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# Van den BERG-KESTEN INEQUALITY FOR THE POISSON BOOLEAN MODEL FOR CONTINUUM PERCOLATION

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*SUMMARY.* For the Poisson Boolean model of continuum percolation we prove the Van den Berg-Kesten inequality $P(A \square B) \leq P(A)P(B)$ for *any* two events $A$ and $B$ living on a bounded region.

## 1. **Introduction**

For Bernoulli sequences Van den Berg and Kesten (1985) obtained a correlation like inequality for two increasing events and conjectured that it holds for any two events. This conjecture was proved by Reimer (1994). Later developments led to a need to obtain BK-like inequality for continuum percolation models. Bezuidenhout and Grimmett (1991) suggest versions of BK inequality via weak-convergence arguments that require topological conditions on the events. For the Poisson Boolean model Roy and Sarkar (1992) and Sarkar (1994) obtained BK inequality for a very special kind of increasing events and Van den Berg (1996) proved the inequality for any two increasing events. For the Poisson Boolean model of percolation we prove the BK inequality for any two events living on a bounded region. Our proof depends on a conditioning argument and a theorem of Reimer (1994).

## 2. **Preliminaries**

*Discrete case.* This case is motivated by percolation on the integer lattice in $\mathbb{R}^d$ but for all purposes it reduces to the following. Let $P$ be a product

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probability on $Q_n = \{0,1\}^n$. For $x = (x_1, x_2, \dots, x_n)$ in $Q_n$ and $K \subset \{1, 2, \dots, n\}$ we define $K$ - *cylinder about* $x$.

$$[x]_K := \{y \in Q_n : \quad y_i = x_i \ \ \forall\, i \in K\}.$$

For $E, F \subset Q_n$ we define $E \,\square\, F$ (read : E box F) $:= \{x \in Q_n : \exists$ disjoint $K, L \subset \{1, 2, \dots, n\}$ s.t. $[x]_K \subset E$ and $[x]_L \subset F\}$. We say $y \geq x$ if $y_i \geq x_i \ \forall\, i$ and an event $E$ is said to be increasing if $x \in E$ and $y \geq x$ then $y \in E$. Van den Berg and Kesten (1985) proved

$$P(E \,\square\, F) \leq P(E)P(F) \qquad \dots (2.1)$$

for increasing events $E$ and $F$ and conjectured that (2.1) holds for all events $E$ and $F$. Van den Berg and Fiebig (1987) proved the BK inequality for some other special class of events. Finally Reimer (1994) proved the general case by reducing the proof of (2.1) to the following theorem.

THEOREM 2.1. (Reimer). *For* $E$, $F \subset Q_n$

$$|E \,\square\, F| \leq |E \cap F^c|. \qquad \dots (2.2)$$

Here $|A|$ stands for the cardinality of the set $A$, $x^c := (1-x_1, 1-x_2, \dots, 1-x_n)$ is the *antipodal* of $x = (x_1, x_2, \dots, x_n)$ in $Q_n$ and $F^c := \{x^c : x \in F\}$ is the *antipodal complement* of $F$.

We present an outline of the proof based on Gupta (1999) where the reader can find further details.

PROOF. For $u, v \in Q_n$, we define

$$[u, v] := \{x \in Q_n : \quad x_i \in \{u_i, v_i\} \quad \forall\, i\},$$

the *subcube generated by* $u$ *and* $v$. For $u, v \in Q_n$, we define the *butterfly* $B = B_{u,v}$ on $Q_n$ as consisting of the following four associated subcubes of $Q_n$ :

$$\text{Body}(B) := \{u\},\ \text{Tip}(B) := \{v\},\ \text{Red}(B) := [u, v] \text{ and}$$
$$\text{Yel}(B) := [u, v^c].$$

A *flock* $\mathcal{B} = \{B_{u,v}\}$ is a set of butterflies and we define

$$\text{Body}(\mathcal{B}) := \bigcup_{B \in \mathcal{B}} \text{Body}(B), \text{Tip}(\mathcal{B}) := \bigcup_{B \in \mathcal{B}} \text{Tip}(B),$$

$$\text{Red}(\mathcal{B}) := \bigcup_{B \in \mathcal{B}} \text{Red}(B) \text{ and } \ \text{Yel}(\mathcal{B}) := \bigcup_{B \in \mathcal{B}} \text{Yel}(B)$$

Reimer's *Butterfly Theorem* , see Section 3 of Gupta (1999) for a proof, says that if the butterflies in a flock $\mathcal{B}$ have distinct bodies, i.e.,

$$|\mathcal{B}| = |\text{Body}(\mathcal{B})| \qquad \dots (2.3)$$

then

$$|\mathcal{B}| \leq |R \cap Y^c|, \qquad \ldots (2.4)$$

where $R = \text{Red}(\mathcal{B})$ and $Y = \text{Yel}(\mathcal{B})$.

