A characterization of finite symplectic polar spaces of odd prime order

Binod Kumar Sahoo[1]
and
N. S. Narasimha Sastry

Statistics and Mathematics Unit
Indian Statistical Institute
8th Mile, Mysore Road
R. V. College Post
Bangalore-560059, India

E-mails: binodkumar@gmail.com, nsastry@isibang.ac.in

Abstract. A sufficient condition for the representation group for a nonabelian representation (Definition 1.1) of a finite partial linear space to be a finite p-group is given (Theorem 2.9). We characterize finite symplectic polar spaces of rank r at least two and of odd prime order p as the only finite polar spaces of rank at least two and of prime order admitting nonabelian representations. The representation group of such a polar space is an extraspecial p-group of order p^{1+2r} and of exponent p (Theorems 1.5 and 1.6).

1. Introduction

A *point-line geometry* is a pair $S = (P, L)$ consisting of a nonempty 'point-set' P and a nonempty 'line-set' L of subsets of P of size at least 2. S is a *partial linear space* if any two distinct points x and y are contained in at most one line. Such a line, if it exists, is written as xy, x and y are said to be *collinear* and written as $x \sim y$. If x and y are not collinear we write $x \nsim y$. The graph with vertex set P, two distinct points being *adjacent* if they are collinear in S, is the *collinearity graph* $\Gamma(P)$ of S. We write $d(x, y)$ to denote the distance between two vertices x and y in $\Gamma(P)$. For $x \in P$ and $A \subseteq P$, we define $x^{\perp} = \{x\} \cup \{y \in P : x \sim y\}$ and $A^{\perp} = \bigcap_{x \in A} x^{\perp}$. S is *nondegenerate* if $P^{\perp}$ is empty. A subset of P is a *subspace* of S if any line containing at least two of its points is contained in it. The empty set, singletons, the lines and P are all subspaces of S. For a subset X of P the *subspace* $\langle X \rangle$ *generated* by X is the intersection of all subspaces of S containing X. A subspace is *singular* if each pair of its distinct points is collinear. A *geometric hyperplane* of S is a subspace of S different from P, that meets every line nontrivially.

1.1. Representations of partial linear spaces.

Let p be a prime. Let $S = (P, L)$ be a partial linear space of order p, that is, each line has $p+1$ points. (Note that, usually, order of a generalized polygon means something else, see [20], Section 1.3, p. 387).

Definition 1.1. *(Ivanov [12], p. 305) A representation of S is a pair (R, ψ), where R is a group and ψ is a mapping from the set of points of S into the set of subgroups of order p in R, such that the following hold:*

(i) R is generated by the subgroups $\psi(x), x \in P$.

(ii) For each line $l \in L$, the subgroups $\psi(x)$, $x \in l$, are pairwise distinct and generate an elementary abelian p-subgroup of order p^2.

The group R is then called the *representation group*. The representation (R, ψ) is *faithful* if ψ is injective. For each $x \in P$, we fix a generator r_x of $\psi(x)$ and denote by R_ψ the union of the subgroups $\langle r_x \rangle, x \in P$. A representation (R, ψ) of S is *abelian* or *nonabelian* according as R is abelian or not. Unlike here, 'nonabelian representation' in [12] means that 'the representation group is not necessarily abelian'. A representation (R_1, ψ_1) of S is a *cover* of the representation (R_2, ψ_2) of S if there exist an automorphism β of S and a group homomorphism $\varphi : R_1 \longrightarrow R_2$ such that $\psi_2(\beta(x)) = \varphi(\psi_1(x))$ for every $x \in P$. Further, if φ is an isomorphism then the two representations (R_1, ψ_1) and (R_2, ψ_2) are *equivalent*.

We now indicate various possibilities for the representation group. Embeddings of partial linear spaces (like projective spaces, polar spaces, generalized polygons, etc.) of order p in projective spaces over the field F_p of order p are all examples of abelian representations. The representation group is the corresponding vector space considered as an abelian group. Every representation of a projective space is faithful (by Definition 1.1(ii)) and the representation group of a finite projective space of dimension m over F_p is an elementary abelian group of order p^{m+1}. However, a representation of a generalized quadrangle need not be faithful. For example, let $S = (P, L)$ be a $(2, 1)$-generalized quadrangle, let P_1, P_2, P_3 be three triads partitioning P and let $R = \{1, r_1, r_2, r_3\}$ be the Klein four group. Define $\psi : P \longrightarrow R$ by $\psi(x) = \langle r_i \rangle$ if $x \in P_i$. Then (R, ψ) is an abelian representation which is not faithful.

Root group geometries are some examples of nonabelian representations of partial linear spaces. Let H be a finite simple group of Lie type defined over F_p. Let $\mathcal{G} = (P, L)$ be the root group geometry of H. That is, the 'point set' P is the collection of all (long) root subgroups of H. Two distinct root subgroups $x, y \in P$ are *collinear* if they generate an elementary abelian subgroup of order p^2 and each subgroup of order p in it is a member of P. The 'line' xy is the set of $p + 1$ subgroups of order p in $\langle x, y\rangle$. The identity map defines a representation of $\mathcal{G}$ in H and H is a representation group of $\mathcal{G}$. Note that if H is of type E_6, E_7 or E_8, then $\mathcal{G}$ is a parapolar space (see [4], p. 75); if it is of type G_2 or ${}^3D_4$, then $\mathcal{G}$ is a generalized hexagon with parameters (p, p) and (p, p^3) respectively (see ([6], p. 322 and 328) for p odd and ([7], Lemma 2.2, p. 2) for $p = 2$); if it is type F_4 or ${}^2E_6$, then $\mathcal{G}$ is a metasymplectic space (see Section 4, [6]); and if it is of type ${}^2F_4$, then $\mathcal{G}$ is a (2,8)-generalized octagon (see [19]). For a discussion of root group geometries including the classical ones, see [5] and [10], Chapter 4.

The following example shows that the representation group for a nonabelian representation of a finite partial linear space could be infinite.

