_1}(r, y, z), \ldots, \frac{\partial f}{\partial z_d}(r, y, z) \right),$$

$D_{(y,z)} f = (\nabla_y f, \nabla_z f) =$ gradient of f in (y, z)-variables. We may also write $x = (y, z)$. $\mathbb{M}_d(\mathbb{R})$ denotes the space of all $d \times d$ matrices with real entries. The superscript N (resp. U) will be used to indicate that the discussion is in the context of Nash (resp. utopian) equilibrium.

2 The Set Up and HJB Equations

In this section we describe the constrained d-person differential game in the orthant. Two notions of optimality, viz., Nash and utopian equilibria are discussed. Corresponding systems of HJB equations and the conditions at the boundary are derived for the finite horizon problem. If the Hamiltonian, the optimal control and the value function are sufficiently regular, it is also shown that the value function is a constrained viscosity solution to the HJB equation.

The hypotheses are more general than alluded to in Sect. 1. The drift and the reflection field can be time, space and control dependent. $R_{ij}, i \neq j$ can also take positive values.

$G := \{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$ denotes the d-dimensional positive orthant. We have two functions $b : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, $R : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ called respectively the *drift* and the *reflection field*; denote $b(s, y, z) = (b_1(s, y, z), \ldots, b_d(s, y, z))$ and $R(s, y, z) = ((R_{ij}(s, y, z)))_{1 \leq i,j \leq d}$. We make the following assumptions:

(A1) For $1 \leq i \leq d, b_i$ are bounded measurable; also $(y, z) \mapsto b_i(t, y, z)$ are Lipschitz continuous, uniformly in t; let $|b_i(t, y, z)| \leq \tilde{\beta}_i, 1 \leq i \leq d, \tilde{\beta} = (\tilde{\beta}_1, \ldots, \tilde{\beta}_d)$.

(A2) For $1 \leq i, j \leq d$, R_{ij} are bounded measurable; also $(y, z) \mapsto R_{ij}(t, y, z)$ are Lipschitz continuous, uniformly in t. Moreover $R_{ii} \equiv 1$ for all i (this is a suitable normalization).

(A3) For $i \neq j$ there exist constants W_{ij} such that $|R_{ij}(t, y, z)| \leq W_{ij}$. Set $W = ((W_{ij}))$ with $W_{ii} \equiv 0$; we assume that $\sigma(W) < 1$, where $\sigma(W)$ denotes the spectral radius of W.

Remark 2.1 Recall the situation considered in Sect. 1. For $i \neq j$ let $R_{ij} \equiv -W_{ij} \leq 0$. If $\sum_{j \neq i} |R_{ji}| = \sum_{j \neq i} W_{ji} = 1$ for all i, then $(0, 0, \ldots, 0)$ can be a trap for the system, as all the companies need money at the same time to avoid ruin, and there is no scope for getting it from external sources. From the above it is clear that 1 is an eigenvalue of W in this case. Thus (A3) ensures that such a contingency does not arise and hence we have an *open system*.

When (A3) holds, note that

$$(I - W)^{-1} = I + W + W^2 + W^3 + \cdots \tag{2.1}$$

is a matrix of nonnegative terms. We shall choose and fix $\beta_i > 0$, $1 \le i \le d$ such that

$$0 \le ((I - W)^{-1}\tilde{\beta})_i \le \beta_i, \quad i = 1, 2, \dots, d, \tag{2.2}$$

where $\tilde{\beta}$ is as in (A1).

Fix the terminal time $T > 0$. For $s \in [0, T]$, $y, z \in \bar{G}$, $u(\cdot) = (u_1(\cdot), \dots, u_d(\cdot))$ such that $0 \le u_i(\cdot) \le \beta_i$, $1 \le i \le d$, set

$$y(t) = y + \int_s^t u(r)dr, \tag{2.3}$$

$$\begin{aligned} z(t) &:= z(t; s, y, z, u(\cdot)) \\ &= z + \int_s^t b(r, y(r), z(r))dr + \int_s^t R(r, y(r), z(r))u(r)dr \end{aligned} \tag{2.4}$$

equivalently, for $1 \le i \le d, t \ge s$

$$y_i(t) = y_i + \int_s^t u_i(r)dr, \tag{2.5}$$

$$\begin{aligned} z_i(t) = z_i &+ \int_s^t b_i(r, y(r), z(r))dr + \int_s^t u_i(r)dr \\ &+ \sum_{j \ne i} \int_s^t R_{ij}(r, y(r), z(r))u_j(r)dr. \end{aligned} \tag{2.6}$$

Clearly $y_i(\cdot) \ge 0$, well defined and nondecreasing. By the standard assumptions (A1), (A2) the integral equation (2.4) has a unique solution. We shall treat $(y(\cdot), z(\cdot))$ as state of the system. The pair of (2.3), (2.4), or equivalently (2.5), (2.6), forms *state equations*. We shall consider only controls $u(\cdot)$ that take values in the compact set $\prod_{i=1}^d [0, \beta_i]$. For $s \in [0, T]$, $y, z \in \bar{G}$ write

$$\begin{aligned} \mathcal{U}(s, y, z; T) = \{u(\cdot) = (u_1(\cdot), \dots, u_d(\cdot)) : 0 \le u_i(\cdot) \le \beta_i, z_i(\cdot) \ge 0, \\ \text{on } [s, T], 1 \le i \le d\} \end{aligned} \tag{2.7}$$

to denote the set of *feasible controls*; in (2.7) $z_i(\cdot)$ is given by (2.4) or (2.6). The *cost function* for the ith player is given by

$$\begin{aligned} &J_i(s, y, z; T, u(\cdot)) \\ &\quad = g_i(T, y(T), z(T)) \\ &\qquad + \int_s^T L_i(r, y(r), z(r))dr + \int_s^T M_i(r, y(r), z(r))u_i(r)dr, \end{aligned} \tag{2.8}$$

where the three terms on the right side denote respectively terminal cost, running cost and control cost. A control $u^*(\cdot) \in \mathcal{U}(s, y, z; T)$ is called a *utopian equilibrium* in $\mathcal{U}(s, y, z; T)$ if

$$J_i(s, y, z; T, u^*(\cdot)) \leq J_i(s, y, z; T, u(\cdot)) \tag{2.9}$$

for all $u \in \mathcal{U}(s, y, z; T), i = 1, 2, \dots, d$. Similarly a control $u^*(\cdot) = (u_1^*(\cdot), \dots, u_d^*(\cdot)) \in \mathcal{U}(s, y, z; T)$ is called a *Nash equilibrium* in $\mathcal{U}(s, y, z; T)$ if for $i = 1, 2, \dots, d$

$$\begin{aligned} &J_i(s, y, z; T, u^*(\cdot)) \\ &\quad = \inf\{J_i(s, y, z; T, u(\cdot)) : u_{-i} = u^*_{-i}, u \in \mathcal{U}(s, y, z; T)\}. \end{aligned} \tag{2.10}$$

Remark 2.2 Under (A1–A3), by the proof of Theorem 5.1 of [19], $u_i(\cdot) = ((I - W)^{-1}\tilde{\beta})_i$, $1 \leq i \leq d$, is a feasible control. So $\mathcal{U}(s, y, z; T) \neq \phi$ for any $s \in [0, T]$, $y, z \in \bar{G}$. This is a reason why we consider only those controls taking values in $\prod_{i=1}^d [0, \beta_i]$.

For fixed $1 \leq i \leq d$, let $u_{-i}(\cdot) = (u_1(\cdot), \dots, u_{i-1}(\cdot), u_{i+1}, \dots, u_d(\cdot))$ be such that

$$0 \leq u_j(r) \leq \beta_j, \quad 0 \leq r \leq T, \ j \neq i. \tag{2.11}$$

For any $s \in [0, T]$, $y, z \in \bar{G}$, $i = 1, 2, \dots, d$, $u_{-i}(\cdot)$ satisfying (2.11) we shall assume

$$\begin{aligned} &\mathcal{U}(s, y, z; T, u_{-i}(\cdot)) \\ &\quad := \{u_i(\cdot) : 0 \leq u_i(\cdot) \leq \beta_i, (u_i(\cdot), u_{-i}(\cdot)) \in \mathcal{U}(s, y, z; T)\} \neq \phi. \end{aligned} \tag{2.12}$$

A sufficient condition for (2.12) to hold is given in an Appendix.

To derive the system of HJB equations with state constraints, we consider first the case of Nash equilibrium.

For s, y, z, u_{-i} as above, where i is fixed, define the *value function* for ith player by

$$\begin{aligned} &V^{(N,i)}(s, y, z; T, u_{-i}(\cdot)) \\ &\quad = \inf\{J_i(s, y, z; T, (u_i(\cdot), u_{-i}(\cdot))) : u_i \in \mathcal{U}(s, y, z; T, u_{-i})\}. \end{aligned} \tag{2.13}$$

Following the approach given in Sect. I.4 of [13], we get the following *dynamic programming principle*.

Theorem 2.3 (i) *Assume* (A1), (A2). *Fix* $1 \leq i \leq d$; *let* $u_{-i}(\cdot)$ *satisfy* (2.11). *Assume* (2.12). *Let* $s \in [0, T]$, $y, z \in \bar{G}$, $u_i \in \mathcal{U}(s, y, z; T, u_{-i})$. *Then for* $s \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned} &V^{(N,i)}(t_1, y(t_1), z(t_1); T, u_{-i}(\cdot)) \\ &\quad \leq V^{(N,i)}(t_2, y(t_2), z(t_2); T, u_{-i}(\cdot)) \\ &\qquad + \int_{t_1}^{t_2} L_i(r, y(r), z(r)) dr + \int_{t_1}^{t_2} M_i(r, y(r), z(r)) u_i(r) dr, \end{aligned} \tag{2.14}$$

where $(y(\cdot), z(\cdot))$ *is the solution to state equation corresponding to the control* (u_i, u_{-i}).

(ii) *Under the above hypotheses,* $u_i^*(\cdot)$ *is optimal in* $\mathcal{U}(s, y, z; T, u_{-i})$, *that is,*

$$V^{(N,i)}(s, y, z; T, u_{-i}(\cdot)) = J_i(s, y, z; T, (u_i^*, u_{-i})),$$

if and only if for any $t \in [s, T]$

$$\begin{aligned} &V^{(N,i)}(s, y, z; T, u_{-i}) - V^{(N,i)}(t, \bar{y}(t), \bar{z}(t); T, u_{-i}) \\ &\quad = \int_s^t L_i(r, \bar{y}(r), \bar{z}(r))dr + \int_s^t M_i(r, \bar{y}(r), \bar{z}(r))u_i^*(r)dr, \end{aligned} \tag{2.15}$$

where $(\bar{y}(\cdot), \bar{z}(\cdot))$ *is the solution to state equation corresponding to the control* (u_i^*, u_{-i}). *Moreover if* (2.15) *holds then for any* $t \in [s, T]$, *the restriction of* $u_i^*(\cdot)$ *to* $[t, T]$ *is optimal in* $\mathcal{U}(t, \bar{y}(t), \bar{z}(t); T, u_{-i})$.

(iii) *Assume* (A1–A3). *Then* $u^*(\cdot) \in \mathcal{U}(s, y, z; T)$ *is a Nash equilibrium if and only if for* $s \le t_1 \le t_2 \le T$, $i = 1, 2, \ldots, d$

$$\begin{aligned} &V^{(N,i)}(t_1, y^*(t_1), z^*(t_1); T, u_{-i}^*(\cdot)) \\ &\quad = V^{(N,i)}(t_2, y^*(t_2), z^*(t_2); T, u_{-i}^*(\cdot)) \\ &\qquad + \int_{t_1}^{t_2} L_i(r, y^*(r), z^*(r))dr + \int_{t_1}^{t_2} M_i(r, y^*(r), z^*(r))u_i^*(r)dr, \end{aligned} \tag{2.16}$$

where $(y^*(\cdot), z^*(\cdot))$ *is the solution to state equation corresponding to* $u^*(\cdot)$. *Moreover, when* (2.16) *holds, for any* $t \in [s, T]$ *the restriction of* $u^*(\cdot)$ *to* $[t, T]$ *is a Nash equilibrium in* $\mathcal{U}(t, y^*(t), z^*(t); T)$.