We observe that $[u, v] = [u]_K$, where $K = \{i : u_i = v_i\}$. Then it is easily seen that for each $a \in E \,\Box\, F$, there exists $b \in Q_n$ such that $[a, b] \subset E$ and $[a, b^c] \subset F$. Choose and fix, for each $a \in E \,\Box\, F$, one such $b \in Q_n$. Then $\mathcal{B} = \{B_{a,b} : a \in E \,\Box\, F\}$ is a flock of butterflies satisfying

$$|\mathcal{B}| = |\text{Body } (\mathcal{B})| = |E \,\Box\, F|, \text{Red}(\mathcal{B}) \subset E$$

and

$$\text{Yel}(\mathcal{B}) \subset F. \qquad \ldots (2.5)$$

Then by (2.3), (2.4) and (2.5) we have,

$$|E \,\Box\, F| \leq |E \cap F^c|.$$

$\Box$

*Continuum case.* In the Poisson Boolean model $(X, \lambda, \rho)$ of continuum percolation each point $x$ of $X$, a Poisson process of density $\lambda > 0$, is the centre of a closed ball of random radius in such a way that different radii are independently distributed according to $\rho$ and are also independent of the process $X$. We denote a typical realisation of this percolation process by $w = \{(x_i, r_i) : i = 1, 2, \ldots\}$. The restriction of the realisation $w$ to a region $K \subset \mathbb{R}^d$ is

$$w_K := \{(x_i, r_i) \in w : x_i \in K\}.$$

For $K \subset \mathbb{R}^d$, we define

$$[w]_K := \{w' : w'_K = w_K\} \qquad \ldots (2.6)$$

We say that an event $A$ lives on $U$ if $w \in A$ implies that any $w'$ such that $w'_U = w_U$ is also in $A$. For $A$ and $B$ living on a bounded region $U$ define

$$A \,\Box\, B := \{w : \exists \ \text{ disjoint regions } K, L \subset U \text{s.t.} \\ [w]_K \subset A \text{ and } [w]_L \subset B\} \qquad \ldots (2.7)$$

For a reformulation of this definition, see (4.3) below.

There are many mathematical constructions of the Poisson Boolean percolation available in the literature, see, e.g., Section 1.4 of Meester and Roy (1994). In the next section we discuss our mathematical framework.

## 3. Mathematical Framework

We fix a Borel subset $U$ of $\mathbb{R}^d$. Let $\nu$ be a finite, not identically zero, non-atomic Borel measure on $U$ and $\rho$ be a Borel probability on $[0, \infty)$. Consider the measure $\theta = \nu \times \rho$ on $(S, \mathcal{S})$ where $S = U \times [0, \infty)$ and $\mathcal{S}$ is its Borel $\sigma$-field.

Let $(\Omega, \mathcal{F}, P)$ be a probability space on which are defined two families of random variables $\{\tau_n, n \geq 1\}$ and $\{\eta_n, n \geq 1\}$ such that

$(i)$ $\{\tau_n, n \geq 1\}$ and $\{\eta_n, n \geq 1\}$ are independent,

$(ii)$ $\tau_n$ are $[0, \infty)$ - valued, independent and identically distributed with $P(\tau_n > t) = e^{-t\theta(S)}$,

and

$(iii)$ $\eta_n := (\xi_n, r_n)$ are $S = U \times [0, \infty)$ - valued, independent and identically distributed with

$$P(\eta_n \in E) = \frac{\theta(E)}{\theta(S)}, \quad E \in \mathcal{S},$$

or equivalently,

$$P(\xi_n \in A, r_n \in B) = \frac{\nu(A)}{\nu(U)}.\rho(B)$$

for Borel subsets $A$ and $B$ of $U$ and $[0, \infty)$ respectively. Let

$$m(w) = \begin{cases} 0 & if \quad \tau_1(w) > 1 \\ \max & \{j : \tau_1(w) + \tau_2(w) + \ldots + \tau_j(w) \leq 1\} \text{ otherwise.} \end{cases}$$