Example 1.2. *Let $S = (P, L)$ be a $(2, 2)$-generalized hexagon. Then S is isomorphic to $H(2)$ (the one admitting an embedding in $O_7(2)$) or its dual $H(2)^*$ (see [20], Theorem 4, p. 402). For each $x \in P$, $H(x) = \{y \in P : d(x, y) < 3\}$ is a geometric hyperplane of S. The subgraph of $\Gamma(P)$ induced on the complement of $H(x)$ in P is connected if $S \simeq H(2)$ and has two components if $S \simeq H(2)^*$ (see [9], section 3). By ([12], Lemma 3.6, p. 310), $H(2)^*$ admits a nonabelian representation whose representation group is infinite. In fact, this representation is the cover of all other representations of $H(2)^*$.*

Our basic tool in this paper (Theorem 2.9) in fact is a sufficient condition on S and on the nonabelian representation of S to ensure that the representation group is a finite p-group.

We refrain from listing several natural questions that suggest themselves regarding the representations and the possible representation groups of finite partial linear spaces. For more on nonabelian representations, see [12].

1.2. **Polar spaces.** A *polar space* [2] here is a nondegenerate point-line geometry $S = (P, L)$ with at least three points per line satisfying the *'one or all'* axiom:

For each point-line pair (x, l), $x \notin l$, x is collinear with one or all points of l.

(see [2], Theorem 4, p. 161 and [22], 7.1, p. 102). *Rank* of S is the supremum of the lengths m of chains $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_m$ of singular subspaces in S. Since L is nonempty, the rank of S is at least two, but could be infinite. A remarkable discovery of Buekenhout and Shult is that a polar space is a partial linear space ([2], Theorem 3, p. 161). A polar space of rank 2 is a *generalized quadrangle* (GQ, for short). That is, it is a nondegenerate partial linear space such that:

Whenever $x \in P, l \in L$ with $x \notin l$, x is collinear with exactly one point of l.

If a finite GQ has a line with at least three points and a point on at least three lines then there exist integers s and t such that each line contains $s+1$ points and each point is on $t+1$ lines ([3], Theorem 7.1, p. 98). In that case we say that it is a (s,t)-GQ.

Building on the work of Veldkamp, Tits classified polar spaces whose rank is finite and at least three [22]. (For polar spaces of possibly infinite rank, see [14].) This implies that a finite polar space of rank $r \geq 3$ and of order p is isomorphic to either the symplectic polar space $W_{2r}(p)$ or one of the orthogonal polar spaces $Q^+_{2r}(p)$, $Q_{2r+1}(p)$ and $Q^-_{2r+2}(p)$. For notation see ([21], p. 329). If $r=2$ the above yield (p,p)-,$(p,1)$-,(p,p)- and (p,p^2)-GQs respectively. We note the number of points of these polar spaces ([21], Theorem 1, p. 330):

$$\begin{aligned}
|W_{2r}(p)| &= (p^{2r}-1)/(p-1); \\
|Q^+_{2r}(p)| &= (p^{r-1}+1)(p^r-1)/(p-1); \\
|Q_{2r+1}(p)| &= (p^{2r}-1)/(p-1); \\
|Q^-_{2r+2}(p)| &= (p^r-1)(p^{r+1}+1)/(p-1).
\end{aligned}$$

The following inductive property of these spaces is important for us (see [3], section 6.4, p. 90).

Lemma 1.3. *Let S be one of the above polar spaces of finite rank $r \geq 3$ and let x,y be two noncollinear points. Then $\{x,y\}^\perp$ is a polar space of rank $r-1$ and is of the same type as S.*

Finite GQs are classified only for $s = 2,3$ (see [20], 5.1, p. 401). See [16] for several examples of finite GQs. In [15], Kantor studied finite (p,t)-GQs S with $t \geq 2$ admitting a rank 3 automorphism group G on points and proved that one of the following holds: (i) $t = p^2-p-1$ and $p^3 \nmid |G|$; (ii) $G \cong PSp(4,p)$ or $P\Gamma U(4,p)$ and S is one of the natural GQs associated with these groups; (iii) $p=2$, $G = Alt(6)$ and S is the GQ associated with $PSp(4,2)$ ([15], Theorem 1.1). This paper started with a search for new finite (p,t)-GQs embedded in groups and resulted in a characterization of finite symplectic polar spaces $W_{2r}(p)$ of rank $r \geq 2$ for odd primes p (Theorems 1.5 and 1.6).

1.3. **Extraspecial p-groups and Hall-commutator formula.** A finite p-group G is *extraspecial* if its Frattini subgroup $\Phi(G)$, the commutator subgroup G' and the center $Z(G)$ coincide and have order p. An extraspecial p-group is of order p^{1+2m} for some integer $m \geq 1$, has exponent at most p^2 if p is odd and 4 if $p=2$, and the maximum of the orders of its abelian subgroups is p^{m+1} (see [8], section 20, p. 78,79). We denote by p^{1+2m}_+ an extraspecial p-group of order p^{1+2m} if its exponent is p when p is odd and the abelian subgroups of order p^{m+1} are elementary abelian when $p=2$. Note that p^{1+2}_+ is isomorphic to the group of 3×3 upper triangular matrices with entries from F_p and 1 on the diagonal. For more on extraspecial p-groups, see ([11], section 3, p. 127 and Appendix 1, p. 141).

For elements g_1, g_2 in a group, we write $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ and $g_1^{g_2} = g_2^{-1} g_1 g_2$. We repeatedly use the following Hall's commutator formula ([8], 7.2, p. 22), mostly without mention.

Lemma 1.4. *Let G be a group. Then for $g_1, g_2, g_3 \in G$,*

(*i*) $[g_1 g_2, g_3] = [g_1, g_3]^{g_2} [g_2, g_3]$*;*
(*ii*) $[g_1, g_2 g_3] = [g_1, g_3][g_1, g_2]^{g_3}$.

1.4. **Statement of main results.** In this paper we prove:

Theorem 1.5. *Let $S = (P, L)$ be a finite polar space of rank $r \geq 2$ and of prime order p. If S admits a nonabelian representation (R, ψ) then:*

(*i*) *p is odd;*
(*ii*) $R = p_+^{1+2r}$*;*
(*iii*) *S is isomorphic to $W_{2r}(p)$.*

Theorem 1.6. *$W_{2r}(p)$, $r \geq 2$, admits a nonabelian representation. Any two such representations are equivalent.*

In Section 2 we prove a sufficient condition for a nonabelian representation group to be a p-group (Theorem 2.9) which is crucial here and also in [18]. In Section 3 we prove Theorem 1.5(i) and that $R \simeq p_+^{1+2m}$ for some $m \geq 1$. In Section 4 we prove Theorem 1.5 when the rank is two. Finally, in Section 5 we prove Theorem 1.5 for the general rank and Theorem 1.6.

