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Optimal Main Effect Plans with Non-Orthogonal Blocks

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Abstract

Work on obtaining optimal main effect plans in non-orthogonal blocks was initiated recently by Mukerjee, Dey and Chatterjee (2002), who gave a set of sufficient conditions for a main effect plan to be universally optimal under possible non-orthogonal blocking and also suggested a construction procedure for obtaining such block designs. Their method is however, not applicable for all factorials. In this paper, a new construction procedure is given for situations where the procedure of Mukerjee *et al.* (2002) is inapplicable.

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1 Introduction and Preliminaries

Fractional factorial plans are of substantial recent interest due to their wide applicability, particularly in industrial experimentation and quality control work. A major part of the existing work concerns optimal plans in the absence of blocks and, the available results on such plans in block designs mostly center around orthogonal blocking. In the context of a main effect plan with n factors $F_1, \ldots, F_n$, a common technique for achieving optimal plans with orthogonal blocking is to start with an orthogonal array of strength two, having $n+1$ columns, and then to identify one of these columns with the blocking factor and the remaining columns with $F_1, \ldots, F_n$. As a result, this method allocates all the levels of each F_i equally often in each block, a necessary condition for orthogonal blocking. Thus, this method is applicable only when the block size is an integral multiple of the number of levels of each F_i, a requirement that is not always possible to meet. Mukerjee, Dey and Chatterjee (2002) initiated work on the problem of finding optimal main effect plans with possibly non-orthogonal blocking. With reference to a general factorial set up, Mukerjee *et al.* (2002) obtained sufficient conditions for a main effect plan to be universally optimal (and hence in particular, A-, D- and E-optimal) under possibly non-orthogonal blocking and, also gave a construction procedure using generalized Youden designs in combination with orthogonal arrays.

The construction procedure of Mukerjee *et al.* (2002) however is not applicable in all situations. For instance, the method will fail if one or more factors have $m \geq 4$ levels and the block size is two. For the method of Mukerjee *et al.* (2002) to work in such a set up, a generalized Youden design with $m(\geq 4)$ symbols, m columns and two rows is required, but such a generalized Youden design is nonexistent. To overcome this difficulty, we propose an alternative method of construction leading to universally optimal plans with non-orthogonal blocking. The proposed method can be viewed as a generalization of the procedure of Mukerjee *et al.* (2002) in the sense that their procedure is a special case of our method. We also show that in some cases, it is possible to obtain block designs with small block sizes if we do not insist on universal optimality and are satisfied with a weaker optimality criterion, like E-optimality.

Consider a factorial experiment involving n factors $F_1, \ldots, F_n$ at $m_1, \ldots, m_n (\geq 2)$ levels respectively. A typical treatment combination is denoted by the n-tuple $j_1 \ldots j_n$, where $0 \leq j_i \leq m_i - 1, 1 \leq i \leq n$. For $1 \leq i \leq n$, let $\boldsymbol{\tau}_i = (\tau_{i0}, \ldots, \tau_{i,m_i-1})'$ be the $m_i \times 1$ vector of fixed effects corresponding to the levels of F_i. With reference to an $m_1 \times \cdots \times m_n$ factorial, let $\mathcal{D}(b,k)$ be the class of all fractions laid out in a block design involving b (≥ 2) blocks of size k (≥ 2) each. For any plan $d \in \mathcal{D}(b,k)$, let N_{id} be the $m_i \times b$ incidence matrix of the levels of F_i versus the blocks, $1 \leq i \leq n$.

Under a fixed effects additive linear model, Mukerjee *et al.* (2002) proved the following basic result, which we state for future reference.

THEOREM 1. *If there exists a plan $d_0 \in \mathcal{D}(b,k)$ such that*

(a) the bk treatment combinations in d_0, written as rows, form an orthogonal array of strength two,

(b) for $1 \leq i \leq n$, N_{id_0} is the incidence matrix of a balanced block design in m_i treatments or, symbols and b blocks, and

(c) for $1 \leq i \neq t \leq n$, $N_{id_0}{N_{td_0}}'$ has all elements equal,

then d_0 is universally optimal in $\mathcal{D}(b,k)$ for inference on each $\boldsymbol{\tau}_i$ $(1 \leq i \leq n)$.

In particular, such a plan will be D-, A- and E-optimal for complete sets of orthonormal contrasts representing the main effect of each F_i.