A standard fact is that (see, e.g., Ikeda and Watanabe, 1981)

$\eta_i(w) = (\xi_i(w), r_i(w))$, $i = 1, 2, \ldots, m(w)$ is a realisation of a Poisson point process on $S = U \times [0, \infty)$ with intensity measure $\theta = \nu \times \rho$. Consequently, $(i)$ $\xi_i$, $\quad i = 1, 2, \ldots, m(w)$ is a realisation of a Poisson point process on $U$ with intensity measure $\nu$,

$(ii)$ $r_i$, which may be interpreted as the radii of closed balls around $\xi_i$, $\quad i = 1, 2. \ldots,$ are independent and identically distributed as $\rho$, and

$(iii)$ the radii are independent of the Poisson point process on $U$ mentioned in $(i)$ above.

The above provides an adequate framework for all the events living on a region $U$ for the Poisson Boolean model $(X, \lambda, \rho)$ described in Section 2 if we take $U \subseteq \mathbb{R}^d$ to be a bounded Borel set and $\nu$ to be $\lambda$-times the Lebesgue measure on $U$. Define, for $E \in \mathcal{S}$,

$$X(w, E) := \#\{i \leq m(w) : \eta_i(w) \in E\}. \qquad \ldots (3.1)$$

Let $\mathcal{N}$ be the class of those finite counting measures on $(S, \mathcal{S})$ which assign mass at most one to singletons. We endow $\mathcal{N}$ with the $\sigma$-field $\mathcal{B}$ generated by sets $\{N \in \mathcal{N} : N(E) = k\}$, $E \in \mathcal{S}$, $k = 0, 1, 2, \ldots.$

The non-atomicity of $\theta$ implies that, with probability 1, $X(w,\cdot) \in \mathcal{N}$. Also support$(X(w,\cdot)) = \{\xi_i(w) : i = 1,2,\ldots,m(w)\}$. It easily follows that the map $w \longmapsto X(w,\cdot)$ is measurable and induces a probability $P_\theta$ on $(\mathcal{N},\mathcal{B})$. We may regard the above percolation process as an $\mathcal{N}$-valued random variable defined on $(\Omega,\mathcal{F},P)$. However in the following we take $(\mathcal{N},\mathcal{B},P_\theta)$ as our probability space and the identity map on this space as our percolation process $X$.

## 4. Proof of Van den Berg-Kesten Inequality

We denote by $\{P_\theta : \theta \in \Theta\}$ the class of all probability measures on $(\mathcal{N},\mathcal{B})$ where $\Theta = \{\theta = \nu \times \rho : \nu$ is a not identically zero, non-atomic measure on $U$ and $\rho$ is a probability on $[0,\infty)\}$, and $P_\theta$ is as described in the previous section.

Lemma 4.1. *Consider the statistical structure*

$$(\mathcal{N}\times\mathcal{N}, \mathcal{B}\otimes\mathcal{B}, \{P_\theta \times P_\theta : \theta \in \Theta\})$$

*and let* $X(N_1,N_2) := N_1$, $Y(N_1,N_2) := N_2$ *and* $Z(N_1,N_2) := N_1 + N_2 = X(N_1,N_2) + Y(N_1,N_2)$. *Then, with probability* $1, Z \in \mathcal{N}$ *and*

*(a)* $Z$ *is a sufficient statistic for* $\{P_\theta \times P_\theta : \theta \in \Theta\}$, *and*

*(b) the conditional distribution of* $(X,Y)$ *given* $Z = N_0$ *is uniform on the set*

$$T_{N_0} = \{(N_1,N_2) : \quad N_1 \le N_0, \quad N_2 = N_0 - N_1\}. \qquad \ldots(4.1)$$

Here $N_1 \le N_0$ means $N_1(E) \le N_0(E)\ \forall\, E \in \mathcal{S}$, or equivalently, support$(N_1) \subset$ support$(N_0)$.