2. Initial Results

Let $S = (P, L)$ be a partial linear space. We assume that $\Gamma(P)$ is connected and that with each $x \in P$ is associated a geometric hyperplane $H(x)$ in S containing x. Consider the following conditions on S:

($C1$) If $y \in H(x)$ then $x \in H(y)$.
($C2$) The subgraph $\Gamma(H'(x))$ of $\Gamma(P)$ induced on the complement $H'(x)$ of $H(x)$ in P is connected.
($C3$) If $y \in H'(x)$ then there exist lines l_1 and l_2 containing x and y respectively such that for each $w \in l_1$, $H(w)$ intersects l_2 at exactly one point. Further, this correspondence is a bijection from l_1 to l_2.
($C4$) The graph $\Sigma(P)$ with vertex set P in which two points x and y are adjacent if $y \in H'(x)$ is connected.

Example 2.1. *Let $S = (P, L)$ be a polar space of rank $r \geq 2$. Then $\Gamma(P)$ is connected. For each $x \in P$, associate the geometric hyperplane $x^\perp$ of S. Then ($C1$), $\cdots$, ($C4$) hold.*

Example 2.2. *Let $S = (P, L)$ be a near $2n$-gon, $n \geq 2$, admitting quads (see [1]). We assume that each line of S contains at least three points. By definition, $\Gamma(P)$ is connected. For each $x \in P$, associate the geometric hyperplane $H(x) = \{y \in P : d(x, y) < n\}$ of S. Clearly $(C1)$ holds. The second corollary to ([1], Theorem 3, p. 155) implies that $(C2)$ holds. Now, ([1], Theorem 2, p. 151) implies that if $d(x, y) = n$, $x, y \in P$ and l_1 is any line containing x, then there exists a line l_2 containing y such that $(C3)$ holds. This also implies that if $u \sim v$, $u, v \in P$, then there exists $w \in P$ such that $d(u, w) = d(v, w) = n$. So u, w, v is a path in $\Sigma(P)$. Then connectedness of $\Sigma(P)$ follows from that of $\Gamma(P)$. Thus $C(4)$ holds.*

We study nonabelian representations of finite polar spaces of order p here (Theorems 1.5 and 1.6) and that of near hexagons of order two and admitting quads in [18].

Remark 2.3. *If $S = (P, L)$ is a generalized $2n$-gon and $H(x), x \in P$, is as in Example 2.2, then $(C2)$ need not hold, see Example 1.2.*

Let (R, ψ) be a representation of S. For $x, y \in P$, define $u_{xy} = [r_x, r_y]$. *Throughout this section we assume that*

$$u_{xy} = 1 \text{ whenever } x \in P \text{ and } y \in H(x).$$

Proposition 2.4. *Assume that $(C1)$ and $(C2)$ hold in S. Then the following hold:*

(*i*) *If $u_{vw} = 1$ for $v, w \in P$ with $v \in H'(w)$, then $r_w \in Z(R)$.*
(*ii*) *If $a \in P$ and $r_a \in Z(R)$, then $r_c \in Z(R)$ for every $c \sim a$.*

Proof. (*i*) Let $y \in H'(w)$, $y \sim v$ and $vy \cap H(w) = \{x\}$. Then $u_{wy} = 1$ because $x \notin \{v, y\}$ and $u_{wx} = u_{vw} = 1$. Now, connectedness of $\Gamma(H'(w))$ implies that $u_{wz} = 1$ for every $z \in H'(w)$. Since $u_{wz} = 1$ for $z \in H(w)$ also, $r_w \in Z(R)$.

(*ii*) By definition, $H(a) \subsetneq P$. Let $b \in H'(a)$. By $(C1)$, $a \in H'(b)$. By (*i*), $r_b \in Z(R)$ because $u_{ab} = 1$. Now, $ac \cap H(b)$ is a singleton. Since each line contains at least 3 points, there exists a point z in $ac \cap H'(b)$ different from a. Now, $b \in H'(z)$ by $(C1)$ and $u_{bz} = 1$. So, $r_z \in Z(R)$ by (*i*) again. So the subgroup generated by $\psi(ac)$ is contained in $Z(R)$ and $r_c \in Z(R)$. □

Corollary 2.5. *Assume that $(C1)$ and $(C2)$ hold in S. If R is nonabelian then the following hold:*

(*i*) *$u_{xy} \neq 1$ whenever $x, y \in P$ and $y \in H'(x)$.*
(*ii*) *$R_\psi \cap Z(R) = \{1\}$.*
(*iii*) *If $x \sim y$ then $y \in H(x)$.*
(*iv*) *If $H(x) \neq H(y)$ for each pair of noncollinear points x and y, then ψ is faithful.*

Proof. (*i*) follows from Proposition 2.4 and the connectedness of $\Gamma(P)$. (*ii*) and (*iii*) follow from (*i*). We now prove (*iv*). Suppose that $\langle r_x \rangle = \langle r_y \rangle$ for distinct x, y in P. Then $x \nsim y$ by Definition 1.1(*ii*). By (*i*), $u \in H(x)$ if and only if $u \in H(y)$. So $H(x) = H(y)$, a contradiction. □

Proposition 2.6. *Assume that* $(C3)$ *holds in* S. *Then for* $x, y \in P$, $[u_{xy}, r_x] = [u_{xy}, r_y] = 1$. *If* $u_{xy} \neq 1$ *then* u_{xy} *is of order* p *and* $\langle r_x, r_y \rangle = p_+^{1+2}$.

