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EFFICIENCIES OF TWO-STAGE PROCEDURES FOR SELECTING THE NORMAL POPULATION WITH LARGEST MEAN

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SUMMARY. Various selection procedures are available for selecting the normal population which has the largest mean, when the variances are known and equal. However, a general comparison of these procedures for all parameter combinations is not available in the literature. In this paper, we compare two two-stage selection procedures with the standard one-stage procedure by deriving general algebraic expressions for their efficiencies. We obtain necessary and sufficient conditions on the parameters for these efficiencies to be more than unity. For illustration, we compute the efficiencies for some parameter choices. Our results will help experimenters to decide which of these procedure will be appropriate in a particular situation and also assist in choosing the different parameters required for implementing the procedure in an efficient way.

1. Introduction

The problem of selecting the normal population with the largest population mean has received considerable attention over the years. Bechhofer (1954) proposed a single-stage procedure using the indifference zone appoach. Subsequently, several authors have proposed different procedures for this problem. Among these are the two-stage procedures of Tamhane and Bechhofer (1977, 1979), and Miescke and Sher (1980) and some multi-stage and fully sequential procedures. The question of comparing these procedures is important in order to recommend to the experimenter the appropriate method to use in a particular real-life setting.

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Bechhofer and Goldsman (1989) compared some of these procedures and other fully sequential procedures using Monte Carlo sampling experiments for some chosen parameter choices. Using numerical estimates of certain performance characteristics, they assessed the different procedures for the particular parameter combinations considered by them. However, it seems that a general comparison of these procedures for all parameter combinations is not available in the literature.

In this paper, for general parameter combinations, we algebraically compare the above two two-stage procedures and the one-stage procedure by first obtaining expressions for the efficiencies of the two-stage procedures compared to the one-stage procedure. Using these expressions, we obtain necessary and sufficient conditions under which a two-stage procedure is more efficient than the one-stage procedure. These conditions could be used by the experimenter to decide which of the above procedures will be suitable in a particular situation and also to appropriately choose values for the constants needed to implement the procedure in an efficient way. For illustration, the efficiency values actually attained by a two-stage procedure are computed for certain parameter choices. These show that a two-stage procedure may be highly efficient compared to the one-stage procedure, depending on the values of the constants. We restrict our study to comparisons between one-stage and two-stage procedures only and do not consider fully sequential procedures since, as explained in Bechhofer and Goldsman(1989), truly sequential experimentation is not a feasible possibility in many real-life situations and in such cases the one-stage and two-stage procedures play important roles.

Consider $k \geq 2$ normal populations, denoted by $\pi_1, \cdots, \pi_k$, say, with unknown means $\mu_1, \cdots, \mu_k$ respectively and a common known variance σ^2. Let $\mu = (\mu_1, \cdots, \mu_k)$. Without loss of generality, let the ordered values of the μ_i be denoted by $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_{k-1} < \mu_k$. The values of the μ_j $(j = 1, \ldots, k)$ are unknown and the pairing of the μ_j with the π_i is unknown. The objective of the experiment is to select the population associated with μ_k. Let $PCS(\mu, \mathcal{P})$ be the probability of correct selection (PCS) for a procedure $\mathcal{P}$, when the true mean is μ. The procedures considered in this paper are based on the indifference zone approach which requires that for given δ and P^*, the probability of correct selection satisfies

$$PCS(\mu, \mathcal{P}) \geq P^* \text{ whenever } \mu_k - \mu_{k-1} \geq \delta. \tag{1}$$

The quantities $\delta, P^*,\ 0 < \delta < \infty,\ 1/k < P^* < 1$ are constants specified before the experimentation begins.

For the sake of completeness, we briefly describe the three selection procedures under study. For detailed information on these we refer to Bechhofer and Goldsman(1989) and the cited papers.