Proof (i) Let $s \le t_1 \le t_2 \le T$ and $\hat{u}_i(\cdot) \in \mathcal{U}(t_2, y(t_2), z(t_2); T, u_{-i}(\cdot))$. Define $\tilde{u}_i(r) = u_i(r), t_1 \le r < t_2, \tilde{u}_i(r) = \hat{u}_i(r), t_2 \le r \le T$. By uniqueness of solution to state equations (2.3), (2.4) it follows that $\tilde{u}_i(\cdot) \in \mathcal{U}(t_1, y(t_1), z(t_1); T, u_{-i}(\cdot))$ (this is a switching condition). By definition of the value function we have

$$\begin{aligned} &V^{(N,i)}(t_1, y(t_1), z(t_1); T, u_{-i}(\cdot)) \\ &\quad \le J_i(t_1, y(t_1), z(t_1); T, (\tilde{u}_i(\cdot), u_{-i}(\cdot))) \\ &\quad = J_i(t_2, y(t_2), z(t_2); T, (\tilde{u}_i(\cdot), u_{-i}(\cdot))) \\ &\qquad + \int_{t_1}^{t_2} L_i(r, y(r), z(r))dr + \int_{t_1}^{t_2} M_i(r, y(r), z(r))u_i(r)dr \end{aligned}$$

where we have used the fact that the solutions corresponding to the controls (u_i, u_{-i}) and $(\tilde{u}_i, u_{-i})$ agree up to t_2. Taking infimum over $\hat{u}_i(\cdot) \in \mathcal{U}(t_2, y(t_2), z(t_2); T, u_{-i}(\cdot))$ we get (2.14).

(ii) From the proof of part (i) it is clear that optimality is achieved if and only if equality holds in (2.14); this proves (ii).

(iii) This is a easy consequence of part (ii) and the definition of Nash equilibrium. □

Fix $i, u_{-i}(\cdot)$ satisfying (2.11); assume (2.12). It is convenient to define, for $s \in [0, T]$, $y, z \in \mathbb{R}^d$, $c \in [0, \infty)$, $\mathbb{R}^{2d}$-valued vector by

$$f^{(N,i)}(s, (y, z), c) := \begin{pmatrix} \begin{pmatrix} c \\ u_{-i}(s) \end{pmatrix} \\ b(s, y, z) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & R(s, y, z) \end{pmatrix} \begin{pmatrix} 0 \\ \begin{pmatrix} c \\ u_{-i}(s) \end{pmatrix} \end{pmatrix}, \quad (2.17)$$

where $(c, u_{-i}(s)) := (u_1(s), \ldots, u_{i-1}(s), c, u_{i+1}(s), \ldots, u_d(s))$, the square matrix on r.h.s. is of order $2d$, and the scalar by

$$C^{(N,i)}(s, (y, z), c) = L_i(s, y, z) + M_i(s, y, z)c. \quad (2.18)$$

It is to be kept in mind that $u_{-i}(\cdot)$ acts as a parameter. In this notation state equations (2.3), (2.4) can be written as

$$d \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = f^{(N,i)}(t, (y(t), z(t)), u_i(t))dt, \quad t > s \quad (2.19)$$

with initial value $(y(s), z(s)) = (y, z)$, and the cost function for ith player as

$$\begin{aligned} &J_i(s, y, z; T, (u_i(\cdot), u_{-i}(\cdot))) \\ &\quad = g_i(T, y(T), z(T)) + \int_s^T C^{(N,i)}(r, (y(r), z(r)), u_i(r))dr. \end{aligned} \quad (2.20)$$

Next the *Hamiltonian* $H^{(N,i)}$ (for ith player in case of Nash equilibrium) is defined by

$$\begin{aligned} &H^{(N,i)}(s, (y, z), p) \\ &\quad = \sup\{[-\langle p, f^{(N,i)}(s, (y, z), c)\rangle - C^{(N,i)}(s, (y, z), c)] : 0 \le c \le \beta_i\} \end{aligned} \quad (2.21)$$

for $s \ge 0$, $y, z \in \mathbb{R}^d$, $p \in \mathbb{R}^{2d}$.

Assume that u_{-i} is right continuous, and that $V^{(N,i)}$ is continuously differentiable. Assume also that b, R are continuous in t.

Let $s \in [0, T)$, $y, z \in G$ (that is, interior point). Let $0 \le c \le \beta_i$. Then there is $u_i(\cdot) \in \mathcal{U}(s, y, z; T, u_{-i}(\cdot))$ which is right continuous and $\lim_{r \downarrow s} u_i(r) = c$. This can be seen as follows. As $z \in G$ is an interior point, for the control (c, u_{-i}) one can run the state equation till τ, where $\tau = \inf\{t \ge s : z(t) \notin \bar{G}\}$. Choose $t_0 \in (s, \tau \wedge T)$; clearly $z(t_0) \in \bar{G}$. By (2.12) there is $v_i(\cdot) \in \mathcal{U}(t_0, y(t_0), z(t_0); T, u_{-i}(\cdot))$. Take $u_i(r) = c$, if $s \le r < t_0$, $u_i(r) = v_i(r)$, $t_0 \le r \le T$.

Denote $V(s, y, z) = V^{(N,i)}(s, y, z; T, u_{-i}(\cdot))$, $f(s, (y, z), c) = f^{(N,i)}(s, (y, z), c)$, $C(s, (y, z), c) = C^{(N,i)}(s, (y, z), c)$. By (2.14) in Theorem 2.3(i) we get

$$\partial_0 V(s, y, z) + \langle D_{(y,z)} V(s, y, z), f(s, (y, z), c)\rangle + C(s, (y, z), c) \ge 0, \quad (2.22)$$

where $\partial_0 = \frac{\partial}{\partial s}$ and $D_{(y,z)}$ = gradient in the (y, z)-variables. As $c \in [0, \beta_i]$ is arbitrary, we get

$$\begin{aligned} &\inf\{[\partial_0 V(s, y, z) + \langle D_{(y,z)} V(s, y, z), f(s, (y, z), c)\rangle + C(s, (y, z), c)] : 0 \le c \le \beta_i\} \\ &\quad \ge 0. \end{aligned} \quad (2.23)$$

If $u_i(\cdot) \in \mathcal{U}(s, y, z; T, u_{-i}(\cdot))$ is optimal, and right continuous at s, then by Theorem 2.3(ii), note that equality holds in (2.22) with c replaced by $u_i(s)$. Therefore V satisfies

$$\begin{aligned}&\inf\{[\partial_0 V(s,y,z) + \langle D_{(y,z)}V(s,y,z), f(s,(y,z),c)\rangle + C(s,(y,z),c)] : c \in [0,\beta_i]\}\\&\quad = 0.\end{aligned} \tag{2.24}$$

With the Hamiltonian introduced in (2.21), we write (2.24) in the conventional (but equivalent) form as

$$-\partial_0 V(s,y,z) + H^{(N,i)}(s,(y,z),D_{(y,z)}V(s,y,z)) = 0, \tag{2.25}$$

for $(s,y,z) \in [0,T) \times G \times G$. In (2.25) $u_{-i}(\cdot)$ acts as a parameter. Equation (2.25) is a *Hamilton-Jacobi-Bellman equation*.

Recall that we admit only those controls such that $z_k(\cdot) \geq 0$ for all k. This restriction implies a condition at the boundary, and leads to what is referred to as a "problem with state constraints" in the literature. Soner [24] was the first to consider such problems; one may consult [3] or [13] for detailed discussions. We describe below the state constraint in our context.

Since $y(t) \in \bar{G}$ automatically, note that $u_i(\cdot)$ is feasible if and only if $(y(t), z(t)) \in \bar{G} \times \bar{G}$ for all t; hence $\bar{G} \times \bar{G}$ is taken as the state space. For $y, z \in \mathbb{R}^d$ denote $I_y = \{j : y_j = 0\}$, $I_z = \{k : z_k = 0\}$. Clearly $(y,z) \in \partial(G \times G) \Leftrightarrow I_y \cup I_z \neq \phi$. For $(y,z) \in \partial(G \times G)$ let $\mathcal{N}_{(y,z)} =$ set of all unit inward normals at (y,z). It is not difficult to see that $n \in \mathcal{N}_{(y,z)} \Leftrightarrow n$ is a convex combination of e_j, e_{d+k} with $j \in I_y, k \in I_z$ where e_ℓ's denote unit vectors in $\mathbb{R}^{2d}$.

Let $(y,z) \in \partial(G \times G)$. Suppose there is a feasible control $u_i(\cdot)$ with some initial data and t such that $(y(t), z(t)) = (y,z)$ where $(y(\cdot), z(\cdot))$ is the solution to the state equation (2.19) corresponding to the control $u_i(\cdot)$. Since $(y(t), z(t))$ is a boundary point note that $(y(r), z(r)) \in \bar{G} \times \bar{G}$ for all $r \geq t \Leftrightarrow \langle \binom{y(r)-y(t)}{z(r)-z(t)}, n\rangle \geq 0$ for all $n \in \mathcal{N}_{(y,z)}, r \geq t$. Hence $\langle d\binom{y(t)}{z(t)}, n\rangle \geq 0$ for all $n \in \mathcal{N}_{(y,z)}$. So by (2.19) we get

$$\langle f^{(N,i)}(t,(y,z),u_i(t)), n\rangle \geq 0, \quad \forall n \in \mathcal{N}_{(y,z)}. \tag{2.26}$$

Note that $u_i(\cdot), u_j(\cdot), j \neq i$ are always nonnegative (even for $i, j \notin I_y$). So by (2.17), the "boundary condition" (2.26) is essentially

$$b_i(t,y,z) + u_i(t) + \sum_{\ell \neq i} R_{i\ell}(t,y,z)u_\ell(t) \geq 0 \tag{2.27}$$

if $i \in I_z$, and

$$b_k(t,y,z) + R_{ki}(t,y,z)u_i(t) + u_k(t) + \sum_{\ell \neq i,k} R_{k\ell}(t,y,z)u_\ell(t) \geq 0, \quad k \neq i,\ k \in I_z \tag{2.28}$$

again remembering that $i, u_j(\cdot), j \neq i$ are fixed.

Let $(y,z)\in\partial(G\times G)$. Let $u_i(\cdot)\in\mathcal{U}(s,y,z;T,u_{-i}(\cdot))$ be optimal and right continuous. Denoting $V^{(N,i)}$, $f^{(N,i)}$, $C^{(N,i)}$ respectively by V, f, C, as $V\in C^1([0,T]\times\bar{G}\times\bar{G})$, f is continuous, by the dynamic programming principle we get

$$\partial_0 V(s,y,z)+\langle D_{(y,z)}V(s,y,z), f(s,(y,z),u_i(s))\rangle + C(s,(y,z),u_i(s))=0. \tag{2.29}$$

Assume that the Hamiltonian is continuous. As (2.25) holds on $[0,T)\times G\times G$, it is now clear that it is true on $[0,T]\times\bar{G}\times\bar{G}$ as well. Consequently by (2.29) we now get, denoting $H^{(N,i)}$ by H,

$$\begin{aligned} &H(s,(y,z),D_{(y,z)}V(s,y,z)) \\ &\quad= -\langle D_{(y,z)}V(s,y,z), f(s,(y,z),u_i(s))\rangle - C(s,(y,z),u_i(s)). \end{aligned} \tag{2.30}$$

Now by the definition of H with $p=D_{(y,z)}V(s,y,z)-\gamma n$, for any $\gamma\geq 0$, $n\in\mathcal{N}_{(y,z)}$ by (2.26), (2.30) we get

$$H(s,(y,z),D_{(y,z)}V(s,y,z)-\gamma n)\geq H(s,(y,z),D_{(y,z)}V(s,y,z)). \tag{2.31}$$

Thus the state constraint (2.26), which is essentially (2.27), (2.28), implies that the implicit inequality (2.31) has to be satisfied by $D_{(y,z)}V^{(N,i)}$ at a boundary point. The heuristics above on state constraints have been influenced by the discussion on pp. 102–103 of [13].

It is well known that "viscosity solutions" is the appropriate framework to treat Hamilton-Jacobi-Bellman equations; in particular "constrained viscosity solutions" form the proper context to take care of problems with state constraints. Bardi and Capuzzo-Dolcetta [3] and Fleming and Soner [13] have very nice treatment of viscosity solutions to HJB equations when the Hamiltonian is continuous.