2 Optimal Block Designs

We now describe a construction procedure satisfying the conditions of Theorem 1. Suppose there exists an orthogonal array $L_{b_0}(m_1 \times \cdots \times m_n)$ of strength two having b_0 rows and n columns such that its ith column involves m_i symbols $0, 1, \ldots, m_i - 1$ $(1 \leq i \leq n)$. Denote this array by $L = (l_{si})$, where $1 \leq s \leq b_0,\ 1 \leq i \leq n$. Furthermore, for $1 \leq i \leq n$, consider a k-resolvable balanced block design ξ_i, involving $p_i m_i$ blocks, each of size k and m_i symbols $0, 1, \ldots, m_i - 1$, where $p_i \geq 1$ is an integer. Note that Mukerjee *et al.* (2002) covered only the special case $p_i = 1$. Let p be the least common multiple of $p_1, \ldots, p_n$. Take p/p_i copies of ξ_i and call it S_i, $1 \leq i \leq n$. For $1 \leq i \leq n$, let the resolvable groups of blocks of S_i be $S_{i1}, \ldots, S_{ip}$, each group having m_i blocks. Since $S_{ij},\ 1 \leq j \leq p$ has each symbol replicated k times, from Das and Dey (1989) it follows that it is possible to rearrange the symbols within the blocks of S_{ij}, so that viewing S_{ij} as a $k \times m_i$ array, say S^*_{ij}, each symbol occurs once in each row of the array, $1 \leq i \leq n,\ 1 \leq j \leq p$. Denote the columns of S^*_{ij} by $\boldsymbol{S}^*_{ij}(h)$ $(0 \leq h \leq m_i - 1)$. With reference to an $m_1 \times \cdots \times m_n$ factorial, suppose a plan $d_0 \in \mathcal{D}(b, k),\ b = pb_0$, is constructed such that for $1 \leq j \leq p, 1 \leq s \leq b_0$, the k treatment combinations in the $\{(j-1)b_0 + s\}$th block of d_0 are given by the rows of the $k \times n$ array

$$A_{js} = [\boldsymbol{S}^*_{1j}(l_{s1}), \ldots, \boldsymbol{S}^*_{nj}(l_{sn})]. \tag{2.1}$$

We then have the following result.

THEOREM 2. *The plan d_0, constructed as above, is universally optimal in $\mathcal{D}(b, k)$ for inference on every $\boldsymbol{\tau}_i$ $(1 \leq i \leq n)$.*

PROOF. We need to verify conditions (a) - (c) of Theorem 1. This verification proceeds on the lines of the proof of Theorem 2 in Mukerjee *et al.* (2002) by using (2.1) and noting the following:

(A) Conditions (a) and (c) of Theorem 1 follow from the fact that L is an orthogonal array of strength two and for each $i, j,\ 1 \leq i \leq n, 1 \leq j \leq p$, in S^*_{ij}, each symbol occurs once in each row.

(B) Condition (b) of Theorem 1 follows from the following facts: (i) for any fixed i and j, each of $\boldsymbol{S}^*_{ij}(0), \ldots, \boldsymbol{S}^*_{ij}(m_i - 1)$ appears b_0/m_i times in the collection $\{\boldsymbol{S}^*_{ij}(l_{si})\},\ 1 \leq j \leq p, 1 \leq s \leq b_0$, (ii) for $1 \leq i \leq n$, the columns of $\cup_{j=1}^{p} S^*_{ij}$ form a balanced block design on m_i symbols and $m_i p$ blocks each of size k and thus, for $1 \leq i \leq n$, N_{id_0} is the incidence matrix of a balanced block design. This completes the proof. □

As in Mukerjee *et al.* (2002), at least one more factor can be added to the plan d_0 of Theorem 2, retaining optimality. Let B be an orthogonal array $L_k(m_{n+1} \times \cdots \times m_{n+g})$ of strength two, if $g > 1$ and of strength one, if $g = 1$. With reference to an $m_1 \times \cdots \times m_n \times m_{n+1} \times \cdots \times m_{n+g}$ factorial, now suppose a plan $d^* \in \mathcal{D}(b, k)$ is constructed such that, for $1 \leq s \leq b$, the k treatment combinations in the $\{(j-1)b_0 + s\}$th block of d^* are given by the rows of the $k \times (n+g)$ array

$$A^*_{js} = [A_{js}, B], \tag{2.2}$$

where A_{js} is as given by (2.1). We then have the following result whose proof is similar to that of Theorem 2.

THEOREM 3. *The plan d^*, constructed as above, is universally optimal in $\mathcal{D}(b, k)$ for inference on every $\boldsymbol{\tau}_i$ $(1 \leq i \leq n+g)$.*

We now show how the above construction works when $p_i \neq 1$ and $k = 2$ or, 3. For $m_i = 4, k = 2$, the least value of p_i is 3 and one can take

$$S^*_{i1} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{bmatrix}, \quad S^*_{i2} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{bmatrix}, \quad S^*_{i3} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 2 \end{bmatrix}. \tag{2.3}$$

Similarly, for $m_i = 5, k = 2$, the least value of p_i is 2 and one can take

$$S^*_{i1} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}, \quad S^*_{i2} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}. \tag{2.4}$$