Single stage procedure of Bechhofer (1954), (Procedure $\mathcal{P}_1$): n independent observations are taken in a single stage from each π_i, and let X_{ij}, $i = 1, \ldots, k, j = 1, \ldots, n$ denote the observations from π_i. Let $\bar{x}_i = (x_{i1} + \ldots + x_{in})/n$ be the k sample means, $i = 1, \ldots, k$. Let $\bar{x}_{[k]} = \max\{\bar{x}_1, \ldots, \bar{x}_k\}$. The population that yielded $\bar{x}_{[k]}$ is selected as the one associated with μ_k.

It is easy to see that,

$$PCS(\mu, \mathcal{P}_1) = \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} \Phi(\frac{\sqrt{n}(x - \mu_i)}{\sigma}) d\Phi(\frac{\sqrt{n}(x - \mu_k)}{\sigma}). \tag{2}$$

where Φ denotes the c.d.f. of the standard normal variate.

Two-stage procedure of Tamhane and Bechhofer (1977, 1979), (Procedure $\mathcal{P}_2$): Implementation of this procedure involves the use of three predetermined constants, n_1, n_2, h, where n_1, n_2 are integers and $h > 0$. The selection is done in two stages if necessary.

STAGE 1 : A common number n_1 of independent observations $x_{ij}^{(1)}$, $j = 1, \ldots, n_1$ are taken from $\pi_1, \cdots, \pi_k$ and the k first-stage sample means $\bar{x}_i = \sum_{j=1}^{n^1} x_{ij}^{(1)}/n_1$, $i = 1, \ldots, k$ are computed. Let $\bar{x}_{[k]}^{(1)} = \max\{\bar{x}_1^{(1)}, \ldots, \bar{x}_k^{(1)}\}$. Determine the subset I of $\{1, \cdots, k\}$ where $I = \{i \mid \bar{x}_i^{(1)} \geq \bar{x}_{[k]}^{(1)} - h\}$, $h > 0$. Let π_I denote the associated subset of $\pi_1, \cdots, \pi_k$. If π_I consists of only one population, stop sampling and select the population yielding $\bar{x}_{[k]}^{(1)}$ as the one associated with μ_k. Otherwise proceed to stage 2.

STAGE 2 : Take n_2 additional independent observations $x_{ij}^{(2)}$, $j = 1, \ldots, n_2$ from each population in π_I. Compute the cumulative sample means $\bar{x}_i = (\sum_{j=1}^{n_1} x_{ij}^{(1)} + \sum_{j=1}^{n_2} x_{ij}^{(2)})/(n_1 + n_2)$, for $i \epsilon I$. Then select the population that yields $\max\{\bar{x}_i \mid i \epsilon I\}$ as the one associated with μ_k.

Two-stage procedure of Miescke and Sher (1980), (Procedure $\mathcal{P}_3$): $\mathcal{P}_3$ is identical with $\mathcal{P}_2$ except that finally the decision is taken on the basis of only the second stage mean $\bar{x}_i^{(2)} = \sum_{j=1}^{n_2} x_{ij}^{(2)}/n_2$. The population that yields $\max\{\bar{x}_i^{(2)} \mid i \epsilon I\}$ is selected as the one associated with μ_k.

It is clear that $\mathcal{P}_2$ is always better than $\mathcal{P}_3$, but it is not clear if $\mathcal{P}_3$ is better than $\mathcal{P}_1$ or not. $\mathcal{P}_3$ may be used if for some unforeseen reasons the

first stage data gets lost. We show later that even in that case, under certain parameter choices, $\mathcal{P}_3$ may perform better than $\mathcal{P}_1$.

Let $S^* =$ cardinality of (I). Define $S = \begin{cases} 0 & \text{if } S^* = 1 \\ S^* & \text{if } S^* > 1. \end{cases}$

Then the sample size required by $\mathcal{P}_1$ is kn and the total sample size required by each of $\mathcal{P}_2$ and $\mathcal{P}_3$ is $T = kn_1 + n_2 S$.

When the true mean vector is μ, let the Average Sample Number (ASN) for a procedure $\mathcal{P}$ be denoted by $ASN(\mathcal{P}, \mu)$. Then, $ASN(\mathcal{P}_1, \mu) = kn$, $ASN(\mathcal{P}_2, \mu) = ASN(\mathcal{P}_3, \mu) = kn_1 + n_2 E(S) = km$, say, where $E(S)$ denotes the expectation of S at μ.