Proof. The non-atomicity of $\theta$ and the independence of $X$ and $Y$ imply that $Z \in \mathcal{N}$. For a partition $E_i$, $i = 1,2,\ldots,k$ of $S$ it is easily checked that

$$P_\theta \times P_\theta\ \{X(E_i) = l_i, Y(E_i) = m_i, \quad i = 1,2,\ldots,k \mid\mid Z\}$$

$$= \prod_{i=1}^{k} \left\{ \binom{l_i + m_i}{m_i} \left(\frac{1}{2}\right)^{l_i+m_i} 1_{Z(E_i)=l_i+m_i} \right\}.$$

The proof of (b) can be completed by usual extension arguments. That the conditional distribution is same for all $\theta \in \Theta$ implies (a). $\square$

Fix $N_0 \in \mathcal{N}$. Let $n = |\ \text{support}(N_0)|$; say, support$(N_0) = \{z_1,z_2,\ldots,z_n\}$. Now $N_1 \le N_0$ and $N_2 = N_0 - N_1$ imply that the supports of $N_1$ and $N_2$ are disjoint and their union is $\{z_1,z_2,\ldots,z_n\}$. Thus the conditional distribution of $(X,Y)$ given $Z = N_0$ assigns, independently, each $z_i$ to either support$(N_1)$ or support$(N_2)$ with probability $\frac{1}{2}$ each.

For $N \le N_0$ define

$$x_i^N = \begin{cases} 1 & \text{if} \quad \eta_i \in \text{ support}(N) \\ 0 & \text{if} \quad \eta_i \notin \text{ support}(N), \end{cases}$$

then $x^N = (x_1^N, x_2^N, \ldots, x_n^N) \in Q_n = \{0,1\}^n, x^{N_0} = (1,1,\ldots,1)$ and if $N_1 \leq N_0, N_2 = N_0 - N_1$ then $x^{N_1}$ is antipodal to $x^{N_2}$ in $Q_n$. Thus $T_{N_0}$ can be identified with

$$\hat{T}_{N_0} = \{(x, x^c) : x \in Q_n\}. \qquad \ldots (4.2)$$

Now fix $A, B \subset \mathcal{N}$ . For $N \in \mathcal{N}$ and $K \subset U$ define

$$[N]_K := \{N' \in \mathcal{N} : N'(E) = N(E) \ \ \forall \ E \in \mathcal{S}, E \subset K \times [0,\infty)\},$$

and

$$\begin{aligned} A \,\square\, B := \quad & \{N \in \mathcal{N} : \exists \text{disjoint } K, L \subset U \text{ s.t.} \\ & [N]_K \subset A \text{ and } [N]_L \subset B\}. \end{aligned} \qquad \ldots (4.3)$$

To ensure the $P_\theta$-measurability of $A \,\square\, B$ we assume that $K$ and $L$ are finite unions of rational rectangles, see Section 6. Now let $N_0 \in \mathcal{N}$ and define

$$\hat{A} = \{N : N \in A, N \leq N_0\}, \quad E = \{x^N : N \in \hat{A}\},$$

$$\hat{B} = \{N : N \in B, N \leq N_0\}, \quad F = \{x^N : N \in \hat{B}\},$$

and

$$\widehat{A \,\square\, B} = \{N : N \in A \,\square\, B, \ N \leq N_0\}. \qquad \ldots (4.4)$$

We have the following lemma.

LEMMA 4.2. $|E \,\square\, F| \geq |\widehat{A \,\square\, B}|$.

PROOF. Suppose $N \in \widehat{A \,\square\, B}$, i.e., $N \in A \,\square\, B$ and $N \leq N_0$. Fix disjoint $K, L \subset U$ s.t. $[N]_K \subset A$ and $[N]_L \subset B$. Thus $[N]_K \cap (N \leq N_0) \subset \hat{A}$ and $[N]_L \cap (N \leq N_0) \subset \hat{B}$. Consequently, $[x^N]_{K^*} \subset E$ and $[x^N]_{L^*} \subset F$ where $K^* = \{i : \xi_i \in K\}$ and $L^* = \{i : \xi_i \in L\}$. Hence $x^N \in E \,\square\, F$. $\square$

THEOREM 4.1. $P_\theta(A \,\square\, B) \leq P_\theta(A) P_\theta(B)$.