We now rephrase some key definitions from [3, 13] in our context. Assume that the Hamiltonian $H^{(N,i)}$ given by (2.21) is continuous.

(i) A continuous function u is said to be a *viscosity subsolution* to the HJB equation (2.25) on $[0,T)\times G\times G$ if for any $s\in[0,T)$, $y\in G$, $z\in G$, any C^1-function w such that $(u-w)$ has a local maximum at (s,y,z) one has

$$-\partial_0 w(s,y,z)+H^{(N,i)}(s,(y,z),D_{y,z}w(s,y,z))\leq 0.$$

(ii) A continuous function u is said to be *viscosity supersolution* to (2.25) on $[0,T]\times\overline{G}\times\overline{G}$ if for any $s\in[0,T]$, $y\in\overline{G}$, $z\in\overline{G}$, any C^1-function w such that $(u-w)$ has a local minimum in $[0,T]\times\overline{G}\times\overline{G}$ at (s,y,z) one has

$$-\partial_0 w(s,y,z)+H^{(N,i)}(s,(y,z),D_{y,z}w(s,y,z))\geq 0.$$

(iii) If u satisfies both (i), (ii) above then it is called a *constrained viscosity solution* to the HJB equation (2.25) on $[0,T]\times\overline{G}\times\overline{G}$.

Note that in (i), (ii) above we use the same Hamiltonian function $H^{(N,i)}$. This is in contrast to the situation in Definition 3.2 later, where we deal with discontinuous Hamiltonian; in Definition 3.2, we will need to consider lower and upper semicontinuous envelopes of the Hamiltonian, respectively, for the subsolution and the supersolution.

We need an elementary lemma first, whose proof is given for the sake of completeness.

Lemma 2.4 *Let $D \subseteq \mathbb{R}^k$ be a convex set (not necessarily smooth or bounded). For $\xi \in \partial D$ let $\mathcal{N}_\xi$ denote the set of unit inward normal vectors at ξ. Let g be a C^1-function and $x \in \partial D$ such that $g(x) = \min\{g(x') : x' \in \bar{D}\}$. Then $\nabla g(x) = \gamma n$ for some $n \in \mathcal{N}_x, \gamma \geq 0$.*

Proof We may assume that $|\nabla g(x)| \neq 0$. Suppose the result is not true. Then there is $x' \in \bar{D}$ such that $\langle \frac{\nabla g(x)}{|\nabla g(x)|}, (x' - x)\rangle < 0$ (since D is convex, $x \in \partial D$ we have $n \in \mathcal{N}_x \Leftrightarrow |n| = 1$ and $\langle \xi - x, n\rangle \geq 0$ $\forall \xi \in \bar{D}$). Clearly $x' \neq x$; put $\ell = \frac{x'-x}{|x'-x|}$. Then $\langle \nabla g(x), \ell\rangle < 0$.

Now, by the mean value theorem, for any $0 < r < 1$ there is $r' \in (0, r)$ such that

$$g(x + r\ell) = g(x) + r\langle \nabla g(x + r'\ell), \ell\rangle.$$

Since g attains minimum at x (over $\bar{D}$) it follows then that $\langle \nabla g(x + r'\ell), \ell\rangle \geq 0$. Letting $r \downarrow 0$, as g is C^1, we get $\langle \nabla g(x), \ell\rangle \geq 0$ which is a contradiction. This proves the lemma. □

Theorem 2.5 *Assume* (A1), (A2); *let b, R be continuous in t, y, z. Fix $i, u_{-i}(\cdot)$ satisfying* (2.11); *also let $u_{-i}(\cdot)$ be continuous. Suppose the Hamiltonian $H^{(N,i)}$ given by* (2.21) *is continuous. Assume* (2.12). *Assume that the value function $V^{(N,i)}(\cdot,\cdot,\cdot; T, u_{-i}(\cdot)) \in C^1([0,T] \times \bar{G} \times \bar{G})$, and that there is a right continuous optimal control $u_i(\cdot)$ in $\mathcal{U}(s, y, z; T, u_{-i}(\cdot))$ for any s, y, z. Then $V^{(N,i)}$ is a constrained viscosity solution to HJB equation* (2.25) *on* $[0,T] \times \bar{G} \times \bar{G}$ *with terminal value $g_i(T,\cdot,\cdot)$.*

Proof First observe that our hypotheses ensure that the arguments given in heuristic discussion above are valid. By the definition of constrained viscosity solution, we need to show that $V \equiv V^{(N,i)}$ is a viscosity subsolution on $[0,T) \times G \times G$, and is a viscosity supersolution on $[0,T] \times \bar{G} \times \bar{G}$.

Let $(s, y, z) \in [0,T) \times G \times G$. Let w be a C^1-function such that $(V - w)$ has a local maximum at (s, y, z). Then $\partial_0 w(s, y, z) = \partial_0 V(s, y, z)$ if $s > 0$, $-\partial_0 w(s, y, z) \leq -\partial_0 V(s, y, z)$ if $s = 0$. Since (y, z) is an interior point, we have $D_{(y,z)} w(s, y, z) = D_{(y,z)} V(s, y, z)$. Therefore, as V satisfies (2.25)

$$\begin{aligned}
&-\partial_0 w(s, y, z) + H^{(N,i)}(s, (y, z), D_{(y,z)} w(s, y, z)) \\
&\quad \leq -\partial_0 V(s, y, z) + H^{(N,i)}(s, (y, z), D_{(y,z)} V(s, y, z) = 0.
\end{aligned}$$

Thus V is a viscosity subsolution on $[0,T) \times G \times G$. In a similar way it can be shown that it is a viscosity supersolution on $[0,T) \times G \times G$.

Remains to consider the case when $s \in [0,T], (y, z) \in \partial(G \times G)$. Let w be a C^1-function such that $(V - w)$ has a local minimum (in $[0,T] \times \bar{G} \times \bar{G}$) at (s, y, z). It is then easily verified that $-\partial_0 w(s, y, z) \geq -\partial_0 V(s, y, z)$. Also there is a ball B around (y, z) such that

$$V(s, y, z) - w(s, y, z) = \min\{V(s, y', z') - w(s, y', z') : (y', z') \in B \cap (\bar{G} \times \bar{G})\}.$$

As $B \cap (\bar{G} \times \bar{G})$ is a convex set in $\mathbb{R}^{2d}$, by Lemma 2.4

$$D_{(y,z)}[V(s,y,z) - w(s,y,z)] = \gamma n, \quad \text{for some } n \in \mathcal{N}_{(y,z)}, \gamma \geq 0.$$

As (y,z) is an interior point of B, note that n is an inward normal to $(G \times G)$ at (y,z). Hence

$$D_{(y,z)} w(s,y,z) = D_{(y,z)} V(s,y,z) - \gamma n.$$

Consequently by (2.31) and (2.25) we now obtain

$$\begin{aligned} &-\partial_0 w(s,y,z) + H^{(N,i)}(s,(y,z), D_{(y,z)} w(s,y,z)) \\ &\quad \geq -\partial_0 V(s,y,z) + H^{(N,i)}(s,(y,z), D_{y,z} V(s,y,z)) = 0. \end{aligned}$$

Thus V is a viscosity supersolution to (2.25) on $[0,T] \times \bar{G} \times \bar{G}$, completing the proof. □

Remark 2.6 Note that if $u_{-i}(\cdot)$ continuous, and the various coefficients are also continuous, then the Hamiltonian $H^{(N,i)}$ is continuous. So in case a Nash equilibrium can be achieved in the class of continuous controls we have the following. Suppose $u^*(\cdot) = (u_1^*(\cdot), \ldots, u_d^*(\cdot))$ is a Nash equilibrium. In addition assume that $u^*(\cdot)$ is continuous and $(s,y,z) \mapsto V^{(N,i)}(s,y,z;T,u_{-i}^*(\cdot)) = J_i(s,y,z;T,u^*(\cdot))$ is in $C^1([0,T] \times \bar{G} \times \bar{G})$ for each $i = 1,2,\ldots,d$. Then by Theorem 2.5 $(s,y,z) \mapsto V^{(N,i)}(s,y,z;T,u_{-i}^*(\cdot)$ is a constrained viscosity solution to the HJB equation (2.25) with $u_{-i}(\cdot)$ replaced by $u_{-i}^*(\cdot)$, for each $1 \leq i \leq d$. So we will have a system of interrelated HJB equations involving continuous Hamiltonians with state constraints. However, in general the value function will not be smooth; nor can one hope to have a continuous Nash equilibrium. Moreover the Hamiltonian will not be continuous in general as we will see later. An interesting question is when can one hope to have a Nash equilibrium in the class of continuous, or at least piecewise continuous controls; in such a case we may not need to go beyond the class of continuous Hamiltonians. We do not have an answer. This may perhaps be related to similar question concerning the solution to the Skorokhod problem, in view of Section "The SP Connection" in an Appendix.

We now briefly indicate the HJB equations in the case of utopian equilibrium; note that $\prod_{i=1}^d [0,\beta_i]$ can be taken as the control set. As before $\bar{G} \times \bar{G}$ is the state space. For $s \in [0,T]$, $y,z \in \mathbb{R}^d$, $u \in \prod_{i=1}^d [0,\beta_i]$ define $\mathbb{R}^{2d}$-valued vector by

$$f^{(U)}(s,(y,z),u) = \begin{pmatrix} u \\ b(s,y,z) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & R(s,y,z) \end{pmatrix} \begin{pmatrix} 0 \\ u \end{pmatrix}. \tag{2.32}$$

So the state equations (2.3), (2.4) become

$$d\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = f^{(U)}(t,(y(t),z(t)),u(t))dt \tag{2.33}$$

with initial value $(y(s), z(s)) = (y, z)$ corresponding to the control $u(\cdot)$. Define a scalar function for $i = 1, \ldots, d$ by

$$C^{(U,i)}(s, (y, z), u) = L_i(s, y, z) + M_i(s, y, z)u_i. \tag{2.34}$$

Note that $C^{(N,i)}$ given by (2.18) and $C^{(U,i)}$ above differ in their domain of definition; so $C^{(U,i)}(s, (y, z), u) = C^{(N,i)}(s, (y, z), u_i)$. Cost function J_i for the ith player, corresponding to the control $u(\cdot)$, is the same as before. The *value function* for the ith player is

$$V^{(U,i)}(s, y, z; T) = \inf\{J_i(s, y, z; T, u(\cdot)) : u(\cdot) \in \mathcal{U}(s, y, z; T)\}. \tag{2.35}$$

The *Hamiltonian* is, for $s \in [0, T]$, $y, z \in \mathbb{R}^d$, $p \in \mathbb{R}^{2d}$, given by

$$\begin{aligned} &H^{(U,i)}(s, (y, z), p) \\ &\quad = \sup\left\{[-\langle p, f^{(U)}(s, (y, z), u)\rangle - C^{(U,i)}(s, (y, z), u)] : u \in \prod_{i=1}^{d}[0, \beta_i]\right\}. \end{aligned} \tag{2.36}$$

In a manner analogues to the earlier discussion, the HJB equation in this context is seen to be

$$-\partial_0 v(s, y, z) + H^{(U,i)}(s, (y, z), D_{(y,z)} v(s, y, z)) = 0. \tag{2.37}$$

The state constraint once again leads to a "boundary condition" which is the analogue of (2.26). Together with the HJB equation, this in turn implies the implicit inequality (2.31) for the Hamiltonian $H^{(U,i)}$ under suitable regularity.

Using Remark 2.2, and proceeding as in the proofs of Theorems 2.3 and 2.5 with obvious modifications we get the following result.

Theorem 2.7 (i) *Assume* (A1–A3). *A control $u^*(\cdot) \in \mathcal{U}(s, y, z; T)$ is a utopian equilibrium if and only if for $s \le t_1 \le t_2 \le T$, $1 \le i \le d$,*

$$\begin{aligned} &V^{(U,i)}(t_1, y^*(t_1), z^*(t_1); T) \\ &\quad = V^{(U,i)}(t_2, y^*(t_2), z^*(t_2); T) \\ &\qquad + \int_{t_1}^{t_2} L_i(r, y^*(r), z^*(r))dr + \int_{t_1}^{t_2} M_i(r, y^*(r), z^*(r))u_i^*(r)dr, \end{aligned} \tag{2.38}$$

where $y^*(\cdot), z^*(\cdot)$ is the solution to state equation corresponding to the control $u^*(\cdot)$.