To compare two procedures at a configuration μ, we compare the probabilities of correct selection of these procedures when n, n_1, n_2 are such that both procedures have the same ASN. Let $PCS(\mathcal{P}, \mu, N)$ be the PCS at μ for procedure $\mathcal{P}$ with ASN N. Then, as a measure of the efficiency of a procedure, we can use the following criterion.

Definition 1 The Asymptotic Relative Efficiency (ARE) of procedure $\mathcal{P}_i$ with respect to another procedure $\mathcal{P}_j$ is given by

$$e_{ij} = \lim_{N \to \infty} \frac{\log(1 - PCS(\mathcal{P}_i, \mu, N))}{\log(1 - PCS(\mathcal{P}_j, \mu, N))}.$$

Note that $\log(1 - PCS(\mathcal{P}_i, \mu, N)), \log(1 - PCS(\mathcal{P}_j, \mu, N))$ are negative and so, $\mathcal{P}_i$ is more efficient than $\mathcal{P}_j$ if $e_{ij} > 1$.

In Section 2, we derive algebraic expressions for e_{21} and e_{31} and obtain conditions on the parameters under which these ARE's will be strictly greater than unity. To derive these results, we first obtain expansions for $1 - PCS$ for these three procedures. In Section 3, we show how the efficiency values may be calculated. The values of e_{21} are computed for some parameter choices to illustrate how the efficiency varies with the parameters.

2. Main Results

To derive expressions for e_{21} and e_{31}, we need the asymptotic expansions of the probabilities of correct selection of $\mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$.

Let $a = \frac{n_1}{m}$ and $b = \frac{n_2}{m}$. We first state the expansions as three theorems below. The proofs of the theorems are given later in this section.

Theorem 2.1. *For given μ and σ^2, as $n \to \infty$,*

$$\log\{1 - PCS(\mu, \mathcal{P}_1)\} = -\frac{n}{4\sigma^2}(\mu_k - \mu_{k-1})^2 + o(n).$$

THEOREM 2.2. *For given μ and σ^2, as $m \to \infty$,*

$$\log\{1 - PCS(\mu, \mathcal{P}_3)\} = m \max\left\{-\frac{a}{4\sigma^2}(\mu_k - \mu_{k-1} + h)^2, -\frac{b}{4\sigma^2}(\mu_k - \mu_{k-1})^2\right\} + o(m).$$

THEOREM 2.3. *For given μ and σ^2, as $m \to \infty$,*

$$\log\{1 - PCS(\mu, \mathcal{P}_2)\} = m \max\left\{-\frac{a}{4\sigma^2}(\mu_k - \mu_{k-1} + h)^2, -\frac{a+b}{4\sigma^2}(\mu_k - \mu_{k-1})^2\right\} + o(m).$$

Using Definition 1 and the expansions as given in Theorems 2.1, 2.2 and 2.3, it follows that taking $kn = km$, the expressions for e_{21} and e_{31} are :

$$e_{21} = \min\left\{\frac{a(\mu_k - \mu_{k-1} + h)^2}{(\mu_k - \mu_{k-1})^2}, \ (a+b)\right\} \tag{3}$$

$$e_{31} = \min\left\{\frac{a(\mu_k - \mu_{k-1} + h)^2}{(\mu_k - \mu_{k-1})^2}, b\right\}. \tag{4}$$

From (3) and (4), we may now derive necessary and sufficient conditions on the proportions a and b for e_{21} and e_{31} to be greater than unity.