PROOF. Using notations of (4.2) and (4.4) we have

$$\begin{aligned} & P_\theta \times P_\theta \ (A \times B || Z = N_0) \\ & \quad = \quad P_\theta \times P_\theta \ \{(N_1, N_2) : N_1 \in \hat{A}, N_2 \in \hat{B}, N_1 \leq N_0, N_2 = N_0 - N_1 || Z = N_0\} \\ & \quad = \quad \frac{|E \cap F^c|}{2^n} \text{ by Lemma (4.1)} \\ & \quad \geq \quad \frac{|E \,\square\, F|}{2^n} \text{ by (2.2)} \\ & \quad \geq \quad \frac{|\widehat{A \,\square\, B}|}{2^n} \text{ by Lemma (4.2)} \\ & \quad \geq \quad P_\theta \times P_\theta \{(A \,\square\, B) \times \mathcal{N} || Z = N_0\}. \end{aligned}$$

Consequently,

$$P_\theta \times P_\theta(A \times B) \geq P_\theta(A \,\square\, B),$$

or equivalently,

$$P_\theta(A) P_\theta(B) \geq P_\theta(A \,\square\, B). \qquad \square$$

REMARK. The BK inequality can be shown to hold even in the case where $\theta$ is not of the form $\nu \times \rho$ but any finite measure on $S = U \times [0, \infty)$ with its $U$-marginal non-atomic.

## 5. **Examples and Applications**

In this section we give some examples and applications of Theorem 4.1. In subsections 5a, 5b and 5c we provide examples of the validity of the BK inequality in cases not accessible via earlier literature. Finally in subsection 5d we briefly describe how our techniques can be adapted to prove the BK inequality in some special cases of the Random-Connection Model.

5a. Consider Poisson Boolean percolation in the plane. Fix a bounded region $U \subset \mathbb{R}^2$. Let us say that a Poisson ball $B$ is isolated in $U$ if there exists $G$, a finite union of open rational rectangles, such that $B \subset G \subset U$ and $G$ does not intersect any other Poisson ball. Let $A$ be the event that there exists an isolated ball in $U$. Then $A \,\square\, A$ is the event that there are at least two isolated balls in $U$. By Theorem 4.1,

$$P(A \,\square\, A) \leq [P(A)]^2. \qquad \ldots (5.1)$$

It may be noticed that the event $A$ is neither increasing nor decreasing. We would like to say that the verification of the above inequality through direct calculations is not entirely trivial. This example was suggested by Damien G. White.

5b. For $p = (p_1, p_2, \cdots, p_n), 0 \leq p_i \leq 1$, let $P_p$ denote the product probability on $Q_n = \{0,1\}^n$ with $i^{th}$-component space having mass $p_i$ at 1 and $1 - p_i$ at 0. Clearly, for any $A \subset Q_n, P_p(A) = P_{p^c}(A^c)$ where $p^c = (1-p_1, 1-p_2, \cdots, 1-p_n)$. This duality implies that if the BK inequality holds for all increasing events $A, B$ then it as well holds for all decreasing events $A, B$.

There is no such apparent duality in the continuum case. Thus the result of Van den Berg (1996) for increasing events does not immediately lead to the same for decreasing events. However, Theorem 4.1 says that the inequality is valid provided both the decreasing events live on a bounded region $U$. This provides a rich class of examples where the BK inequality is available.

5c. Let us consider Continuum Percolation $(X, \lambda, \rho)$ in the plane. Here $X$ is a Poisson process with density $\lambda$ and $\rho$ is the radius distribution. We assume that $\rho$ is supported on the interval, say, $[\frac{1}{2}, 1]$. Let

T : = the rectangle with corners (20,5), (20,−5),(−20,−5), and (−20,5).

R : = the rectangle with corners (20,5), (20,−5), (15,−5), and (15,5).

M : = the rectangle with corners (5,5), (5,$-$5), ($-$5,$-$5), and ($-$5,5).
as illustrated in the figure 1 below.

Let

A be the event that there exists a N-S connection on T, that is, some point on the northern border of T is connected to some point on the southern border.

B be the event that there does not exist an E-W connection on T

C be the event that there does not exist an E-W connection on R and

D be the event that there exists a N-S connection on M. Then it is easily seen that,

$(i)$ A and B are not independent,

$(ii)$ $C \cap D \subset (A \,\square\, B) \subsetneq A \cap B$

and

$$0 < P(C \cap D) \leq P(A \,\square\, B) \leq P(A \cap B). \qquad \ldots (5.1)$$

It may be noticed that the result of Van den Berg (1996) is not applicable to this example.