(ii) *For fixed i, suppose $H^{(U,i)}$ is continuous, $V^{(U,i)} \in C^1([0, T] \times \bar{G} \times \bar{G})$ and that the optimal control is continuous. Then $V^{(U,i)}$ is a constrained viscosity solution to the HJB equation* (2.37).

3 Viscosity Solutions

In this section we take a closer look at the value functions being viscosity solutions to the appropriate HJB equations derived in Sect. 2, as well as the question of uniqueness.

We consider the case of Nash equilibrium first. As already mentioned in Remark 2.6, the hypotheses of Theorem 2.5 are too strong. In particular, by (2.17), (2.21) note that the Hamiltonian $H^{(N,i)}$ depends on $u_{-i}(\cdot)$ and hence can be discontinuous in general. So even to define viscosity solutions we need to introduce semicontinuous envelopes of the Hamiltonian. Moreover, as we shall see uniqueness need not hold.

Fix $1 \le i \le d, u_{-i}(\cdot)$ satisfying (2.11). We continue to assume (A1–A3), (2.12). Recall that the HJB equation is

$$-\partial_0 v(s,y,z) + H^{(N,i)}(s,(y,z),D_{(y,z)}v(s,y,z)) = 0, \tag{3.1}$$

where $H^{(N,i)}$ is given by (2.21).

For notational convenience write $H = H^{(N,i)}$. For $t \in [0,T], y, z \in \bar{G}, p \in \mathbb{R}^{2d}$ set

$$\begin{aligned} &H_*(t,(y,z),p) \\ &\quad = \liminf\{H(t',(y',z'),p') : (t',(y',z'),p') \to (t,(y,z),p) \\ &\qquad\qquad \text{in } [0,T] \times \bar{G} \times \bar{G} \times \mathbb{R}^{2d}\} \\ &\quad = \lim_{\theta \downarrow 0} \inf\{H(t',(y',z'),p') : |(t',(y',z'),p') - (t,(y,z),p)| \le \theta, \\ &\qquad\qquad 0 \le t' \le T, y', z' \in \bar{G}\} \end{aligned} \tag{3.2}$$

which is the *lower semicontinuous envelope*, and

$$\begin{aligned} &H^*(t,(y,z),p) \\ &\quad = \limsup\{H(t',(y',z'),p') : (t',(y',z'),p') \to (t,(y,z),p) \\ &\qquad\qquad \text{in } [0,T] \times \bar{G} \times \bar{G} \times \mathbb{R}^{2d}\} \\ &\quad = \lim_{\theta \downarrow 0} \sup\{H(t',(y',z'),p') : |(t',(y',z'),p') - (t,(y,z),p)| \le \theta, \\ &\qquad\qquad 0 \le t' \le T, y', z' \in \bar{G}\} \end{aligned} \tag{3.3}$$

which is the *upper semicontinuous envelope*.

Lemma 3.1 *Let b, R, L_i, M_i be bounded and continuous; let $f^{(N,i)}, C^{(N,i)}$ be defined by* (2.17), (2.18) *respectively. For $0 \le t \le T$, $y, z \in \bar{G}$, $p \in \mathbb{R}^{2d}$, $0 \le c \le \beta_i$ denote*

$$h(t,(y,z),p;c) = -\langle p, f^{(N,i)}(t,(y,z),c)\rangle - C^{(N,i)}(t,(y,z),c)$$

and

$$h_*(t,(y,z),p;c)=\liminf\{h(t',(y',z'),p';c):(t',(y',z'),p')\to(t,(y,z),p)\},$$
$$h^*(t,(y,z),p;c)=\limsup\{h(t',(y',z'),p';c):(t',(y',z'),p')\to(t,(y,z,p)\}.$$

Here $c\in[0,\beta_i]$ acts as a parameter. Then

$$H_*(t,(y,z),p)=\max\{h_*(t,(y,z),p;0),h_*(t,(y,z),p;\beta_i)\},$$
$$H^*(t,(y,z),p)=\max\{h^*(t,(y,z),p;0),h^*(t,(y,z),p;\beta_i)\}.$$

Proof For notational convenience write $\xi=(t,(y,z),p)\in[0,T]\times\overline{G}\times\overline{G}\times\mathbb{R}^{2d}$. Observe that $h(\xi;c)=\Psi(\xi)c+\Phi(\xi)$, where

$$\Psi(\xi)=-\left(p_i+\sum_{\ell=1}^{d}p_{d+\ell}R_{\ell i}(t,y,z)+M_i(t,y,z)\right)$$

is a bounded continuous function, and $\Phi(\cdot)$ is a bounded measurable function. As h is linear in c, for fixed ξ

$$H(\xi)=\sup\{h(\xi;c):0\le c\le\beta_i\}=\Phi(\xi)+\max\{0,\Psi(\xi)\beta_i\}.$$

For ξ fixed, let $H_*(\xi)=\liminf\{H(\xi'):\xi'\to\xi\}$. Then there is $\xi''\to\xi$ such that $\Phi(\xi'')+\max\{0,\Psi(\xi'')\beta_i\}$ converges to $H_*(\xi)$. By continuity of $\Psi(\cdot)$ we have $\max\{0,\Psi(\xi'')\beta_i\}\to\max\{0,\Psi(\xi)\beta_i\}$. Hence $\Phi(\xi'')$ converges to $H_*(\xi)-\max\{0,\Psi(\xi)\beta_i\}$. It now follows that $\lim\Phi(\xi'')=\liminf\{\Phi(\xi'):\xi'\to\xi\}$; if not, using continuity of $\Psi(\cdot)$ we can easily get a contradiction to the definition of $H_*(\xi)$. Thus

$$\begin{aligned}H_*(\xi)&=\liminf\{\Phi(\xi'):\xi'\to\xi\}+\max\{0,\Psi(\xi)\beta_i\}\\&=\max\{[\liminf\Phi(\xi')],[(\liminf\Phi(\xi'))+\Psi(\xi)\beta_i]\}\\&=\max\{h_*(\xi;0),h_*(\xi;\beta_i)\}\end{aligned}$$

where the last equality follows once again using continuity of Ψ. The second assertion of the lemma is proved similarly. □

Definition 3.2 (a) A locally bounded function v is said to be a *viscosity subsolution* to the discontinuous HJB equation (3.1) on $[0,T)\times G\times G$ if for any $s\in[0,T)$, $y\in G$, $z\in G$, any C^1-function w such that $(v-w)$ has a local maximum at (s,y,z) one has

$$-\partial_0 w(s,y,z)+H_*(s,(y,z),D_{y,z}w(s,y,z))\le 0. \tag{3.4}$$

(b) A locally bounded function v is said to be a *viscosity supersolution* to (3.1) on $[0,T]\times\bar{G}\times\bar{G}$ if for any $s\in[0,T]$, $y\in\bar{G}$, $z\in\bar{G}$, any C^1-function w such that $(v-w)$ has a local minimum in $[0,T]\times\bar{G}\times\bar{G}$ at (s,y,z) one has

$$-\partial_0 w(s,y,z)+H^*(s,(y,z),D_{y,z}w(s,y,z))\ge 0. \tag{3.5}$$

(c) If v satisfies both (a), (b) above then it is called a *constrained viscosity solution* to (3.1) on $[0, T] \times \bar{G} \times \bar{G}$.

Note [3] (see Remark V.4.2 and Exercise V.4.1) very briefly discusses viscosity solution of a discontinuous HJB equation. However we do not know of any other instance of a discontinuous HJB equation with state constraints.

If the Hamiltonian is continuous, then it is clear that $H_* = H = H^*$ and hence the above definition is basically the same as the one given in Sect. 2.

Theorem 3.3 *Let $i, u_{-i}(\cdot)$ be fixed such that $0 \le u_j(\cdot) \le \beta_j, j \ne i$. Assume that b, R, L_i, M_i, g_i are bounded continuous. Assume that (2.12) holds. Suppose the value function $V^{(N,i)}$, defined by (2.13), is a bounded continuous function on $[0, T] \times \bar{G} \times \bar{G}$. Then $(s, y, z) \mapsto V^{(N,i)}(s, y, z; T, u_{-i}(\cdot))$ is a constrained viscosity solution to the discontinuous HJB equation (3.1) on $[0, T] \times \bar{G} \times \bar{G}$ with terminal value $g_i(T, \cdot, \cdot)$.*

Proof Our proof is influenced by the proofs of Proposition III.2.8, pp. 104–106, and Theorem IV.5.7, p. 278 of [3]. For simplicity of notation we shall drop the superscripts N, i.

Subsolution: Let $(s, y, z) \in [0, T) \times G \times G$. Let w be a C^1-function such that $(V - w)$ has a local maximum at (s, y, z). Take $u_i(\cdot) \equiv c$, where c is an arbitrary point in $[0, \beta_i]$, and denote by $y(\cdot), z(\cdot)$ the solution to the state equation (2.19) corresponding to the control $u(\cdot) = (u_i(\cdot), u_{-i}(\cdot))$ with $y(s) = y, z(s) = z$. As (s, y, z) is an interior point note that $z(s') \in G$ for all s' sufficiently close to s with $s' > s$. As $(V - w)$ has a local maximum at (s, y, z), by (2.14) in Theorem 2.3 (dynamic programming principle), continuity of $y(\cdot), z(\cdot)$ and (2.18) we get

$$\begin{aligned} w(s, y, z) - w(s', y(s'), z(s')) &\le V(s, y, z) - V(s', y(s'), z(s')) \\ &\le \int_s^{s'} C(r, y(r), z(r)) dr. \end{aligned}$$

As w is C^1, by state equation (2.19) and (2.17)

$$\begin{aligned} &w(s, y, z) - w(s', y(s'), z(s')) \\ &= -\int_s^{s'} \frac{d}{dr} w(r, y(r), z(r)) dr \\ &= -\int_s^{s'} [\partial_0 w(r, y(r), z(r)) + \langle D_{(y,z)} w(r, y(r), z(r)), f(r, (y(r), z(r)), c)\rangle] dr. \end{aligned}$$

Consequently we get

$$\begin{aligned} -\int_s^{s'} &[\partial_0 w(r, y(r), z(r)) + \langle D_{(y,z)} w(r, y(r), z(r)), f(r, (y(r), z(r)), c)\rangle \\ &+ C(r, (y(r), z(r)), c)] dr \le 0 \end{aligned}$$

for all s' sufficiently close to s with $s' > s$. Hence

$$\liminf_{r \downarrow s}[-\partial_0 w(r, y(r), z(r)) + h(r, (y(r), z(r)), D_{(y,z)} w(r, y(r), z(r)); c)] \leq 0$$

where $h(\cdots)$ is an in Lemma 3.1. As $y(\cdot), z(\cdot), \partial_0 w, D_{(y,z)} w$ are continuous it now follows that

$$-\partial_0 w(s, y, z) + h_*(s, (y, z), D_{(y,z)} w(s, y, z); c) \leq 0 \tag{3.6}$$

for any $0 \leq c \leq \beta_i$, where $h_*(\cdots)$ is as in Lemma 3.1. Now use (3.6) and Lemma 3.1 to get the required conclusion (3.4).