Proof. Let $x \in P$, $y \in H'(x)$ and l_1, l_2 be lines as in $(C3)$. Let x, a, u be three pairwise distinct points in l_1 and y, b, v be points in l_2 such that $y \in H(a)$, $b \in H(x)$ and $v \in H(u)$. By $(C3)$, y, b, v are pairwise distinct. Write $r_x = r_a^i r_u^j$, $r_y = r_v^k r_b^m$ for some i, j, k, m, $(1 \leq i, j, k, m \leq p-1)$. Now,

$$u_{xy} = [r_a^i r_u^j, r_y] = [r_u^j, r_y] = [r_u^j, r_v^k r_b^m] = [r_u^j, r_b^m] = [r_x r_a^{-i}, r_b^m] = [r_a^{-i}, r_b^m].$$

Since $[r_a^{-i}, r_b^m] = [r_b^m, r_a^i]^{r_a^{-i}}$,

$$\begin{aligned} u_{xy} &= [r_b^m, r_a^i]^{r_a^{-i}} = [r_y r_v^{-k}, r_a^i]^{r_a^{-i}} = [r_v^{-k}, r_a^i]^{r_a^{-i}} = [r_v^{-k}, r_u^{-j} r_x]^{r_a^{-i}} \\ &= [r_v^{-k}, r_x]^{r_a^{-i}} = [r_b^m r_y^{-1}, r_x]^{r_a^{-i}} = [r_y^{-1}, r_x]^{r_a^{-i}} = [r_y^{-1}, r_x]. \end{aligned}$$

So $u_{xy} r_y^{-1} = r_x^{-1} r_y^{-1} r_x = r_y^{-1} \left[r_y^{-1}, r_x\right] = r_y^{-1} u_{xy}$. Thus $[u_{xy}, r_y] = 1$. Similarly, $u_{yx} = \left[r_x^{-1}, r_y\right]$. This, together with $\left[r_y, r_x^{-1}\right] = \left[r_x^{-1}, r_y\right]^{-1} = u_{yx}^{-1} = u_{xy}$ implies that $[u_{xy}, r_x] = 1$. Now, $\left[r_x^i, r_y\right] = [r_x, r_y]^i = u_{xy}^i$ for all $i \geq 0$. So $u_{xy}^p = 1$ and $\langle r_x, r_y \rangle = p_+^{1+2}$. □

Proposition 2.7. *Assume that* $(C1), \cdots, (C4)$ *hold in* S. *Then* $R' \leq Z(R)$ *and* $|R'| \leq p$.

Proof. For $x, y \in P$, let $U_{xy} = \langle u_{xy} \rangle$. Let a, b be adjacent in $\Gamma(H'(x))$ and $ab \cap H(x) = \{c\}$. Now $r_b = r_a^i r_c^j$ for some i, j, $1 \leq i, j \leq p-1$. Since $[r_x, r_c] = 1$, we have

$$u_{xb} = [r_x, r_b] = \left[r_x, r_a^i r_c^j\right] = \left[r_x, r_a^i\right] = [r_x, r_a]^i = u_{xa}^i.$$

So $U_{xb} = U_{xa}$. This, together with $(C2)$, implies that U_{xy} is independent of the choice of y in $H'(x)$. Since $u_{xy} = u_{yx}^{-1}$, we have $U_{xy} = U_{yx}$. So, if $x, y \in P$ with $y \in H'(x)$, then $U_{xy} = U_{yx}$. Now, by $(C4)$, U_{xy} is independent of the edge $\{x, y\}$ in $\Sigma(P)$. We denote this common subgroup by U.

We now show that $U \leq Z(R)$. Let $x \in P$ and $y \in H'(x)$. We show that $[u_{xy}, r_z] = 1$ for each $z \in P$. We may assume that $z \in H'(x) \cup H'(y)$. In this case it is clear from Proposition 2.6 because $U_{xy} = U_{xz}$ if $z \in H'(x)$. Similarly, if $z \in H'(y)$.

Now, since $R = \langle r_x : x \in P \rangle$, $u_{xy} \in Z(R)$ and $u_{xy} = 1$ if $y \in H(x)$, it follows that $R' = \langle u_{xy} : x \in P, y \in H'(x) \rangle = U$ and is of order at most p (Proposition 2.6). □

Proposition 2.8. *Assume that* $(C1), \cdots, (C4)$ *hold in* S. *If* R *is nonabelian then exponent of* R *is* p *or* 4 *according as* p *is odd or* $p = 2$. *In particular, if* P *is finite then* R *is finite and* $\Phi(R) = R'$.

Proof. Let $r = r_1 r_2 \cdots r_n \in R$, $r_i \in R_\psi$. We use induction on n. Let $r = h r_n$, where $h = r_1 r_2 \cdots r_{n-1}$. Since $R' \subseteq Z(R)$, $r_n^i h = h r_n^i \left[r_n^i, h\right] = h r_n^i [r_n, h]^i$. So $r^{i+1} = h^{i+1} r_n^{i+1} [r_n, h]^{1+2+\cdots+i}$ for all $i \geq 0$. Now, the result follows because by induction $h^p = 1$ if p is odd and $h^4 = 1$ if $p = 2$. Note that if $p = 2$, exponent of R can not be 2 as R is nonabelian.

Now, if P is finite then R/R' and so R are finite and $\Phi(R) = R' \langle r^p : r \in R \rangle = R'$. For $p = 2$, the last equality holds because $r^2 \in R'$ for every $r \in R$. □

We now summarize the above results.

Theorem 2.9. *Let $S = (P, L)$ be a connected partial linear space of prime order p. Suppose that for each $x \in P$ there is associated a geometric hyperplane $H(x)$ containing x such that $(C1), \cdots, (C4)$ hold. Let (R, ψ) be a nonabelian representation of S such that $[\psi(x), \psi(y)] = 1$ for all $x, y \in P$ with $y \in H(x)$. Then the following hold:*

- *(i) If $x, y \in P$ with $y \in H'(x)$, then $[\psi(x), \psi(y)] \neq 1$ and $\langle \psi(x), \psi(y) \rangle = p_+^{1+2}$;*
- *(ii) $|R'| = p$, $R' \subseteq Z(R)$, R is a p-group, and exponent of R is p or 4 according as p is odd or $p = 2$.*

Further, $R_\psi \cap Z(R) = \{1\}$; ψ is faithful if $H(x) \neq H(y)$ whenever $x \nsim y$; and R is finite with $R' = \Phi(R)$ if P is finite.