5d. *(Random-Connection model).* We are given a Poisson point process $X$ on $\mathbb{R}^d$ and a so-called connection function $g : [0, \infty) \to [0, 1]$, which is assumed to be non-increasing. In the random-connection model, for all distinct pairs of points $x_i$ and $x_j$ of the point process $X$, we insert an edge between them with probability $g(|x_i - x_j|)$ independently of all other pairs of points of $X$, where $|\cdot|$ denotes the usual Euclidean distance.

Let us fix a Borel $U \subset \mathbb{R}^d$ and consider a Poisson point process on $S = U \times [0, \infty) \times [0, \infty) \times \cdots$ with intensity measure $\theta = \nu \times \rho \times \rho \times \cdots$ such that $\nu$ is a finite, not identically zero, non-atomic Borel measure on $U$ and $\rho$ is a probability on $[0, \infty)$ with distribution function $F$. Given a realisation, say $(\xi_i, r_{i1}, r_{i2}, \cdots), i = 1, 2, \cdots,$ of this process we insert an edge between $\xi_i$ and $\xi_j$

iff

$$|\xi_i - \xi_j| \leq r_{ij} + r_{ji} \qquad \ldots (5.2)$$

It is easily seen that, for this random-connection model,

$$g(x) = P(r_{ij} + r_{ji} \geq x) = 1 - (F * F)(x), \qquad \ldots (5.3)$$

where $F * F$ stands for the convolution of $F$ with itself. It may be observed that the corresponding analogue of Theorem 4.1 holds. Thus, for the random-connection model with $g$ given by (5.3), the BK inequality holds for events $A$ and $B$ living on $U$.

In a similar way, by considering Poisson point process on an enlarged space $S$, one can prove the validity of BK inequality for other connection functions $g$, e.g.,

$$g(x) = \Pi_{i=1}^{k}\{1 - (F_i * F_i)(x)\},$$

where $F_1, F_2, \ldots, F_k$ are probability distribution functions. We hope to return to this problem.

## 6. Measurability of $A \,\Box\, B$

We briefly outline a proof of the fact that for any $A, B \in \mathcal{B}$, $A \,\Box\, B$ is measurable for any probability $P_\theta$ on $(\mathcal{N}, \mathcal{B})$. We first observe that $(\mathcal{N}, \mathcal{B})$ is a *standard Borel space* – see Parthasarathy (1967), p. 133 for the definition. Indeed $\mathcal{N}_k = \{N \in \mathcal{N} : N(S) = k\}$ can be identified with the set of distinct $k$-tuples in the cartesian product $S^k$. As $\mathcal{N}_k \in \mathcal{B}$ and $\mathcal{N} = \bigcup_{k \geq 0} \mathcal{N}_k$, it is not difficult to see that $(\mathcal{N}, \mathcal{B})$ is standard Borel. If $K \subset S$ is Borel and $A \in \mathcal{B}$ then

$$\{(N, N') : N'(E) = N(E) \forall E \in \mathcal{S}, E \subseteq K \times [0, \infty], N \in A, N' \notin A\}$$

is a Borel subset of $\mathcal{N} \times \mathcal{N}$ and its projection to the first coordinate is precisely

$$\{N : N \in A, [N]_K \not\subset A\}.$$

Thus this set is *analytic* (see Parthasarathy (1967), p. 16) and hence measurable w.r.t. any probability $P_\theta$ on $(\mathcal{N}, \mathcal{B})$. In particular for any $A, B \in \mathcal{B}$ and Borel sets $K, L \subset S$ the set

$$\{N : [N]_K \subset A \text{ and } [N]_L \subset B\}$$

is $P_\theta$-measurable. Note that in our framework $S = U \times [0, \infty)$. Denote by $\mathcal{C}$ subsets of $S$ of the form $V \times [0, \infty)$ where $V$ is a rational rectangle intersected with $U$. Running $K, L$ over the countable class of disjoint pairs of finite unions of sets in $\mathcal{C}$ we observe that $A \,\Box\, B$, given by (4.3), is $P_\theta$-measurable.

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