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GOODNESS OF YATES-RAO PROCEDURE FOR RECOVERY OF INTER-BLOCK INFORMATION*

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SUMMARY. The paper is concerned with the estimation problem for a proper block design with recovery of inter-block information for which numerous procedures are available in the literature with very little known about their relative merits. A desirable property, termed here as goodness, of any such procedure is the natural requirement that it should not, under any circumstances, lead to loss of efficiency compared to the usual procedure without recovery of inter-block information. So far, this has been theoretically investigated only in the case of the Yates-Rao procedure and yielded fragmentary results. The purpose of the present paper is to supercede these results by providing the necessary and sufficient condition for goodness of the Yates-Rao procedure in the case of any proper block design excluding the rare cases where the application of the procedure without truncation of the estimated variance ratio can give rise to a negative weight for the inter-block estimator.

1. **Introduction**

Yates (1939, 1940) initiated the subject of recovery of inter-block information in the context of a cubic lattice (1939) and a BIBD (1940). His procedure was adopted to PBIBD by Nair (1944) and modified and extended by Rao (1947) to all proper block designs. The difference between Yates' and Rao's approach has been discussed by Sprott (1956) and Fraser (1957). A general approach synthesizing the two based on a canonical reduction to minimal sufficient statistics is given in Roy and Shah (1962) and Bhattacharya (1981). Both Yates' and Rao's procedures are based on estimates of intra-block and inter-block error variances by the ANOVA method to which numerous alternatives have been developed by others (see e.g., Tocher (1952), Roy and Shah (1962), Hartley and Rao (1967), Cunningham and Henderson (1968) as corrected in Thompson (1969), Nelder

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∗ This paper is dedicated to the memory of late Professor C.G. Khatri.

(1968), Patterson and Thompson (1971) and Bhattacharya (1981)). However, the motivation for recovery of inter-block information is to improve upon the customary intra-block analysis. It is, therefore, natural to require that the additional information from inter-block analysis should be used in such a way that under no possible circumstances it leads to estimators which would be worse than what we could obtain without using it. A procedure which satisfies this natural requirement will be referred to as good in this paper. Ad-hoc procedures which are good under appropriate conditions have been developed by many (see e.g., Graybill and Deal (1959), Seshadri (1963a,b), Shah (1964), Brown and Cohen (1974), Khatri and Shah (1974), Bhattacharya (1978, 1980, 1981) and Kubokawa (1987)) but very little has been done so far to examine the goodness of procedures based on various well-known general principles.

Yates (1939) was quite aware of the point raised above as is evident from his attempt to establish the goodness of his procedure by numerical integration in his very first paper on the subject. The procedure considered by him in this connection was, however, a simplified version of that commonly associated with his name and ignored the treatment component of the adjusted block sum of squares. For the selected design and selected parameter values, his calculations indicated that the procedure considered by him was good. Simulation studies conducted by some authors (namely, El-Shaarawi *et al.* (1975), Khatri and Shah (1975)) indicated similar results in the case of large designs considered in those studies not only for the Yates-Rao procedure but also for two other procedures, namely the maximum likelihood procedure (Roy and Shah (1962) and Hartley and Rao (1967)) and its modification due to Patterson and Thompson (1971), both derived as marginal procedures in El-Shaarawi *et al.* (1975). However, numerical or simulation studies are not quite satisfactory for the purpose since, leaving aside the question of accuracy, such studies can be carried out only for selected designs at selected parameter points.

Some attempts to resolve the problem theoretically has been made by Shah (1964), Bhattacharya (1978, 1981) and Kubokawa (1987) in the case of Yates-Rao procedure. While Shah (1964) considered the subclass of D_1-class designs (defined in that paper) given by linked block designs introduced by Youden (1951), Bhattacharya (1978) considered the complimentary subclass within D_1-class designs. Shah (1964) showed that for linked block designs (including symmetrical BIBD's), Yates-Rao procedure is good if and only if number of blocks exceeds six. There are many designs for which this condition fails (e.g., the symmetrical BIBD with four treatments and three replications; several others, which are not BIBD's can be found in Roy and Laha (1956)). For the complimentary subclass of D_1-class designs, Bhattacharya (1978) obtained separately (i) a sufficient condition and (ii) a necessary condition, which together resolved the problem for all asymmetrical BIBD's listed in Fisher and Yates (1963) with the exception of one, namely the BIBD with parameters $v = 5$, $b = 10$, $k = 2$, $r = 4$. His work revealed one more BIBD (asymmetrical BIBD with parameters $v = 4,\ b = 6,\ k = 2,\ r = 3$) for which the Yates-Rao procedure is not good. Bhattacharya (1981) unified and extended the earlier results to all proper block designs excluding the rare class for which either application of the Yates-Rao procedure without truncation of the estimated variance ratio can give rise to a negative weight for the inter-block estimator or the sum of the degrees of freedom for the inter-block error sum of squares and the dimension of the vector space of all treatment contrasts estimable from both intra-block and inter-block analyses is less than three. Kubokawa (1987) improved the sufficient condition in Bhattacharya (1978) in the case of a BIBD and resolved the question of goodness of the Yates-Rao procedure favourably in the case of the exceptional BIBD mentioned above.