Also, for $m_i = 5, k = 3$, the least value of p_i is 2 and one can take

$$S^*_{i1} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & 0 & 1 \end{bmatrix}, \quad S^*_{i2} = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 0 & 1 \\ 3 & 4 & 0 & 1 & 2 \end{bmatrix}. \tag{2.5}$$

EXAMPLE. We describe the construction of an optimal main effect plan for a $5^6 \times 2$ factorial in $\mathcal{D}(50, 2)$. In the set up of Theorem 3, take $n = 6, g = 1, m_1 = \cdots = m_6 = 5, m_7 = 2, b_0 = 25$ and $k = 2$, so that $p_i = 2,\ 1 \leq i \leq 6$ and $b = pb_0 = 50$. Also take $L \equiv L_{25}(5^6)$ displayed below (in transposed form).

$$L_{25}(5^6) = \begin{bmatrix} 01234 & 12340 & 23401 & 34012 & 40123 \\ 01234 & 23401 & 40123 & 12340 & 34012 \\ 00000 & 11111 & 22222 & 33333 & 44444 \\ 01234 & 01234 & 01234 & 01234 & 01234 \\ 01234 & 34012 & 12340 & 40123 & 23401 \\ 01234 & 40123 & 34012 & 23401 & 12340 \end{bmatrix}' .$$

Now using S^*_{i1} and S^*_{i2} of (2.4), the fifty blocks can be obtained. For example, using S^*_{i1} and the first five rows of $L_{25}(5^6)$ one gets five blocks, each of size two, as shown below. The remaining blocks are obtained similarly, using S^*_{i1} and S^*_{i2}.

Block 1	Block 2	Block 3	Block 4	Block 5
0000000	1101110	2202220	3303330	4404440
1111111	2212221	3313331	4414441	0010001

If one does not demand universal optimality for all the factors and is satisfied with a weaker optimality criterion like E-optimality for one or more factors, one can obtain optimal main effect plans in non-orthogonal blocks with small block sizes in some cases. To that end, we have the following result, whose proof is similar to that of Theorem 1. A similar result has also been obtained by Bagchi and Bose (2004).

THEOREM 4. *Suppose there exists a plan $d_1 \in \mathcal{D}(b,k)$ such that*

(a) the bk treatment combinations in d_1, written as rows, form an orthogonal array of strength two,

(b′) for $1 \le i \le n$, N_{id_1} is the incidence matrix of an equireplicate ϕ_i-optimal block design in the class of all designs with m_i treatments or, symbols and b blocks each of size k, where $\phi_i(\cdot)$ is a non-increasing optimality criterion, and

(c) for $1 \le i \ne t \le n$, $N_{id_1}N'_{td_1}$ has all elements equal.

Then d_1 is ϕ_i-optimal in $\mathcal{D}(b,k)$ for inference on $\boldsymbol{\tau}_i$, $1 \le i \le n$.

A construction procedure, satisfying the conditions of Theorem 4 can be developed on lines similar to that given in (2.1) and (2.2). For example, in the set up of Theorem 3, let $n = 5, g = 1, m_1 = 4, m_2 = \cdots = m_6 = 2$ and consider a group divisible design with 4 symbols and 4 blocks each of size two; here, columns are blocks.

$$d = \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{array}$$

Note that the design d is a cyclic design. Following the earlier notation, we can take $S_{11}^* = [d]$, and for $2 \leq i \leq 5$, $S_{i1}^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Also take $L \equiv L_8(4 \times 2^4)$ as shown below:

$$L_8(4 \times 2^4) = \begin{bmatrix} 0011 & 2233 \\ 0101 & 0101 \\ 0110 & 0110 \\ 0101 & 1010 \\ 0110 & 1001 \end{bmatrix}'.$$

Using $L_8(4 \times 2^4)$, S_{i1}^*, $1 \leq i \leq 5$ as above and, following a replacement procedure similar to (2.1) and (2.2), we get a main effect plan for a 4×2^5 experiment, split into 8 blocks of size two each, which is shown below:

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
000000	011110	101010	110100	200110	211000	301100	310010
111111	100001	210101	201011	311001	300111	010011	001101

Since in $L_8(4 \times 2^4)$, each symbol in the 4-symbol column appears twice, on replacing the symbols of the 4-symbol column by the columns of S_{11}^* to get the blocks, each column of S_{11}^* gets repeated twice in the final design and N_{1d_1} is the incidence matrix of the block design $[d, d]$. Now, $[d, d]$ is a group divisible design with parameters, in the usual notation, $v = 4, b = 8, k = 2, r = 4, m = 2 = n, \lambda_1 = 0, \lambda_2 = 2$. Such a design is known to be E-optimal in the class of all designs with 4 treatments and 8 blocks each of size two (*cf.* Jacroux (1983)). Thus the design shown above is E-optimal for inference on $\boldsymbol{\tau}_1$ and is universally optimal for inference on $\boldsymbol{\tau}_i$, $2 \leq i \leq 6$.

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