Remark 2.10. *For $p = 2$, Theorem 2.9(ii) is a consequence of ([12], Lemma 3.5, p. 310) where Ivanov did not assume $(C3)$. Our proof of Proposition 2.7 is similar to that of ([13], Lemma 2.2, p. 526).*

Corollary 2.11. *Let S and (R, ψ) be as in Theorem 2.9. If P is finite then (R, ψ) is the cover of a representation (R_1, ψ_1) of S where R_1 is extraspecial or $p = 2$ and $Z(R_1)$ is cyclic of order 4.*

Proof. If $Z(R)$ is elementary abelian (this is the case if p is odd), write $Z(R) = R'T$, $R' \cap T = \{1\}$ for some subgroup T of $Z(R)$. Let $R_1 = R/T$. Then R_1 is extra special. Define ψ_1 from P to R_1 by $\psi_1(x) = \langle r_x T \rangle$, $x \in P$. Since $r_x \notin Z(R)$, $\langle r_x T \rangle$ is a subgroup of R_1 of order p for each $x \in P$. Then (R_1, ψ_1) is a nonabelian representation of S and (R, ψ) is a cover of (R_1, ψ_1).

If $Z(R)$ is not elementary abelian, then $p = 2$. Write $Z(R) = \langle a \rangle K$, $\langle a \rangle \cap K = \{1\}$ where $K \leq Z(R)$ and a is of order 4. Since $r^2 \in R'$ for every $r \in R$, it follows that $R' = \langle a^2 \rangle$. Now taking $R_1 = R/K$, the above argument completes the proof. □

3. Nonabelian Representation Group of a Polar Space

If a polar space of rank $r \geq 2$ and of order p admits a faithful abelian representation then the polar space is necessarily classical (for rank 2 case, see [17], 4.4.8, p. 76) and the representation is, up to a projective linear transformation, a standard one. The following proposition shows that a polar space of finite rank and of order p admits a nonabelian representation only if p is odd. For any representation (R, ψ) of S, Definition 1.1(ii) implies that $[r_x, r_y] = 1$ if $y \in x^\perp$. By Example 2.1, all the results of the previous section hold.

Proposition 3.1. *Let $S = (P, L)$ be a polar space of finite rank $r \geq 2$ and of order three. Then every representation of S is abelian.*

Proof. Let (R, ψ) be a representation of S. By Lemma 1.3, there exists a chain of subspaces $Q_0 = P \supsetneq Q_1 \supsetneq Q_2 \supsetneq \cdots \supsetneq Q_{r-2}$ such that Q_i is a polar space of rank $r - i$. Thus Q_{r-2} is a

$(2,t)$-GQ. Let $x, y \in Q_{r-2}$, $x \nsim y$, and T be a $(2,1)$-GQ in Q_{r-2} containing x and y. Such a T exists because each line has 3 points. Let $\{x,y\}^{\perp} = \{a,b\}$ in T. For $u \sim v$, we define $u * v \in P$ by $uv = \{u, v, u*v\}$. In T, since $[r_b, r_y] = [r_b, r_x] = 1$ and $r_{(a*x)*(b*y)} = r_{(a*y)*(b*x)}$, it follows that $r_x r_y = r_y r_x$. Now, Corollary 2.5(i) completes the proof. □

For the rest of this paper we assume that p *is an odd prime.*

Let $S = (P, L)$ be a polar space of finite rank $r \geq 2$ and of order p and (R, ψ) be a nonabelian representation of S. Note that if $r \geq 3$, then finiteness of P and that of r are equivalent. However, if S is a GQ with $s+1$ points per line, then finiteness of P is not known except when $s = 2, 3, 4$ (see [3], p.100). The rest of this section is devoted to prove that R is extraspecial if P is finite.

Lemma 3.2. *ψ is faithful and $[r_x, r_y] \neq 1$ if $x \nsim y$.*

Proof. This follows from Corollary 2.5(i) and (iv). □

Given a line l and two distinct points a and b on it, we write

$$\psi(l) = \left\{ \langle r_a \rangle, \langle r_b \rangle, \langle r_a r_b \rangle, \langle r_a^2 r_b \rangle, \cdots, \langle r_a^{p-1} r_b \rangle \right\}.$$

Let $x, y \in P$, $x \nsim y$ and $u, v \in \{x,y\}^{\perp}$, $u \nsim v$. Then $[r_x, r_y] \neq 1$ and $[r_u, r_v] \neq 1$. Let $l_0 = xu$, $l_1 = vy$, $m_0 = xv$ and $m_1 = uy$. Consider the lines l_0 and l_1. By 'one or all' axiom, each point of l_0 is collinear with exactly one point of l_1 and vice-versa. Let $l_0 = \{x, u, x_1, x_2, \cdots, x_{p-1}\}$ and $\langle r_{x_i} \rangle = \langle r_x^i r_u \rangle$ for $1 \leq i \leq p-1$. Let $x_i \sim v_i$ in l_1. Then $l_1 = \{v, y, v_1, v_2, \cdots, v_{p-1}\}$. Replacing the generator r_v by r_v^j for some j $(2 \leq j \leq p-1)$, if necessary, we may assume that $\langle r_{v_1} \rangle = \langle r_v r_y \rangle$. So $[r_x r_u, r_v r_y] = 1$. Then $\left[r_x^i r_u, r_v^i r_y\right] = 1$ for all $i \geq 0$ because $R' \subseteq Z(R)$. By Lemma 3.2, $\left[r_x^i r_u, r_v^j r_y\right] \neq 1$ if $i \neq j$. So $\langle r_{v_i} \rangle = \langle r_v^i r_y \rangle$. Let m_{i+1} be the line such that $\psi(m_{i+1}) = \langle r_x^i r_u, r_v^i r_y \rangle$, $1 \leq i \leq p-1$.

Let $z \in m_i \setminus (l_0 \cup l_1)$ and $w \in m_j \setminus (l_0 \cup l_1)$ for $i \neq j$, $0 \leq i, j \leq p$. If $i = 0$, then $\langle r_z \rangle = \langle r_x^{k_1} r_v \rangle$ and if $i > 0$ then $\langle r_z \rangle = \left\langle \left(r_x^{i-1} r_u\right)^{k_1} \left(r_v^{i-1} r_y\right) \right\rangle$ for some k_1, $1 \leq k_1 \leq p-1$. Similarly, $\langle r_w \rangle = \langle r_x^{k_2} r_v \rangle$ or $\left\langle \left(r_x^{j-1} r_u\right)^{k_2} \left(r_v^{j-1} r_y\right) \right\rangle$ for some k_2, $1 \leq k_2 \leq p-1$, according as $j = 0$ or $j > 0$. Now, from $R' \subseteq Z(R)$, the identity $[r_x, r_y] = [r_v, r_u]$ (a consequence of $[r_x r_u, r_v r_y] = 1$) and the fact that each point of m_i is collinear with exactly one point of m_j for $i \neq j$ (a consequence of 'one or all' axiom), the following lemma is straight forward.