Supersolution: Let $(s, y, z) \in [0, T] \times \bar{G} \times \bar{G}$ (it could be a boundary point). Let w be a C^1-function such that $(V - w)$ has a local minimum in $[0, T] \times \bar{G} \times \bar{G}$ at (s, y, z). Let $\epsilon > 0, s' \in (s, T]$. Take $\eta = \epsilon(s' - s)$. By definition (2.13) of the value function there exists $\overline{u}_i(\cdot)$, possibly depending on ϵ, s' such that $(\overline{u}_i(\cdot), u_{-i}(\cdot)) \in \mathcal{U}(s, y, z)$ and

$$\begin{aligned}
V(s, y, z) &\geq J_i(s, y, z; T, (\bar{u}_i, u_{-i})) - \eta \\
&= \int_s^{s'} C(r, (\bar{y}(r), \bar{z}, (r)), \bar{u}_i(r))dr \\
&\quad + J_i(s', \bar{y}(s'), \bar{z}(s'); T, (\bar{u}_i, u_{-i})) - \epsilon(s' - s) \\
&\geq \int_s^{s'} C(r, (\bar{y}(r), \bar{z}(r)), \bar{u}_i(r))dr + V(s', \bar{y}(s'), \bar{z}(s')) - \epsilon(s' - s),
\end{aligned}$$

where $(\bar{y}(\cdot), \bar{z}(\cdot))$ denotes the solution to state equation (2.19) corresponding to the control $(\bar{u}_i(\cdot), u_{-i}(\cdot))$, and we have used (2.8), (2.18). Hence

$$V(s, y, z) - V(s', \bar{y}(s'), \bar{z}(s')) \geq \int_s^{s'} C(r, (\bar{y}(r), \bar{z}(r)), \bar{u}_i(r))dr - \epsilon(s' - s)$$

for all s' sufficiently close to s with $s' > s$. Note that $(\bar{y}(r), \bar{z}(r)) \in \bar{G} \times \bar{G}$ for all $r \in [s, T]$ as $(\bar{u}_i, u_{-i})$ is a feasible control. As $(V - w)$ has a local minimum in $[0, T] \times \bar{G} \times \bar{G}$ at (s, y, z), now an argument similar to the one in the first part of the proof, using dynamic programming principle, state equation, and continuity of $\bar{y}(\cdot), \bar{z}(\cdot)$, gives

$$\begin{aligned}
&\int_s^{s'} [-\partial_0 w(r, \bar{y}(r), \bar{z}(r)) + h(r, (\bar{y}(r), \bar{z}(r)), D_{(y,z)} w(r, \bar{y}(r), \bar{z}(r)); \bar{u}_i(r))]dr \\
&\quad \geq -\epsilon(s' - s).
\end{aligned}$$

Therefore by the definition of the Hamiltonian

$$\begin{aligned}
&\int_s^{s'} [-\partial_0 w(r, \bar{y}(r), \bar{z}(r)) + H(r, (\bar{y}(r), \bar{z}(r)), D_{(y,z)} w(r, \bar{y}(r), \bar{z}(r)))]dr \\
&\quad \geq -\epsilon(s' - s').
\end{aligned}$$

So, with $\epsilon > 0$ fixed, for any $s' \in (s, T]$ sufficiently close to s, there is a feasible control $\bar{u}_i(\cdot)$ such that

$$\sup_{s \le r \le s'} \{-\partial_0 w(r, \bar{y}(r), \bar{z}(r)) + H(r, (\bar{y}(r), \bar{z}(r)), D_{(y,z)} w(r, \bar{y}(r), \bar{z}(r)))\} \ge -\epsilon. \tag{3.7}$$

Note that the solution to the state equation is Lipschitz continuous in r, with the Lipschitz constant independent of the control. Hence given any small neighbourhood N of (s, y, z), there exists $s' \in (s, T]$ sufficiently close to s such that $(r, y(r), z(r)) \in N \cap ([s, T] \times \bar{G} \times \bar{G})$ for any $r \in [s, s']$ and any feasible control $u_i(\cdot)$. Consequently (3.7) now implies that

$$\begin{aligned} &\sup\{-\partial_0 w(r', y', z') + H(r', (y', z'), D_{(y,z)} w(r', y', z')) : \\ &\quad (r', y', z') \in N \cap ([s, T] \times \bar{G} \times \bar{G})\} \ge -\epsilon \end{aligned}$$

whence it follows that

$$-\partial_0 w(s, y, z) + H^*(s, (y, z), D_{(y,z)} w(s, y, z)) \ge -\epsilon. \tag{3.8}$$

As $\epsilon > 0$ is arbitrary (3.8) implies (3.5), completing the proof. □

Uniqueness cannot be expected to hold in general as the following counterexample indicates.

Example 3.4 Take $d = 2$. Let $A, B \subset [0, T]$ be subsets such that (i) $A \cup B = [0, T]$, $A \cap B = \phi$; (ii) both A and B are dense in $[0, T]$; (iii) $m(A) > 0$, $m(B) > 0$ where m is the one dimensional Lebesgue measure. Let $K_1 < K_2$ be constants. Define $u_2(s) = K_1 I_A(s) + K_2 I_B(s), 0 \le s \le T$. Let $b \equiv (0, 0), R_{12} \equiv R_{21} \equiv 0$, $R_{11} \equiv R_{22} \equiv 1, M_1 \equiv 1, L_1 \equiv 0$. So f is independent of (y, z) and is given by $f(s, c) = (c, u_2(s), c, u_2(s)) \in \mathbb{R}^2 \times \mathbb{R}^2$ and the Hamiltonian, for $s \in [0, T], p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4$ by

$$\begin{aligned} H(s, p) &= \sup\{-\langle p, f(s, c)\rangle - c : 0 \le c \le \beta_1\} \\ &= \sup\{-u_2(s)(p_2 + p_4) - c(1 + p_1 + p_3) : 0 \le c \le \beta_1\}. \end{aligned}$$

If $1 + p_1 + p_3 \ge 0$ then clearly

$$H(s, p) = \begin{cases} -K_1(p_2 + p_4), & \text{if } s \in A, p \in \mathbb{R}^4, \\ -K_2(p_2 + p_4), & \text{if } s \in B, p \in \mathbb{R}^4. \end{cases}$$

Consequently, as A, B are dense in $[0, T]$,

$$H_*(s, p) = -K_2(p_2 + p_4), \quad \text{if } p_2 + p_4 > 0, \tag{3.9}$$

$$H^*(s, p) = -K_1(p_2 + p_4), \quad \text{if } p_2 + p_4 > 0 \tag{3.10}$$

For a smooth function $(s, y, z) \mapsto v(s, y, z)$ such that $1 + \frac{\partial v}{\partial y_1} + \frac{\partial v}{\partial z_1} \geq 0$ and $\frac{\partial v}{\partial y_2} + \frac{\partial v}{\partial z_2} > 0$ on $[0, T] \times \bar{G} \times \bar{G}$, note that

$$\begin{aligned} &-\partial_0 v(s, y, z) + H_*(s, D_{(y,z)} v(s, y, z)) \\ &\quad = -\frac{\partial v}{\partial s}(s, y, z) - K_2 \frac{\partial v}{\partial y_2}(s, y, z) - K_2 \frac{\partial v}{\partial z_2}(s, y, z) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &-\partial_0 v(s, y, z) + H^*(s, D_{(y,z)} v(s, y, z)) \\ &\quad = -\frac{\partial v}{\partial s}(s, y, z) - K_1 \frac{\partial v}{\partial y_2}(s, y, z) - K_1 \frac{\partial v}{\partial z_2}(s, y, z). \end{aligned} \tag{3.12}$$

For the linear first order p.d.e.

$$\left(-\frac{\partial v}{\partial s} - K \frac{\partial v}{\partial y_2} - K \frac{\partial v}{\partial z_2}\right)(s, y, z) = 0, \quad 0 < s < T,\ y, z \in G, \tag{3.13}$$

with terminal value

$$v(T, y, z) = 1 - e^{-y_2} e^{-z_2}, \quad y, z \in \bar{G} \times \bar{G} \tag{3.14}$$

the solution is given by

$$v(s, y, z) = 1 - e^{-2K(T-s)} e^{-y_2} e^{-z_2}, \quad 0 \leq s \leq T,\ y, z \in \bar{G}. \tag{3.15}$$

(Note that the general solution to (3.13) can be written in the form $\varphi(Ks - y_2, y_2 - z_2)$ where φ is an arbitrary C^1-function.) Observe that $\frac{\partial v}{\partial y_2} + \frac{\partial v}{\partial z_2} > 0$. In view of this define two functions

$$v_1(s, y, z) = 1 - e^{-2K_1(T-s)} e^{-y_2} e^{-z_2}, \tag{3.16}$$

$$v_2(s, y, z) = 1 - e^{-2K_2(T-s)} e^{-y_2} e^{-z_2} \tag{3.17}$$

for $0 \leq s \leq T$, $y, z \in \bar{G}$. Note that $v_1(T, y, z) = v_2(T, y, z) = 1 - e^{-y_2} e^{-z_2}$. By (3.11–3.17) it is clear that

$$-\partial_0 v_1(s, y, z) + H^*(s, D_{(y,z)} v_1(s, y, z)) = 0, \tag{3.18}$$

$$-\partial_0 v_2(s, y, z) + H_*(s, D_{(y,z)} v_2(s, y, z)) = 0. \tag{3.19}$$

Since $K_1 < K_2$, (3.11), (3.12), (3.16), (3.17) imply

$$-\partial_0 v_1(s, y, z) + H_*(s, D_{(y,z)} v_1(s, y, z)) < 0, \tag{3.20}$$

$$-\partial_0 v_2(s, y, z) + H^*(s, D_{(y,z)} v_2(s, y, z)) > 0 \tag{3.21}$$

on $[0, T] \times \bar{G} \times \bar{G}$. Now using (3.18), (3.20) arguing as in Theorem 2.5 it can be shown that v_1 is a constrained viscosity solution to (3.1) on $[0, T] \times \bar{G} \times \bar{G}$. Similarly (3.19), (3.21) lead to showing that v_2 is also a constrained viscosity solution to (3.1) on $[0, T] \times \bar{G} \times \bar{G}$. Clearly v_1, v_2 are both bounded and Lipschitz continuous.

In the above, take $A = A_0 \cup A_1$ where A_0 is a Cantor set of positive Lebesgue measure and A_1 a countable dense set in $[0, T]$. Then there does not exist any function $\hat{u}(\cdot)$ on $[0, T]$ such that $u_2(\cdot) = \hat{u}(\cdot)$ a.s. and $m(\hat{D}) = 0$ where $\hat{D}$ is the set of discontinuities of $\hat{u}(\cdot)$. Because, if so, then for $a \cdot a \cdot s \in A_0$ there exist $s_n \in B$ with $s_n \to s$ and $K_1 = u_2(s) = \hat{u}(s) = \lim_n \hat{u}(s_n) = \lim_n u(s_n) = \lim_n K_2 = K_2$ which is a contradiction.

We next consider the case of utopian equilibrium. When i is fixed, the usual definitions work, uniqueness holds, and problem is somewhat easier; but it should be kept in mind that to get a utopian equilibrium d control problems should attain their minima at the same control.

For each fixed $1 \le i \le d$ we have a control problem with controls taking values in $\prod_{i=1}^{d}[0, \beta_i]$. The HJB equation in this case is

$$-\partial_0 v(s, y, z) + H^{(U,i)}(s, (y, z), D_{(y,z)}v(s, y, z)) = 0, \tag{3.22}$$

where $H^{(U,i)}$ is given by (2.36). If the data b, R, L_i, M_i are continuous in all the variables, the Hamiltonian $H^{(U,i)}$ is also continuous. So the definition of viscosity solution given in Definition 3.2 and the usual definition given in [3, 13] and recalled in Sect. 2 coincide.

Theorem 3.5 *In addition to* (A1–A3) *assume that b, R are continuous in the time variable as well. Let g_i, L_i, M_i be bounded continuous. Assume that $(s, y, z) \mapsto V^{(U,i)}(s, y, z; T)$, defined by* (2.35), *is a bounded continuous function on $[0, T] \times \bar{G} \times \bar{G}$. Then $V^{(U,i)}$ is a constrained viscosity solution to* (3.22) *on $[0, T] \times \bar{G} \times \bar{G}$.*

Proof Along lines similar to the proof of Theorem 3.3. Because of continuity of $f^{(U)}$ and $H^{(U,i)}$, the proof is much simpler; for example there is no need for Lemma 3.1. In fact the approach given in the proofs of Proposition III.2.8 and Theorem IV.5.7 of [3] can be more directly adapted. □

We now address the question of uniqueness. It is well known that this involves proving a comparison result. Our approach below is inspired by the proofs of Theorem III.3.7 and IV.5.8 of [3], of course, with some crucial deviations/modifications.

We denote $x = (y, z) \in \mathbb{R}^d \times \mathbb{R}^d$ and $\langle x \rangle = \langle (y, z) \rangle = [1 + \sum_{i=1}^{d} y_i^2 + \sum_{j=1}^{d} z_j^2]^{1/2}$; also $e_{2d} = (1, 1, \ldots, 1, 1) \in \mathbb{R}^{2d}$ with all the coordinates equal to 1. The following lemma is the analogue of Lemma III.2.11 of [3], and the proof is similar.