THEOREM 2.4. *$\mathcal{P}_3$ is more efficient than $\mathcal{P}_1$ if and only if*

$$1 < b < \frac{kh(2\mu_k - 2\mu_{k-1} + h)}{(\mu_k - \mu_{k-1} + h)^2 E(S)}. \tag{5}$$

PROOF. From (4), $e_{31} > 1$ if and only if

$$a > \left(\frac{\mu_k - \mu_{k-1}}{\mu - \mu_{k-1} + h}\right)^2 \text{ and } b > 1. \tag{6}$$

However, since ARE's are calculated when the ASN's are made equal, we have $n = m$ and so $k = ak + bE(S)$. Hence, condition (6) is equivalent to $k > k\left(\frac{\mu_k - \mu_{k-1}}{\mu_k - \mu_{k-1} + h}\right)^2 + bE(S)$ and $b > 1$. Hence, $e_{31} > 1$ if and only if $1 < b < \frac{kh(2\mu_k - 2\mu_{k-1} + h)}{(\mu_k - \mu_{k-1} + h)^2 E(S)}$. Hence the theorem. □

THEOREM 2.5. *$\mathcal{P}_2$ is more efficient than $\mathcal{P}_1$ if and only if*

$$a > \left(\frac{\mu_k - \mu_{k-1}}{\mu_k - \mu_{k-1} + h}\right)^2. \tag{7}$$

PROOF. Clearly

$$a + b > a + bE(S)/k = [kan + bnE(S)]/kn = 1.$$

So it follows from (3) that, $e_{21} > 1$ if and only if $a > \left(\frac{\mu_k - \mu_{k-1}}{\mu_k - \mu_{k-1} + h}\right)^2$. Hence the theorem. □

REMARK 2.1. Theorems 2.5 and 2.6 may be used to determine the constants of a procedure to be used in a given situation. Depending on the choice of a and b, $\mathcal{P}_2$ or $\mathcal{P}_3$ may be more efficient or less efficient than $\mathcal{P}_1$. Clearly, for all μ, given h, it will always be possible to choose a satisfying (7) and thus make $\mathcal{P}_2$ more efficient. However, given h, it will not be always possible to choose a and b so that (5) is satisfied.

REMARK 2.2. From (3) and (4) it is clear that $e_{21} > e_{31}$ for all μ and so, $\mathcal{P}_2$ is always more efficient than $\mathcal{P}_3$. So, in our illustration in Section 3, we only consider $\mathcal{P}_1$ and $\mathcal{P}_2$.

REMARK 2.3. For a given choice of n, n_1, h, δ, from (7) it may easily be checked whether $\mathcal{P}_2$ is more efficient than $\mathcal{P}_1$ or not. However, even when $e_{21} > 1$, to judge how this efficiency varies in different situations, e_{21} needs to be computed. In Section 3, we show how e_{21} values are computed and show some illustrative values of e_{21}.

To prove Theorems 2.1-2.3, we need the following result which we state below as Lemma 1. This result can be easily proved using standard techniques of mathematical analysis and so its proof is omitted.

LEMMA 1. *Let $\{f_n\}$ be a sequence of real-valued uniformly bounded continuous functions on R. Assume that $f_n(x)$ uniformly converge to $f(x)$ for each x and $f(x)$ is continuous such that $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exists. Then*

$$\lim_{n\to\infty} \left\{ \int_{-\infty}^{\infty} (f_n(x))^n \phi(x) dx \right\}^{1/n} = \sup_{x \in R} f(x),$$

where ϕ is a continuous probability density function having support on R.

PROOF OF THEOREM 2.1. Writing $\phi(x)$ as the p.d.f. of the standard normal variate, it follows from (2) that

$$\begin{aligned} 1 - PCS(\mu, \mathcal{P}_1) &= \int_{-\infty}^{\infty} \left[1 - \prod_{i=1}^{k-1} \Phi(\frac{\sqrt{n}(x - \mu_i)}{\sigma})\right] \phi\left(\frac{\sqrt{n}(x - \mu_k)}{\sigma}\right) dx \\ &= \int_{-\infty}^{\infty} [f_n(x)]^n \phi(x) dx, \end{aligned}$$

say, where

$$[f_n(x)]^n = \left[1 - \prod_{i=1}^{k-1} \Phi\left(\frac{\sqrt{n}(x-\mu_i)}{\sigma}\right)\right] \sqrt{n} e^{-(n-1)(x-\mu_k)^2/2\sigma^2}.$$