Lemma 3.3. *$z \sim w$ if and only if $k_1 + k_2 = p$.*

Proposition 3.4. *If $a, d \in R_\psi$ then $ad\,[a,d]^{(p-1)/2} \in R_\psi$.*

Proof. Let $a, d \in R_\psi - \{1\}$. Let $x_1, x_2 \in P$ be such that $\langle r_{x_1} \rangle = \langle a \rangle$ and $\langle r_{x_2} \rangle = \langle d \rangle$. We may assume that $x_1 \nsim x_2$. Then $[a,d] \neq 1$ by Lemma 3.2. We show that $\langle ad\,[a,d]^{(p-1)/2} \rangle$ is the image of some element of P. Let $y_1, y_2 \in \{x_1, x_2\}^{\perp}$ be such that $y_1 \nsim y_2$, $\langle r_{y_1} \rangle = \langle b \rangle$ and $\langle r_{y_2} \rangle = \langle c \rangle$. Consider the lines $l_0 = x_1y_1$ and $l_1 = x_2y_2$. Let $z_1 \in l_0$ be such that $\langle r_{z_1}\rangle = \langle ab\rangle$ and let $z_1 \sim z_2 \in l_1$. Replacing the generator c by c^j for some j, if necessary, we may assume that $\langle r_{z_2}\rangle = \langle cd\rangle$. Let $m_0 = x_1y_2$ and $m_1 = z_1z_2$. Let $u \in m_0$ be such that $\langle r_u\rangle = \langle a^{(p-1)/2}c\rangle$. Then $x_1 \neq u \neq y_2$. Let $u \sim v$ in m_1. By Lemma 3.3, $\langle r_v\rangle = \left\langle (ab)^{(p+1)/2}(cd)\right\rangle$. If $y_1 \sim w$ in the line uv, then $\langle r_w\rangle = \left\langle \left(a^{(p-1)/2}c\right)^k (ab)^{(p+1)/2}(cd)\right\rangle$ for some k $(1 \leq k \leq p-1)$. Now $\left[b, \left(a^{(p-1)/2}c\right)^k (ab)^{(p+1)/2}(cd)\right] = 1$. So, $[b,c]^{k+1} = 1$ and $k+1 = p$. The subgroup $\left\langle b^{(p-1)/2}\left(a^{(p-1)/2}c\right)^{p-1}(ab)^{(p+1)/2}(cd)\right\rangle$ is the image of some point of y_1w. But $b^{(p-1)/2}\left(a^{(p-1)/2}c\right)^{p-1}(ab)^{(p+1)/2}(cd) = ad\,[b,c]^{(p+1)/2} = ad\,[a,d]^{(p-1)/2}$. In the last equality we have used $[a,d] = [b,c]^{-1}$, a consequence of $[ab,cd] = 1$. Thus, $ad\,[a,d]^{(p-1)/2} \in R_\psi$. □

Proposition 3.5. *R_ψ is a complete set of coset representatives of R' in R.*

Proof. Let $r_1R' = r_2R'$ for some $r_1, r_2 \in R_\psi$. Since $R' \subseteq Z(R)$, r_1 and r_2 are both trivial or are both nontrivial (Corollary 2.5(ii)). Assume that the later holds and that $r_1 = r_2w$ for some $w \in R'$. Let $x_1, x_2 \in P$ be such that $\langle r_{x_1}\rangle = \langle r_1\rangle$ and $\langle r_{x_2}\rangle = \langle r_2\rangle$. Since $[r_1, r_2] = 1$, either $x_1 = x_2$ or $x_1 \sim x_2$ (Lemma 3.2). If $x_1 \sim x_2$ then $w \neq 1$ by Definition 1.1(ii) and $\langle w\rangle$ would be the image of some point in the line x_1x_2, a contradiction to Corollary 2.5(ii). So $x_1 = x_2$ and $r_1 = r_2^i$ for some i $(1 \leq i \leq p-1)$. Then $r_2^{i-1} = w \in R' \subseteq Z(R)$. Now, Corollary 2.5($ii$) implies that $i = 1$ and so $w = 1$ and $r_1 = r_2$.

Now, let $sR' \in R/R'$. Write $s = r_1r_2\cdots r_k$, $r_i \in R_\psi$. Let $R' = \langle z\rangle$. Since $R' \subseteq Z(R)$, there is some integer j such that $r_1r_2\cdots r_kz^j$ is an element, say r, of R_ψ by Proposition 3.4. Then $sR' = rR'$, completing the proof of the proposition. □

Proposition 3.6. *Assume that P is finite. Then $|R| = p(1 + (p-1)\,|P|)$ and $R = p_+^{1+2m}$ for some $m \geq 1$.*

Proof. Since $|R'| = p$ (Proposition 2.7), the first assertion follows from Proposition 3.5. Also, $R' = Z(R)$ because $R_\psi \cap Z(R) = \{1\}$ and $R' \subseteq Z(R)$. Now, Proposition 2.8 completes the proof. □

Corollary 3.7. *If S is a finite classical polar space of rank $r \geq 2$ admitting a nonabelian representation, then S is isomorphic to $W_{2m}(p)$ or $Q_{2m+1}(p)$.*

Proof. By Proposition 3.6, $|P| = (p^{2m}-1)/(p-1)$ for some $m > 0$. So the corollary follows from the number of points of classical polar spaces (see 1.2). □

By proposition 3.5, S admits a faithful abelian representation with representation group R/R'. Considering R/R' as a vector space over F_p, it has dimension $2m$. Since $Q_{2m+1}(p)$ does not possess faithful abelian $2m$-dimensional representation, the only possibility is that S is isomorphic to $W_{2m}(p)$. We thank the referee for this remark. In the next sections, we prove this fact giving a geometrical argument involving triads of points of a generalized quadrangle.