Lemma 3.6 *Let b, R, L_i, M_i be bounded and Lipschitz continuous in all the variables (including the time variable). Then for $s, s' \in [0, T]$, $x, x', \xi \in \bar{G} \times \bar{G}$, $\sigma > 0$, $\theta > 0$, $\theta' > 0$*

$$\begin{aligned}
&|H^{(U,i)}(s, x, \{\sigma[x - x' + \xi] + \theta x\}) \\
&\quad - H^{(U,i)}(s', x', \{\sigma[x - x' + \xi] - \theta' x'\})| \\
&\quad \le K\sigma|x - x' + \xi| \cdot \{|s - s'| + |x - x'|\} \\
&\qquad + K\theta\langle x\rangle^2 + K\theta'\langle x'\rangle^2 + K\{|s - s'| + |x - x'|\},
\end{aligned} \tag{3.23}$$

where K depends only on bounds and Lipschitz constants of b, R, L_i, M_i.

Theorem 3.7 *Assume* (A3) *and that* b, R, L_i, M_i *are bounded Lipschitz continuous functions in all the variables* (*including the time variable*). *Let* v_1, v_2 *be functions on* $[0,T] \times \bar{G} \times \bar{G}$ *such that*

(a) v_1, v_2 *are bounded uniformly continuous functions on* $[0,T] \times \bar{G} \times \bar{G}$;
(b) v_1 *is a viscosity subsolution to* (3.22) *on* $[0,T) \times G \times G$;
(c) v_2 *is a viscosity supersolution to* (3.22) *on* $[0,T] \times \bar{G} \times \bar{G}$;
(d) $v_1(T,\cdot,\cdot) = v_2(T,\cdot,\cdot)$ *on* $\bar{G} \times \bar{G}$.

Then $v_1 \le v_2$ *on* $[0,T] \times \bar{G} \times \bar{G}$. *In particular, under the above hypotheses, if the value function* $V^{(U,i)}$ *given in Theorem* 3.5 *is bounded uniformly continuous on* $[0,T] \times \bar{G} \times \bar{G}$, *then it is the unique constrained viscosity solution to the HJB equation* (3.22) *in the class of bounded uniformly continuous functions with terminal value* $g_i(T,\cdot,\cdot)$.

Proof Suppose $M \equiv \sup\{v_1(s,y,z) - v_2(s,y,z) : s \in [0,T], y,z \in \bar{G}\} > 0$. For $\delta \in (0,M)$ note that there exists $(\tilde{s},\tilde{x}) = (\tilde{s},\tilde{y},\tilde{z}) \in [0,T] \times \bar{G} \times \bar{G}$ such that $v_1(\tilde{s},\tilde{x}) - v_2(\tilde{s},\tilde{x}) = \delta$. Clearly $s < T$ by (d). Choose $\lambda > 0, \eta > 0, \mu > 0, \nu > 0$ such that

$$2\lambda\langle\tilde{x}\rangle + 2\eta(T-\tilde{s}) + \nu + 2d\mu \le \frac{1}{2}\delta$$

ensuring that

$$2\lambda\langle\tilde{x}\rangle^m + 2\eta(T-\tilde{s}) + \nu + 2d\mu \le \frac{1}{2}\delta, \quad \text{for all } 0 < m \le 1. \tag{3.24}$$

For $(s,x;s',x') \in ([0,T] \times \bar{G} \times \bar{G})^2$ define

$$\Psi_\epsilon(s,x;s',x') = v_1(s,x) - v_2(s',x') - \mu\left|\frac{x-x'}{\epsilon} - e_{2d}\right|^2 - \nu\left|\frac{s-s'}{\epsilon} + 1\right|^2 - \lambda(\langle x\rangle^m + \langle x'\rangle^m) - \eta[(T-s)+(T-s')]. \tag{3.25}$$

We will choose $\epsilon > 0$, $0 < m \le 1$ suitably later; m will be chosen appropriately and fixed, whereas ϵ will be treated as a parameter; so only dependence on ϵ is highlighted in Ψ_ϵ. Since $\Psi_\epsilon \to -\infty$ as $|x| + |x'| \to \infty$ and Ψ_ϵ is continuous, it follows that there exists $(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon)$ such that, by (3.24),

$$\begin{aligned}\Psi_\epsilon(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon) &= \sup\{\Psi_\epsilon(s,x;s',x') : s,s' \in [0,T], x,x' \in \bar{G}\} \\ &\ge \Psi_\epsilon(\tilde{s},\tilde{x};\tilde{s},\tilde{x}) \ge \frac{1}{2}\delta.\end{aligned}$$

Consequently

$$\lambda[\langle x_\epsilon\rangle^m + \langle x'_\epsilon\rangle^m] + \eta[(T-s_\epsilon)+(T-s'_\epsilon)] + \mu\left|\frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d}\right|^2 + \nu\left|\frac{s_\epsilon - s'_\epsilon}{\epsilon} + 1\right|^2 \le \sup v_1 - \inf v_2 \equiv C_0 \tag{3.26}$$

for all $\epsilon > 0$, $0 < m \leq 1$. (Note that $C_0 > 0$, otherwise $M = 0$.) It follows from (3.26) that $x_\epsilon, x'_\epsilon \in \overline{B(0:(C_0/\lambda)^{1/m})}$ for all $\epsilon > 0$.

As Ψ_ϵ has its maximum at $(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon)$

$$\Psi_\epsilon(s_\epsilon, x_\epsilon; s_\epsilon + \epsilon, x_\epsilon - \epsilon e_{2d}) + \Psi_\epsilon(s'_\epsilon - \epsilon, x'_\epsilon + \epsilon e_{2d}; s'_\epsilon, x'_\epsilon) \leq 2\Psi_\epsilon(s_\epsilon, x_\epsilon; s'_\epsilon, x'_\epsilon). \tag{3.27}$$

By mean value theorem $\langle x_\epsilon - \epsilon e_{2d}\rangle^m = \langle x\rangle^m + O(\epsilon)$ as x_ϵ varies over a bounded set. Therefore

$$\begin{aligned}&\Psi_\epsilon(s_\epsilon, x_\epsilon; s_\epsilon + \epsilon, x_\epsilon - \epsilon e_{2d})\\ &\quad = v_1(s_\epsilon, x_\epsilon) - v_2(s_\epsilon + \epsilon, x_\epsilon - \epsilon e_{2d}) - 2\lambda\langle x_\epsilon\rangle^m\\ &\qquad - 2\eta(T - s_\epsilon) + \eta\epsilon + O(\epsilon)\end{aligned} \tag{3.28}$$

and similarly

$$\begin{aligned}&\Psi_\epsilon(s'_\epsilon - \epsilon, x'_\epsilon + \epsilon e_{2d}; s'_\epsilon, x'_\epsilon)\\ &\quad = v_1(s'_\epsilon - \epsilon, x'_\epsilon + \epsilon e_{2d})\\ &\qquad - v_2(s'_\epsilon, x'_\epsilon) - 2\lambda\langle x'_\epsilon\rangle^m - 2\eta(T - s'_\epsilon) - \eta\epsilon + O(\epsilon).\end{aligned} \tag{3.29}$$

Denote by ω the common modulus of continuity of v_1 and v_2; note that ω can be taken to be bounded as v_1, v_2 are bounded uniformly continuous. Now (3.27–3.29) imply

$$\begin{aligned}&\mu\left|\frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d}\right|^2 + \nu\left|\frac{s_\epsilon - s'_\epsilon}{\epsilon} + 1\right|^2\\ &\quad \leq \omega(|s_\epsilon - s'_\epsilon + \epsilon| + |x_\epsilon - x'_\epsilon - \epsilon e_{2d}|) + O(\epsilon).\end{aligned} \tag{3.30}$$

If $\epsilon \leq 1$, then r.h.s. of (3.30) is bounded by a constant independent of ϵ. So from (3.30) we get

$$|x_\epsilon - x'_\epsilon| + |s_\epsilon - s'_\epsilon| \leq K_1\epsilon. \tag{3.31}$$

Plugging (3.31) back into (3.30)

$$\left|\frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d}\right|^2 + \left|\frac{s_\epsilon - s'_\epsilon}{\epsilon} + 1\right|^2 \leq \omega(K_2\epsilon) + O(\epsilon) \tag{3.32}$$

as $\epsilon \downarrow 0$, where K_2 is constant independent of ϵ.

Clearly the ball $B(x + \epsilon e_{2d} : \epsilon) \subset G \times G$ for any $x \in \bar{G} \times \bar{G}$. So by (3.32) it is now easily seen that $(s_\epsilon, x_\epsilon) \in [0, T) \times G \times G$ for any $\epsilon > 0$ such that r.h.s. of (3.32) <1.

With $0 < \epsilon \leq 1$ as above, for $(s, x) = (s, y, z) \in [0, T] \times \bar{G} \times \bar{G}$ define

$$\begin{aligned}w_{1\epsilon}(s, x) = v_2(s'_\epsilon, x'_\epsilon) + \mu\left|\frac{x - x'_\epsilon}{\epsilon} - e_{2d}\right|^2 + \nu\left|\frac{s - s'_\epsilon}{\epsilon} + 1\right|^2\\ + \lambda(\langle x\rangle^m + \langle x'_\epsilon\rangle^m) + \eta[(T - s) + (T - s'_\epsilon)].\end{aligned}$$

Note that $(v_1 - w_{1\epsilon})$ has a maximum at $(s_\epsilon, y_\epsilon, z_\epsilon)$ and that $w_{1\epsilon}$ is C^1. Similarly, for $(s', x') = (s', y', z') \in [0, T] \times \bar{G} \times \bar{G}$ define

$$w_{2\epsilon}(s', x') = v_1(s_\epsilon, x_\epsilon) - \mu\left|\frac{x_\epsilon - x'}{\epsilon} - e_{2d}\right|^2 - \nu\left|\frac{s_\epsilon - s'}{\epsilon} + 1\right|^2 \\ - \lambda[\langle x_\epsilon\rangle^m + \langle x'\rangle^m] - \eta[(T - s_\epsilon) + (T - s')].$$

Clearly $w_{2\epsilon}$ is C^1 and $(v_2 - w_{2\epsilon})$ has a minimum in $[0, T] \times \bar{G} \times \bar{G}$ at $(s'_\epsilon, y'_\epsilon, z'_\epsilon)$. Since v_1 is a viscosity subsolution in the interior and v_2 is a viscosity supersolution to (3.22) in the closure, it can be seen that

$$2\eta + H\left(s_\epsilon, x_\epsilon, \left\{\frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] + m\lambda\langle x_\epsilon\rangle^{m-2}x_\epsilon\right\}\right) \\ - H\left(s'_\epsilon, x'_\epsilon, \left\{\frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] - m\lambda\langle x'_\epsilon\rangle^{m-2}x'_\epsilon\right\}\right) \le 0, \qquad (3.33)$$

where $H = H^{(U,i)}$. Now by Lemma 3.6, (3.26), (3.31), (3.32)

$$\Bigg|H\left(s_\epsilon, x_\epsilon, \left\{\frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] + m\lambda\langle x_\epsilon\rangle^{m-2}x_\epsilon\right\}\right) \\ - H\left(s'_\epsilon, x'_\epsilon, \left\{\frac{2\mu}{\epsilon^2}[x_\epsilon - x'_\epsilon - \epsilon e_{2d}] - m\lambda\langle x'_\epsilon\rangle^{m-2}x'_\epsilon\right\}\right)\Bigg| \\ \le K\frac{2\mu}{\epsilon}\left|\frac{x_\epsilon - x'_\epsilon}{\epsilon} - e_{2d}\right|[|s_\epsilon - s'_\epsilon| + |x_\epsilon - x'_\epsilon|] \\ + Km\lambda[\langle x_\epsilon\rangle^m + \langle x'_\epsilon\rangle^m] + K[|s_\epsilon - s'_\epsilon| + |x_\epsilon - x'_\epsilon|] \\ \le 2KK_1\mu[\omega(K_2\epsilon) + O(\epsilon)]^{1/2} + mKC_0 + KK_1\epsilon. \qquad (3.34)$$

Now choose $m \in (0, 1)$ such that $m < \frac{\eta}{2KC_0}$. Then (3.33), (3.34) imply

$$\frac{3}{2}\eta - KK_1\epsilon - 2KK_1\mu[\omega(K_2\epsilon) + O(\epsilon)]^{1/2} \le 0.$$

But this would contradict $\eta > 0$ for $\epsilon \downarrow 0$. Thus $M = 0$ and hence $v_1 \le v_2$ on $[0, T] \times \bar{G} \times \bar{G}$. □

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Appendix

The appendix has two parts. In the first part we give a sufficient condition for feasible controls to exist in the context of Nash equilibrium. The connection with Skorokhod problem of probability theory is briefly reviewed in the second part. We also give an example to show that Nash equilibrium need not be unique.