Hence, by Lemma 1,

$$\lim_{n\to\infty} \{1 - PCS(\mu, \mathcal{P}_1)^{1/n}\} = \sup_{x \in R} \lim_{n\to\infty} f_n(x),$$

On simplification, it follows that

$$\lim_{n\to\infty} f_n(x)$$
$$= \lim_{n\to\infty} \left[1 - \prod_{1}^{k-1} \int_{-\infty}^{x} \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{n(u-\mu_i)^2}{2\sigma^2}\right\} du\right]^{1/n} \exp\left\{-\frac{(x-\mu_k)^2}{2\sigma^2}\right\}. \quad (8)$$

Let $A_{jn} = \{(X_j > x\}$ and let $p_n = 1 - \prod_1^{k-1} [\int_{-\infty}^{x} \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp\{-\frac{n(u-\mu_i)^2}{2\sigma^2}\} du]$. Then,

$$p_n = P(X_i > x \text{ for some } i) = P(\cup_{j=1}^{k-1} A_{jn}).$$

Since $P(A_{k-1,n}) \le p_n \le (k-1)P(A_{k-1,n})$, $\lim_{n\to\infty} p_n^{\frac{1}{n}} = \lim_{n\to\infty} [P(A_{k-1,n})]^{\frac{1}{n}}$. By Lemma 1,

$$\lim_{n\to\infty} [P(A_{k-1,n})]^{1/n} = \lim_{n\to\infty} \left[\int_x^{\infty} \frac{\sqrt{n}}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{n(u-\mu_{k-1})^2}{2\sigma^2}\right\} du\right]^{1/n} = k(x),$$

where $k(x) = 1$ if $x \le \mu_{k-1}$ and $k(x) = e^{-\frac{(x-\mu_{k-1})^2}{2\sigma^2}}$ if $x > \mu_{k-1}$.

Hence, from (8), using Lemma 1,

$$\lim_{n\to\infty} \{1 - PCS(\mu, \mathcal{P}_1)\}^{1/n}$$
$$= \sup_{x \in R} \left[\exp\left\{-\frac{(x-\mu_k)^2}{2\sigma^2}\right\} k(x)\right] = \exp\left\{-\frac{(\mu_k - \mu_{k-1})^2}{4\sigma^2}\right\}.$$

Theorem 2.1 follows. □

Proof of Theorem 2.2. Let p_{1n}= Prob[k is not in I], p_{2n} = Prob[(k is in I) and ($\bar{x}_k^{(2)}$ not highest in stage 2)],

Then,

$$1 - PCS(\mu \mid \mathcal{P}_3) = c_1 p_{1n} + c_2 p_{2n} = c_3(p_{1n} + p_{2n}) \quad (9)$$

where c_1, c_2, are positive constants bounded away from 0 and ∞, $c_3 \leq \max\{c_1, c_2\}$.

Let p_{3n} = Prob [(k not in I) and ($\bar{x}_{k-1}^{(1)}$ highest in stage 1)], and p_{4n} = Prob [(k in I) and ($\bar{x}_{k-1}^{(2)}$ highest in stage 2)]. Then, $p_{3n} = c_4 p_{1n}$ and $p_{4n} = c_5 p_{2n}$ for some constants $c_4, c_5, 1 \leq c_4, c_5 \leq k-1$. Hence, from (9), for some constant c_6,

$$1 - PCS(\mu \mid \mathcal{P}_3) = c_6(p_{3n} + p_{4n}).$$

Clearly, $p_{3n} = c_7 \text{Prob}(\bar{x}_{k-1}^{(1)} - \bar{x}_k^{(1)} > h) = \text{Prob}\left[Z < -\sqrt{\frac{an}{2\sigma^2}}(\mu_k - \mu_{k-1} + h)\right]$, where Z is a $N(0,1)$ variable.
Now using the inequality,

$$(1/x - 1/x^3)g(x) \leq F(x) \leq g(x)/x, \quad x > 0,$$

where F, g denote respectively the c.d.f and p.d.f of Z, we get,

$$\lim_{n\to\infty} p_{3n}^{1/n} = \exp\left\{-\frac{a}{4\sigma^2}(\mu_k - \mu_{k-1} + h)^2\right\}.$$