4. Rank 2 Case

Let $S = (P, L)$ be a finite (s,t)-GQ. A *triad of points* in S is a triple T of pairwise noncollinear points. An element of $T^{\perp}$ is a *center* of T. A pair of distinct points $\{x, y\}$ in S is *regular* if $x \sim y$ or if $x \nsim y$ and $\left|\{x,y\}^{\perp\perp}\right| = t+1$. A point x is *regular* if $\{x, y\}$ is regular for each $y \in P \setminus \{x\}$. The pair $\{x, y\}$, $x \nsim y$, is *antiregular* if $|z^{\perp} \cap \{x,y\}^{\perp}| \leq 2$ for each $z \in P \setminus \{x, y\}$. A point x is *antiregular* if $\{x, y\}$ is antiregular for each $y \in P \setminus x^{\perp}$. Dually, we define a triad of lines, center of a triad of lines, regularity and antiregularity of a line.

Proposition 4.1. *Let $S = (P, L)$ be a (p, t)-GQ. If S admits a triad of lines with at least 3 centers then every representation of S is abelian.*

Proof. Let $\{l_1, l_2, l_3\}$ be a triad of lines in S with centers m_1, m_2, m_3. Let $\{x_{ij}\} = l_i \cap m_j, 1 \leq i,j \leq 3$. Consider the lines l_1 and l_2. Replacing $r_{x_{11}}$ by $r_{x_{11}}^k$ for some k, if necessary, we may assume that the point a of l_1 with $\langle r_a \rangle = \langle r_{x_{11}} r_{x_{12}} \rangle$ is collinear with the point b with $\langle r_b \rangle = \langle r_{x_{21}} r_{x_{22}} \rangle$. So $[r_{x_{11}} r_{x_{12}}, r_{x_{21}} r_{x_{22}}] = 1$. Then $\left[r_{x_{11}}^i r_{x_{12}}, r_{x_{21}}^i r_{x_{22}}\right] = 1$ for $0 \leq i \leq p-1$. Let $\langle r_{x_{13}} \rangle = \left\langle r_{x_{11}}^i r_{x_{12}} \right\rangle$ and $\langle r_{x_{23}} \rangle = \left\langle r_{x_{21}}^j r_{x_{22}} \right\rangle$ for some i, j, $1 \leq i, j \leq p-1$. If $i \neq j$ then R is abelian (Corollary 2.5(i)). So assume that $i = j$. Let $\langle r_{x_{31}} \rangle = \left\langle r_{x_{11}}^k r_{x_{21}} \right\rangle$ and $\langle r_{x_{33}} \rangle = \left\langle \left(r_{x_{11}}^i r_{x_{12}}\right)^n \left(r_{x_{21}}^i r_{x_{22}}\right) \right\rangle$ for some k, n, $1 \leq k, n \leq p-1$. If $n \neq p-k$, then R is abelian by Lemma 3.3. So, we assume that $\langle r_{x_{33}} \rangle = \left\langle \left(r_{x_{11}}^i r_{x_{12}}\right)^{p-k} \left(r_{x_{21}}^i r_{x_{22}}\right) \right\rangle$. By a similar argument, we assume that $\langle r_{x_{32}} \rangle = \left\langle r_{x_{21}}^{p-k} r_{x_{22}} \right\rangle$. Now, Lemma 3.3 implies that R is abelian because $x_{32} \sim x_{33}$ and $p - k \neq p - (p-k)$. □

Corollary 4.2. *If S admits a nonabelian representation then every line of S is antiregular and no line of S is regular.*

Proposition 4.3. *Let $S = (P, L)$ be a finite (p, t)-GQ. If S admits a nonabelian representation (R, ψ), then $t = p$ and $R = p_+^{1+4}$.*

Proof. We have $|P| = (p+1)(pt+1)$ ([17], 1.2.1, p. 2). So $|R| = p^2\left(t\left(p^2-1\right)+p\right)$ (Proposition 3.6). By Corollary 4.2, $t \geq 2$. So, $p^2\left(t\left(p^2-1\right)+p\right) \geq p^4$. Now, $|R| = p^{2m+1}$ for some integer $m \geq 1$. Thus,

$$t = p\left(p^{2(m-2)} + p^{2(m-3)} + \cdots + p^2 + 1\right).$$

Since $t \leq p^2$ ([17], 1.2.3, p. 3), $m = 2$, $t = p$ and $R = p_+^{1+4}$. □

In $Q_5(p)$ all lines are regular ([17], 3.3.1(i), p 51). So every representation of $Q_5(p)$ is abelian. On the other hand, since p is odd, $W_4(p)$ is not self-dual and is isomorphic to the dual of $Q_5(p)$ ([17], 3.2.1, p. 43). No point of $Q_5(p)$ is regular ([17], 1.5.2(i), p. 13), so no line of $W_4(p)$ is regular. Again, all points of $Q_5(p)$ are antiregular ([17], 3.3.1(i), p. 51), so all lines of $W_4(p)$ are antiregular. We prove

Proposition 4.4. *Let $S = (P, L)$ be a (p, p)-GQ. If S admits a nonabelian representation then S is isomorphic to $W_4(p)$.*

Proof. Since $W_4(p)$ is characterized by the regularity of each of its point ([17], 5.2.1, p. 77), it is enough to show that if $x, y \in P$ and $x \nsim y$ then $\{x, y\}^{\perp\perp}$ contains $\{a, b\}^{\perp}$ for distinct $a, b \in \{x, y\}^{\perp}$. Let (R, ψ) be a nonabelian representation of S. Let $z \in \{a, b\}^{\perp}$ and $w \in \{x, y\}^{\perp}$. We claim that $z \sim w$. Write $H = C_R(r_a) \cap C_R(r_b)$. Then

$$|H| = \frac{|C_R(r_a)|\,|C_R(r_b)|}{|C_R(r_a)\,C_R(r_b)|} = \frac{p^4 p^4}{p^5} = p^3.$$

Let $K = \langle r_x, r_y \rangle$. By Proposition 2.6, $|K| = p^3$. So $K = H$ because $K \leq H$. Then $[r_w, r_z] = 1$ because $[r_w, K] = 1$. So $z \sim w$ by Theorem 2.9(i). □

5. Proof of Theorems 1.5 and 1.6

Proof of Theorem 1.5. By Proposition 3.1, p is an odd prime. By Lemma 1.3 and Proposition 4.4, S is isomorphic to $W_{2r}(p)$. Proposition 3.6 implies that $R = p_+^{1+2r}$. This completes the proof of Theorem 1.5.