A Sufficient Condition for (2.12)

We give a sufficient condition for $\mathcal{U}(s, y, z; T, u_{-i})$ to be nonempty for fixed i, u_{-i} satisfying (2.11). In fact we have something more.

Theorem 4.1 *Assume* (A1–A3); *moreover assume that $b, \mathbb{R}$ are continuous in the time variable as well. In addition let the following hypothesis hold*:

(A4) *For $i = 1, 2, \ldots, d$, $s \in [0, T]$, $y \in \bar{G}$, $z \in \partial G$, $u_j \in [0, \beta_j]$, $j \neq i$ there exists $u_i \in [0, \beta_i]$ with*

$$\left[b_k(s, y, z) + \sum_{\ell=1}^{d} R_{k\ell}(s, y, z) u_\ell \right] > 0 \tag{4.1}$$

for any $k \in I(z)$. Here $\beta = (\beta_1, \ldots, \beta_d)$ is as in (2.11), *and $I(z) = \{k : z_k = 0\}$.*

Then $\mathcal{U}(s, y, z; T, u_{-i}(\cdot))$ is a nonempty weakly compact subset of $L^2[s, T]$ for any $s \in [0, T]$, $y, z \in \bar{G}$, $0 \le u_j(\cdot) \le \beta_j$, $j \neq i$.

Note Compare (4.1) with (2.27), (2.28). We know that (2.27), (2.28) give a necessary boundary condition for $u(\cdot)$ to be feasible. See also Sect. 5, Chap. 4 of [3].

We first prove the following lemma

Lemma 4.2 *Under the above hypotheses $\mathcal{U}(s, y, z; T, u_{-i}(\cdot)) \neq \phi$ if $u_{-i}(\cdot)$ is right continuous. Moreover $u_i(\cdot)$ can also chosen to be right continuous in this case.*

Proof Fix $i, s, y, z, u_{-i}(\cdot)$. We first claim that there exists a right continuous $u_i(\cdot)$ taking values in $[0, \beta_i]$ such that

$$\tau_0 = \inf\{t > s : z(t) \notin G\} > s,$$

where $y(\cdot), z(\cdot)$ is the solution to the state equation corresponding to the control $u(\cdot) = (u_i(\cdot), u_{-i}(\cdot))$ with $y(s) = y, z(s) = z$.

Indeed, if $z \in G$ then $u_i(\cdot)$ can be taken to be any right continuous function on $[s, T]$ taking values in $[0, \beta_i]$. So let $z \in \partial G$; note that $I(z) \neq \phi$. Taking $u_j = u_j(s)$, $j \neq i$, by assumption (A4) choose u_i so that (4.1) holds for all $k \in I(z)$. Set $u(t) = (u_i, u_{-i}(t)), t \ge s$; note that the ith component of $u(\cdot)$ is constant; let $y(\cdot), z(\cdot)$ denote the corresponding solution to the state equation. By continuity of $b_k, R_{kj}, y(\cdot), z(\cdot)$ and right continuity of $u(\cdot)$ at s, note that there exist $\epsilon_0 > 0$, $\eta > 0$ such that for all $t \in [s, s + \epsilon_0]$ we have $I(z(t)) \subseteq I(z)$ and

$$b_k(t, y(t), z(t)) + \sum_{\ell} R_{k\ell}(t, y(t), z(t)) u_\ell(t) \ge \eta$$

for all $k \in I(z)$. So $z_k(t) \ge \eta(t - s) > 0$ for $k \in I(z)$ and $z_\ell(t) > 0$ for $\ell \notin I(z)$, for all $s < t \le s + \epsilon_0$. The claim now follows.

Now put $\tau := \sup\{t \in [s,T]$: there exists a right continuous $u_i(\cdot)$ taking values in $[0,\beta_i]$ and $z(r) \in \bar{G}$ for all $s \le r \le t\}$; here $y(\cdot), z(\cdot)$ denote the solution to the state equation corresponding to $(u_i(\cdot), u_{-i}(\cdot))$. If $\tau < T$ then apply the claim above with $\tau_0, y(\tau_0), z(\tau_0)$ respectively replacing s, y, z; we now get a contradiction to the definition of τ. So $\tau = T$ and hence $z(t) \in \bar{G}$ for all $s \le t \le T$. □

Proof of Theorem 4.1 Fix i, s, y, z; let $u_j(\cdot)$, $j \ne i$ be as in the theorem. For $n \ge 1$ choose $u_{-i}^{(n)}(\cdot) = (u_1^{(n)}(\cdot), \ldots, u_{i-1}^{(n)}(\cdot), u_{i+1}^{(n)}(\cdot), \ldots, u_d^{(n)}(\cdot))$ such that $u_j^{(n)}(\cdot)$ is right continuous, $0 \le u_j^{(n)}(\cdot) \le \beta_j$, and $u_j^{(n)}(\cdot) \to u_j(\cdot)$ in $L^2[s,T]$ as $n \to \infty$, for $j \ne i$. By Lemma 4.2, for $n \ge 1$, there exists $u_i^{(n)}(\cdot)$, right continuous on $[s,T]$, taking values in $[0,\beta_i]$, and if $(y^{(n)}(\cdot), z^{(n)}(\cdot))$ denotes the solution to the state equation corresponding to the control $u^{(n)}(\cdot) = (u_i^{(n)}(\cdot), u_{-i}^{(n)}(\cdot))$ with $y^{(n)}(s) = y, z^{(n)}(s) = z$ then $y^{(n)}(t) \ge 0, z^{(n)}(t) \ge 0$ for all $t \in [s,T]$.

As $\{u_i^{(n)}(\cdot) : n \ge 1\}$ is bounded, by Banach-Alaoglu theorem there exists $u_i(\cdot) \in L^2[s,T]$ such that $u_i^{(n)} \to u_i(\cdot)$ weakly. Put $u(\cdot) = (u_i(\cdot), u_{-i}(\cdot))$; let $y(\cdot), z(\cdot)$ denote the solution to the state equation corresponding to $u(\cdot)$ starting at s, y, z. It can easily be proved that $0 \le u_i(\cdot) \le \beta_i$. Clearly $y_i(\cdot) \ge 0$ and nondecreasing. Using Lipschitz continuity of b, R, uniform boundedness of $u^{(n)}(\cdot), u(\cdot)$ and Gronwall inequality it can be shown that $z^{(n)}(t) \to z(t)$ for all t, and hence that $z(\cdot)$ is $\bar{G}$-valued (cf. proof of Theorem 4.3 of [20]). Thus $\mathcal{U}(s, y, z; T, u_{-i}(\cdot)) \ne \phi$. The same argument implies weak compactness as well. □

The SP Connection

As mentioned in Sect. 1, it has been proved in [19, 20] that the "pushing part" of the solution to the deterministic Skorokhod problem provides Nash/utopian equilibrium under suitable monotonicity conditions. For description of Skorokhod problem, its importance in probability theory and existence and uniqueness of solutions see [12, 14, 17, 19, 23] and the references therein. However to read off the results from [19, 20] in the present context we need the following result, which may be known to experts. As we have not seen an easily accessible proof we include it for the sake of completeness. We shall also use the notation as in [19, 20].

Proposition 4.3 *Let $w \in C([0,\infty) : \mathbb{R})$ be absolutely continuous with derivative $\dot{w}(\cdot)$; assume $w(0) \ge 0$. Let $y^w(\cdot), z^w(\cdot)$ be the solution to the one dimensional Skorokhod problem for $w(\cdot)$. Then $y^w(\cdot), z^w(\cdot)$ are also absolutely continuous and $0 \le \dot{y}^w(\cdot) \le |\dot{w}(\cdot)|, |\dot{z}^w(\cdot)| \le 2|\dot{w}(\cdot)|$ a.s.*

Proof Let $s \ge 0$. Put $\hat{w}(t) = z^w(s) + \int_s^t \dot{w}(r)dr, \hat{y}(t) = y^w(t) - y^w(s), \hat{z}(t) = z^w(t), t \ge s$. Then $\hat{y}(\cdot), \hat{z}(\cdot)$ is the unique solution to the Skorokhod problem for $\hat{w}(\cdot)$ starting at time s. Also put $\tilde{w}(t) \equiv z^w(s)$, $\tilde{y}(t) \equiv 0, \tilde{z}(t) \equiv z^w(s), t \ge s$. Then $\tilde{y}(\cdot), \tilde{z}(\cdot)$ is the unique solution to the Skorokhod problem for $\tilde{w}(\cdot)$ starting at time s.

Note that $\hat{w} - \tilde{w}$ is of bounded variation over $[s,t]$ for any $t \geq s$. So by the lemma of variational distance between maximal functions in Sect. 2 of [23], for any $t \geq s$

$$\mathrm{Var}(y^w : [s,t]) = \mathrm{Var}(\hat{y} - \tilde{y} : [s,t]) \leq \mathrm{Var}(\hat{w} - \tilde{w} : [s,t]) = \int_s^t |\dot{w}(r)| dr \tag{4.2}$$

where $\mathrm{Var}(g : [a,b])$ denotes the total variation of g over $[s,t]$.

As (4.2) holds for every $0 \leq s \leq t$, it follows that $\mathrm{Var}(y^w : d\alpha) = dy^w(\cdot)$ is absolutely continuous. The other assertions are now easy to obtain. □

Let b, R satisfy (A1–A3). Let $w(\cdot)$ be an $\mathbb{R}^d$-valued continuous function on $[0,\infty)$ such that $w(0) \in \bar{G}$. Let $Yw(\cdot)$, $Zw(\cdot)$ denote the solution to the Skorokhod problem in $\bar{G}$ with initial input $w(\cdot)$, drift b and reflection field R as described in [19]. So $Zw(t) \in \bar{G}$ for all t, and $(Yw)_i(\cdot) \geq 0$, nondecreasing, and can increase only when $(Zw)_i(\cdot) = 0$. In view of Proposition 4.3, by the methods/arguments in [19] it follows that if $w(\cdot)$ is absolutely continuous then so are $Yw(\cdot)$, $Zw(\cdot)$. A modification needed here is that the analogue of the metric given in Sect. 3 of [19] be defined in terms of L^1 norm of the derivatives instead of the variational norm/supremum norm of the functions; as the variational norm of an absolutely continuous function is the L^1 norm of the derivative, the arguments of [19] can be easily adapted. In particular taking $w(\cdot) \equiv z \in \bar{G}$, Skorokhod problem for our purposes is the following.

Given $s \geq 0$, $y \in \bar{G}$, $z \in \bar{G}$ consider the problem: Find functions $P(\cdot; s, y, z) \equiv P(\cdot) = (P_1(\cdot), \ldots, P_d(\cdot))$, $Q(\cdot; s, y, z) \equiv Q(\cdot) = (Q_1(\cdot), \ldots, Q_d(\cdot))$ on $[s,\infty)$ satisfying the following:

1. $P_i(t) \geq 0$ a.e. $t \geq s$, $1 \leq i \leq d$;
2. $Q_i(\cdot)$ integrable over every compact interval;
3. $Y(\cdot; s, y, z) \equiv Y(\cdot) = (Y_1(\cdot), \ldots, Y_d(\cdot))$ with

$$Y_i(t) = y_i + \int_s^t P_i(r) dr, \quad t \geq s, \ 1 \leq i \leq d; \tag{4.3}$$

   so $Y_i(\cdot) \geq 0$ and nondecreasing;
4. $Z(\cdot; s, y, z) \equiv Z(\cdot) = (Z_1(\cdot), \ldots, Z_d(\cdot))$ with $Z(\cdot) \in \bar{G}$ and

$$Z_i(t) = z_i + \int_s^t Q_i(r) dr, \quad t \geq s, \ 1 \leq i \leq d; \tag{4.4}$$

5. $Z(\cdot)$ satisfies the Skorokhod equation, viz. for $i = 1, 2, \ldots, d$, $t \geq s$

$$\begin{aligned} Z_i(t) = z_i &+ \int_s^t b_i(r, Y(r), Z(r)) dr + Y_i(t) - y_i \\ &+ \sum_{j \neq i} \int_s^t R_{ij}(r, Y(r), Z(r)) P_j(r) dr; \end{aligned} \tag{4.5}$$

6. $Y_i(\cdot)$ can increase only when $Z_i(\cdot) = 0$, that is, $Z_i(t) P_i(t) = 0$ a.e. t, $1 \leq i \leq d$.