Following the proof of Theorem 2.1, since c_i values are bounded away from 0 and ∞, it follows after some simplification that

$$p_{4n}^{1/n} \to \exp\left\{-\frac{b}{4\sigma^2}(\mu_k - \mu_{k-1})^2\right\}, \quad \text{as } n \to \infty.$$

Hence

$$\begin{aligned}\lim_{n\to\infty} &\log[1 - PCS(\mu \mid \mathcal{P}_3)]^{1/n} \\ &= \max\left\{-\frac{a}{4\sigma^2}(\mu_k - \mu_{k-1} + h)^2,\ -\frac{b}{4\sigma^2}(\mu_k - \mu_{k-1})^2\right\}\end{aligned}$$

and this proves Theorem 2.2. □

Proof of Theorem 2.3. Let p_{5n} = Prob[(k is in I) and ($\bar{x}_k$ not highest in stage 2)] and $p_{6n} = p_{5n}$ + Prob[(k not in I) and ($\bar{x}_k < \max\{\bar{x}_i, i\epsilon I,\ \bar{x}_k\}$)].

So, p_{6n} = Prob $[\bar{x}_k < \max\{\bar{x}_i, i\epsilon I,\ \bar{x}_k\}] = d_1$Prob[($k-1$ is in I) and $\bar{x}_{k-1} = \max\{(\bar{x}_i,\ i\epsilon I,)\ \bar{x}_k\})]$.

Hence, proceeding in a similar manner as in the proof of Theorem 2.2, we can write,

$$1 - PCS(\mu \mid \mathcal{P}_2) = d_2[p_{1n} + p_{5n}] = d_3(p_{1n} + p_{6n})$$

In this case, some computations will show that,

$$(p_{6n})^{1/n} \to \exp\left\{-\frac{1}{4\sigma^2}\frac{n_1+n_2}{n}(\mu_k-\mu_{k-1})^2\right\} = \exp\left\{-\frac{a+b}{4\sigma^2}(\mu_k-\mu_{k-1})^2\right\}.$$

and as before,

$$p_{1n}^{1/n} \to \exp\left\{-\frac{a}{4\sigma^2}(\mu_k-\mu_{k-1}+h)^2\right\}.$$

So it follows that,

$$\begin{aligned}\lim_{n\to\infty} &\log\{1-PCS(\mu \mid \mathcal{P}_2)\}^{1/n} \\ &= \max\left\{-\frac{a+b}{4\sigma^2}(\mu_k-\mu_{k-1})^2,\ -\frac{a}{4\sigma^2}(\mu_k-\mu_{k-1}+h)^2\right\}.\end{aligned}$$

Hence Theorem 2.3 is proved. □

3. Calculation of e_{21}.

Given k, μ, h and m, n_1 and n_2 have to be determined in order to implement $\mathcal{P}_2$. This requires computation of E(S) since $n_2 = km - kn_1/E(S)$. An expression for E(S) at the least favourable configuration(LFC) is given in Lemma 2 below. As shown in Sehr(1988) and Bhandari and Raychaudhuri (1990), the LFC in this context is the slippage configuration given by $\mu = (\theta, \ldots, \theta, \theta+\delta)$ where δ is as in (1).

LEMMA 2. *Let $F_\alpha(x)$ and $f_\alpha(x)$ denote the c.d.f and p.d.f of a $N\left(\alpha, \frac{\sigma^2}{\sqrt{n_1}}\right)$ variable. Then, for given k and h, at the slippage configuration, $E(S)$ is given by:*

$$\begin{aligned}(k-1)\int_{-\infty}^{\infty}&[\{F_\theta(x+h)\}^{k-2}F_{\theta+\delta}(x+h) - \{F_\theta(x-h)\}^{k-2}F_{\theta+\delta}(x-h)]f_\theta(x)dx \\ &+\int_{-\infty}^{\infty}[\{F_\theta(x+h)\}^{k-1} - \{F_\theta(x-h)\}^{k-1}]f_{\theta+\delta}(x)dx.\end{aligned}$$