We prove Theorem 1.6 in Propositions 5.2 and 5.3. In view of Proposition 3.4, we first prove

Proposition 5.1. *Let $G = p_+^{1+2r}$. There exists a set T of coset representatives of $Z(G)$ in G such that if $t_1, t_2 \in T$ then $t_1 t_2 [t_1, t_2]^{(p-1)/2} \in T$. Further, T is unique up to conjugacy in G.*

Proof. Let $Z = Z(G) = \langle z \rangle$ and $V = G/Z$. We consider V as a vector space over F_p. The map $f : V \times V \longrightarrow F_p$ taking (xZ, yZ) to i, where $[x, y] = z^i$ $(0 \leq i \leq p-1)$, is a nondegenerate symplectic bilinear form on V. Write V as an orthogonal direct sum of r hyperbolic planes K_i $(1 \leq i \leq r)$ in V and let H_i be the inverse image of K_i in G. Then H_i is generated by 2 elements x_{i_1} and x_{i_2} such that $[x_{i_1}, x_{i_2}] = z$. Let $A_j = \langle x_{i_j}, 1 \leq i \leq r \rangle$, $j = 1, 2$. Then A_j is an elementary abelian p-subgroup of G of order p^r, $A_j \cap Z = \{1\}$ and $A_1 Z \cap A_2 Z = Z$. Set

$$T = \left\{ xy\,[x, y]^{\frac{p-1}{2}} : x \in A_1, y \in A_2 \right\}.$$

We show that T has the required property. Let $\alpha = xy\,[x, y]^{\frac{p-1}{2}}$, $\beta = uv\,[u, v]^{\frac{p-1}{2}}$ be elements of T where $x, u \in A_1$ and $y, v \in A_2$. If $\alpha Z = \beta Z$, then $u^{-1} x Z = y^{-1} v Z$ and is equal to Z because $A_1 Z \cap A_2 Z = Z$. So $x = u$ and $y = v$ because $A_j \cap Z = \{1\}$. Thus $\alpha Z = \beta Z$ if and only if $x = u, y = v$. So, $|T| = p^{2r}$ and T is a complete set of coset representatives. Since $G' = Z$, a routine calculation shows that $\alpha\beta\,[\alpha, \beta]^{(p-1)/2} = (xu)\,(yv)\,[xu, yv]^{(p-1)/2} \in T$. Thus, T has the stated property.

Now we prove the uniqueness part. In fact, we show that the group of inner automorphisms of G acts regularly on the set $\mathcal{X}$ of all sets of coset representatives of Z in G, each of which is closed under the binary operation $(t_1, t_2) \mapsto t_1 t_2 [t_1, t_2]^{(p-1)/2}$.

Fix an ordered basis $\{v_1 Z, \cdots, v_{2r} Z\}$ for V. Each $T \in \mathcal{X}$ is determined by the sequence $(x_1, \cdots, x_{2r})$, where $T \cap v_i Z = \{x_i\}$. In fact, if $aZ = x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} Z \in V$, where $i_1 < \cdots < i_n$ and $1 \leq j_k \leq p-1$, then $aZ \cap T = \{x_{i_1}^{j_1} \cdots x_{i_n}^{j_n} z^m\}$, where

$$z^m = [x_{i_1}^{j_1}, x_{i_2}^{j_2}]^{(p-1)/2} [x_{i_1}^{j_1} x_{i_2}^{j_2}, x_{i_3}^{j_3}]^{(p-1)/2} \cdots [x_{i_1}^{j_1} \cdots x_{i_{n-1}}^{j_{n-1}}, x_{i_n}^{j_n}]^{(p-1)/2}.$$

Thus, $|\mathcal{X}| \leq p^{2r}$. Further, for $T \in \mathcal{X}$ and $g \in G$, $g^{-1}Tg = T$ implies $g \in Z$. To see this, let $t \in T$ and $g^{-1}tg = t' \in T$. Then, $tZ = g^{-1}tgZ = t'Z$. Since T contains exactly one element from each coset, it follows that $t = t'$ and $g \in C_G(t)$. Thus, $g \in C_G(T) = Z$. Since $|G : Z| = p^{2r}$, $|\mathcal{X}| = p^{2r}$ and G acts transitively on $\mathcal{X}$. □

Proposition 5.2. *$W_{2r}(p)$, $r \geq 2$, admits a nonabelian representation and the representation group is p_+^{1+2r}.*

Proof. Let $G = p_+^{1+2r}$ and T be as in Proposition 5.1. Consider the partial linear space $S = (P, L)$, where $P = \{\langle x \rangle : 1 \neq x \in T\}$ and a line is of the form $\{\langle x \rangle, \langle y \rangle, \langle xy \rangle, \cdots, \langle x^{p-1} y \rangle\}$ for distinct $\langle x \rangle$, $\langle y \rangle$ in P with $[x, y] = 1$. Note that $x^i y \in T$ for each i and $|P| = (p^{2r} - 1)/(p-1)$. We show that S is a polar space of rank r.

Since $T \cap Z(G) = \{1\}$, S is nondegenerate. Let $\langle x \rangle \in P$, $l \in L$ and $\langle x \rangle \notin l$. Then, $\langle x \rangle$ is collinear with one or all points of l because $C_G(x)$ intersects nontrivially with the subgroup H of G generated by the points of l. Note that H is a subgroup of order p^2 and disjoint from $Z(G)$. Rank of S is r because singular subspaces in S correspond to elementary abelian subgroups of G which intersect $Z(G)$ trivially and p^r is the maximum of the orders of such subgroups of G. Thus S is a polar space of rank r.

Clearly G is a representation group of S. So, S is isomorphic to $W_{2r}(p)$ (Theorem 1.5(iii)). □

Proposition 5.3. *Any two representations of $W_{2r}(p)$, $r \geq 2$, are equivalent.*

Proof. Let (R_1, ψ_1) and (R_2, ψ_2) be two representations of $W_{2r}(p)$. By Theorem 1.5(ii), we may assume that $R_1 = R_2 = R$. By Proposition 3.5, each R_{ψ_i} is a set of coset representatives of $Z(R)$ in R. Let $\varphi \in Aut(R)$ be such that $\varphi(R_{\psi_1}) = R_{\psi_2}$ (Proposition 5.1). Define $\beta : P \longrightarrow P$ by $\beta = \psi_2^{-1} \varphi \psi_1$. Now, Lemma 3.2 implies that β is an automorphism of $W_{2r}(p)$. Now, (R, ψ_1) and (R, ψ_2) are equivalent with respect to φ and β. □

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