In such a case we say P, Q (or equivalently Y, Z) solves the Skorokhod problem.

By the above discussion it follows that for any $s \ge 0$, $y, z \in \bar{G}$, the Skorokhod problem has a unique solution whenever b, R satisfy (A1–A3). Moreover, by results of [19] it follows that

$$0 \le P_i(r) \le ((I - W)^{-1}\tilde{\beta})_i \le \beta_i, \quad \text{a.e. } r, \ 1 \le i \le d$$

where $\tilde{\beta}$ is as in (A1) and β as in (2.11). This is one reason to consider only controls taking values in $\prod_{i=1}^{d}[0, \beta_i]$. The functions $Y_i(\cdot)$ or equivalently $P_i(\cdot)$, $1 \le i \le d$ are called the "pushing part" of the solution to the Skorokhod problem.

The following result can now be obtained; while the first part is a consequence of Theorems 4.7, 4.14 of [20], the second part follows from Theorem 5.3 of [19].

Theorem 4.4 (i) *In addition to* (A1–A3), *let b and R satisfy the following conditions.*

(C1) *For* $1 \le i \le d, b_i, R_{ij}$ *are independent of* $z_\ell, \ell \ne i$; *that is,* $b_i(t, y, z) = b_i(t, y, z_i), R_{ij}(t, y, z) = R_{ij}(t, y, z_i)$.

(C2) *For fixed* $1 \le i \le d$, $y_{-i} = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_d)$, $t \ge 0$, $z \in \mathbb{R}^d$

$$b_i(t, (\xi, y_{-i}), z) \ge b_i(t, (\tilde{\xi}, y_{-i}), z),$$
$$R_{ij}(t, (\xi, y_{-i}), z) \ge R_{ij}(t, (\tilde{\xi}, y_{-i}), z), \quad 1 \le j \le d,$$

whenever $\xi \le \tilde{\xi}$; *that is* b_i, R_{ij} *are nonincreasing in* y_i.

(C3) *The functions* $z_i \mapsto b_i(t, y, z_i) = b_i(t, y, z), z_i \mapsto R_{ij}(t, y, z_i) = R_{ij}(t, y, z)$ *are differentiable and*

$$\frac{\partial}{\partial z_i} b_i(t, y, z) \le 0, \qquad \frac{\partial}{\partial z_i} R_{ij}(t, y, z) \le 0, \quad 1 \le i, j \le d.$$

Also for $1 \le i \le d$, *suppose that* $M_i \equiv$ *positive constant,* g_i, L_i *are nonnegative functions independent of* z, *satisfying*

$$g_i(T, (\xi, y_{-i})) \le g_i(T, (\tilde{\xi}, y_{-i})),$$
$$L_i(t, (\xi, y_{-i})) \le L_i(t, (\tilde{\xi}, y_{-i})),$$

whenever $\xi \le \tilde{\xi}, \forall t, y_{-i}$. *Then* $(P_1(\cdot), \dots, P_d(\cdot))$ *is a Nash equilibrium in* $\mathcal{U}(s, y, z; T)$.

(ii) *In addition to* (A1–A3), *assume the hypotheses of Theorem* 5.3 *of* [19]. *Suppose also* $M_i \equiv$ *positive constant,* g_i, L_i *are nonnegative functions independent of* z *satisfying* $g_i(T, \tilde{y}) \le g_i(T, y)$, $L_i(t, \tilde{y}) \le L_i(t, y)$ *whenever* $\tilde{y} \le y$ *for all* $t \ge 0, 1 \le i \le d$. *Then* $(P_1(\cdot), \dots, P_d(\cdot))$ *is a utopian equilibrium in* $\mathcal{U}(s, y, z; T)$. *(Here* $\tilde{y} \le y$ *means* $\tilde{y}_j \le y_j$ *for all* j.)

In the converse direction we have the following result. Note that the conditions on b, R are less stringent than in Theorem 4.8 of [20].

Theorem 4.5 *Let $g_i \equiv 0$, $L_i \equiv 0$, $M_i \equiv 1$ for $1 \le i \le d$. In addition to the hypotheses of Theorem 4.1, assume that $R_{k\ell}(\cdots) \le 0$ for $k \ne \ell$. Fix $s \in [0, T]$, $y \in \bar{G}$, $z \in \bar{G}$. Let $\hat{u}(\cdot) = (\hat{u}_1(\cdot), \dots, \hat{u}_d(\cdot))$ be such that for any $t \in [s, T]$, the restriction of $\hat{u}(\cdot)$ to $[s, t]$ is a Nash equilibrium in $\mathcal{U}(s, y, z; t)$. Let $\hat{y}(\cdot), \hat{z}(\cdot)$ denote the solution to state equation (2.3), (2.4) corresponding to the control $\hat{u}(\cdot)$ with initial value $\hat{y}(s) = y, \hat{z}(s) = z$. Then $\hat{y}(\cdot) - y, \hat{z}(\cdot)$, solves the Skorokhod problem on $[s, T]$, with drift $b(\cdots)$ and reflection field $R(\cdots)$ corresponding to $w(\cdot) \equiv z$.*

Proof Let $\hat{y}(\cdot), \hat{z}(\cdot)$ denote the solution to the state equation corresponding to the control $\hat{u}(\cdot)$. As $\hat{u}(\cdot)$ is feasible it is clear that $\hat{z}(\cdot)$ is $\bar{G}$-valued. So we just need to prove that $\hat{z}_i(\cdot)\hat{u}_i(\cdot) = 0$ a.s. for each i. Put $\hat{h}_i(t) = \int_s^t \hat{z}_i(r)\hat{u}_i(r)dr$, $t \ge s$. We need to prove $\hat{h}_i(\cdot) \equiv 0$, $1 \le i \le d$. Suppose not. Then $\hat{h}_i(t) > 0$ for some $t > s$, $1 \le i \le d$. Hence, by continuity of $\hat{z}_i(\cdot)$, there exist $x > 0, s \le t_0 < \tilde{t}$ such that $\hat{z}_i(t) \ge x > 0$ for all $t_0 \le t \le \tilde{t}$ and

$$m(\{r : \hat{u}_i(r) > 0 \text{ for } r \in [t_0, t]\}) > 0, \quad \forall t_0 \le t \le \tilde{t}. \tag{4.6}$$

We now make a remark concerning (A4). Let $i, u_j, j \ne i, s, y$ be as in (A4). Let $z \in \partial G$ be such that $z_i > 0$; so $R_{ki}(\cdots) \le 0$ for any $k \in I(z)$. Therefore one can take $u_i = 0$ so that (4.1) holds.

Now define $\tilde{u}(\cdot) = (\tilde{u}_i(\cdot), \tilde{u}_{-i}(\cdot))$ by $\tilde{u}_j(\cdot) = \hat{u}_j(\cdot)$, for $j \ne i, \tilde{u}_i(r) = \hat{u}_i(r)$, $s \le r < t_0$ and $\tilde{u}_i(r) = 0, r \ge t_0$. Let $\tilde{y}(\cdot), \tilde{z}(\cdot)$ denote the solution to the state equation corresponding to $\tilde{u}(\cdot)$. Clearly $\tilde{z}(r) = \hat{z}(r)$, $s \le r \le t_0$. In particular $\tilde{z}_i(t_0) \ge x > 0$. Put $t_1 = \inf\{t \ge t_0 : \tilde{z}_i(t) = 0\}$. Clearly $t_1 > t_0$. Now Theorem 4.1 and the remark above used repeatedly give that $\mathcal{U}(s, y, z; t, \hat{u}_{-i}(\cdot)) \ne \phi$ and in fact $\tilde{u}_i(\cdot) \in \mathcal{U}(s, y, z; t, \hat{u}_{-i}(\cdot))$ for any $t \in [t_0, t_1]$. In view of (4.6) it is now clear that

$$\int_s^t \tilde{u}_i(r)dr < \int_s^t \hat{u}_i(r)dr, \quad t_0 < t \le (t_1 \wedge T). \tag{4.7}$$

As (4.7) contradicts our hypothesis that $\hat{u}(\cdot)$ is a Nash equilibrium in $\mathcal{U}_{\bar{\beta}}(s, y, z; t)$ for $t \in [s, T]$, the result now follows. □

Remark 4.6 Theorem 4.4 gives conditions under which the solution to the Skorokhod problem can provide a Nash equilibrium. As there are approximation procedures to solve the Skorokhod problem, this forms a method of getting a system of bounded continuous constrained viscosity solutions to the interlinked family of HJB equations (3.1) with $u_{-i} = P_{-i}$, $1 \le i \le d$. Under the conditions of Theorem 4.4 the pushing part of the solution to Skorokhod problem itself provides a system of constrained viscosity solutions for all t.

We conclude with an example to show that Nash equilibrium need not be unique.

Example 4.7 Let $d = 2$, $b(\cdots) \equiv (1, -1)$, $R_{12}(\cdots) \equiv R_{12} > 0$, $R_{21}(\cdots) \equiv R_{21} > 0$ are constants such that $R_{12}R_{21} < 1$. Let $g_i \equiv 0$, $L_i \equiv 0$, $M_i \equiv 1$, $i = 1, 2$. Take $s = 0$, $y = 0$, $z = 0$. The state equation for the z-part, viz. analogue of (2.6), is given by

$$z_1(t) = t + y_1(t) + R_{12}y_2(t),$$
$$z_2(t) = -t + y_2(t) + R_{21}y_1(t).$$

Solution to the corresponding Skorokhod problem is given by $Y_1(t) \equiv 0$, $Y_2(t) = t$, $Z_1(t) = (1 + R_{12})t$, $Z_2(t) \equiv 0$. By Theorem 4.4 the solution to the Skorokhod problem gives a Nash equilibrium; this also follows from the argument given below.

Let $\lambda_1 \geq 0, 0 \leq \lambda_2 \leq 1$ be such that $\lambda_2 + R_{21}\lambda_1 = 1$. Put $\hat{y}_1(t) = \lambda_1(t)$, $\hat{y}_2(t) = \lambda_2 t$. Fix $\hat{y}_1(\cdot)$. Let $0 \leq y_2(t) < \lambda_2 t$ for some t. Corresponding to $\hat{y}_1(\cdot)$, $y_2(\cdot)$ note that $z_1(t) \geq 0$ but $z_2(t) = -t + y_2(t) + R_{21}\hat{y}_1(t) < 0$.

So with $\hat{y}_1(\cdot)$ fixed, $y_2(\cdot)$ cannot be feasible unless $y_2(t) \geq \lambda_2 t, \forall t$. In an entirely analogous manner with $\hat{y}_2(\cdot)$ fixed, $y_1(\cdot)$ cannot be feasible unless $y_1(t) \geq \lambda_1 t$ for all t. Therefore it follows that for any λ_1, λ_2 as above $(\hat{u}_1(\cdot), \hat{u}_2(\cdot)) \equiv (\lambda_1, \lambda_2)$ gives a Nash equilibrium for each $t \geq 0$. So even Nash equilibrium serving for all t need not be unique. Next note that $(\lambda_1, \lambda_2) = (0, 1)$ as well as $(\lambda_1, \lambda_2) = (\frac{1}{R_{21}}, 0)$ give feasible controls (in fact both are Nash equilibria). So the only possible candidate for utopian equilibrium is $(0, 0)$. But $(0, 0)$ cannot be a feasible control. Hence there is no utopian equilibrium even for a single $t > 0$.

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