PROOF. Let $I_{[\bar{x}_{[k]}-h,\bar{x}_{[k]}]}(x_i) = 1$ if $\bar{x}_{[k]}-h \le \bar{x}_i \le \bar{x}_{[k]}$ and zero otherwise. Then,

$$S^* = \sum_{i=1}^{k} I_{[\bar{x}_{[k]}-h,\bar{x}_{[k]}]}(x_i).$$

Hence, writing $F_i(x)$ and $f_i(x)$ to denote the cdf and pdf of $\bar{x}_i$, $i = 1, \ldots, k$,

$$E(S^*) = \sum_{i=1}^{k}\int_{-\infty}^{\infty} P[x_j \le x+h \text{ for all } j \ne i | x_i = x]f_i(x)dx$$

which simplifies to

$$(k-1)\int_{-\infty}^{\infty}\{F_\theta(x+h)\}^{k-2}F_{\theta+\delta}(x+h)f_\theta(x)dx + \int_{-\infty}^{\infty}\{F_\theta(x+h)\}^{k-1}f_{\theta+\delta}(x)d(x).$$

Now, since $S = S^*$ if $S^* > 1$ and $S = 0$ if $S^* = 1$,

$$E(S) = E(S^*) - \text{Prob}[S^* = 1]. \tag{10}$$

But, from definition of S^*, $\text{Prob}[S^* = 1] = \sum_{i=1}^{k} \text{Prob}(E_i)$, where $E_i = \{(\bar{x}_1, \ldots, \bar{x}_k) : \bar{x}_j \leq \bar{x}_i - h, \text{ for all } j \neq i\}$ and the E_i's are disjoint. Then,

$$\text{Prob}(E_i) = \int_{-\infty}^{\infty} \text{Prob}(E_i|\bar{x}_i = x)f_i(x)dx.$$

So, after some computations it can be shown that

$$\text{Prob}(S^* = 1) = (k-1)\int_{-\infty}^{\infty}\{F_\theta(x-h)\}^{k-2}F_{\theta+\delta}(x-h)f_\theta(x)dx + \int_{-\infty}^{\infty}\{F_\theta(x-h)\}^{k-1}f_{\theta+\delta}(x)d(x).$$

Hence, from (10), the lemma follows.

In a particular situation, using the expression for E(S), for any choice of n_1, E(S) may be calculated from Lemma 2 and hence the corresponding value of n_2 may be obtained. Then, for implementing $\mathcal{P}_2$ with these values of n_1, n_2 and h, e_{21} may be calculated from (3). Some illustrative values of e_{21} for the case $k = 4, m = 50$ are given in Table 1.

REMARK. Table 1 illustrates that $\mathcal{P}_2$ may be less or more efficient than $\mathcal{P}_1$ depending on whether the condition of Theorem 2.6 is satisfied or not. So, the experimenter may be first guided by Theorem 2.6 to decide which procedure will be more efficient in a given situation. However, among the parameter combinations for which $\mathcal{P}_2$ is more efficient according to Theorem 2.6, Table 1 shows that for given δ, h, m, the actual efficiency first increases with the first stage sample size and then decreases. So, in a practical situation, given m, h and δ, it is important to do computations using E(S) as in Table 1 to choose n_1 for maximising the efficiency. This values of n_1 may then be used for implementing the procedure.

TABLE 1. VALUES OF e_{21} AT LFC FOR $k = 4, m = 50$ AND SOME CHOICES OF δ, h AND n_1.

δ	h	n_1	e_{21}
0.2	0.35	9	0.91
		12	1.66
		15	1.50
		20	1.30
		30	1.16
0.2	0.25	9	0.91
		15	1.51
		18	1.74
		20	1.66
		25	1.48
		30	1.34
0.2	0.15	20	1.22
		24	1.47
		30	1.78
		32	1.68
0.15	0.15	23	1.84
		25	1.96
		30	1.70

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