In the present paper we derive the necessary and sufficient condition for goodness of the Yates-Rao procedure applied to any proper block design for which the Yates-Rao procedure without truncation of the estimated variance ratio cannot give rise to a negative weight to the inter-block estimator. Any proper block design for which the additional condition assumed in Bhattacharya (1981) are violated are shown to be not good. The restricted class of designs to which the necessary and sufficient condition applies includes all D_1-class designs and in particular all BIBD's. It also includes most of the PBIB designs listed in Clatworthy (1973).

2. **Preliminaries**

Consider a proper block design with n experimental units, divided into b blocks of size k each for comparing v treatments using design matrices $\boldsymbol{X}$ and $\boldsymbol{Z}$ for treatment and block factors, respectively. Let $\boldsymbol{N}$ and $\boldsymbol{r}$ denote the incidence matrix and replication vector, respectively. Following Tocher (1952) we shall write $\boldsymbol{r}^{\delta}$ ($\boldsymbol{r}^{\delta/2}$) to denote diagonal matrices with r_i ($r_i^{1/2}$) as diagonal elements. δ_{ij} will stand for Kronecker's delta. The customary model for analysis of a block design without recovery of inter-block information is given by

$$\boldsymbol{Y} \sim N(\boldsymbol{X\tau} + \boldsymbol{Z\beta}, \sigma^2\mathbf{I}) \qquad \ldots (2.1)$$

where $\boldsymbol{Y}$ = observed responses from experimental units and $\boldsymbol{\tau}$ = treatment effects, $\boldsymbol{\beta}$ = block effects and σ^2 are unknown. When recovery of inter-block information is contemplated one usually assumes an Eisenhart model III (Eisenhart (1947)) as in Yates (1939, 1940) and Rao (1947) where $\boldsymbol{\beta}$ is a realization of a random vector $\boldsymbol{b}$ such that $\boldsymbol{Y}$ given $\boldsymbol{b} = \boldsymbol{\beta}$ follows (2.1) and $\boldsymbol{b} \sim N(\mathbf{0}, \sigma_*^2\mathbf{I})$. Accordingly, one supplements the information on $\boldsymbol{\tau}$ in (2.1) by

$$\boldsymbol{B} \sim N(\boldsymbol{N'\tau}, k[1 + k\rho_*]\sigma^2\mathbf{I}) \qquad \ldots (2.2)$$

where $\boldsymbol{B}$ = block totals and $\rho_* = \sigma_*^2/\sigma^2$. Alternatively, one replaces (2.1) by

$$\boldsymbol{Y} \sim N(\boldsymbol{X\tau}, \sigma^2[\mathbf{I} + \rho_* \boldsymbol{ZZ}']), \quad \rho_* > 0. \qquad \ldots (2.3)$$

We assume here a generalization [Houtman and Speed (1983)] given by

$$\boldsymbol{Y} \sim N(\boldsymbol{X\tau}, \textstyle\sum_{i=1}^{3} \sigma_i^2 \boldsymbol{P}_i) \qquad \ldots (2.4)$$

where $\boldsymbol{P}_1 = \mathbf{I} - \boldsymbol{ZZ}'/k$, $\boldsymbol{P}_2 = \boldsymbol{ZZ}'/k - \mathbf{11}'/n$ and $\boldsymbol{P}_3 = \mathbf{11}'/n$. Observe that (2.3) is a special case of (2.4) with $\sigma_1^2 = \sigma^2$ and $\sigma_2^2 = \sigma_3^2 = \sigma^2(1 + k\rho_*) > \sigma^2$. For $i = 1, 2$, let $\boldsymbol{X}_i = \boldsymbol{R}_i'\boldsymbol{X}, \boldsymbol{Y}_i = \boldsymbol{R}_i'\boldsymbol{Y}$, where $\boldsymbol{R}_i$ is a semi-orthogonal matrix with column span identical with that of $\boldsymbol{P}_i$ and note that $\boldsymbol{Y}_i \sim N(\boldsymbol{X}'\boldsymbol{\tau}, \sigma_i^2\mathbf{I})$. It can be seen that for the purpose of estimating treatment contrasts (2.1), (2.2) and (2.4) are equivalent to the models of $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$ and ($\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$), referred to as models for intra-block, inter-block and combined analyses. We shall denote these three models by M_1, M_2 and M and the corresponding estimates of a treatment contrast θ by $\tilde{\theta}^{(1)}$, $\tilde{\theta}^{(2)}$ and $\tilde{\theta}$. S_i, e_i will denote the error SS and error df for M_i. Let Θ_1, Θ_2 and Θ denote the set of all treatment contrasts estimable from M_1, M_2 and M respectively and let $\Theta_* = \Theta_1 \bigcap \Theta_2$. Clearly, $\Theta = \text{span}(\Theta_1 \bigcup \Theta_2)$ is the space of all treatment contrasts. Furthermore, $\tilde{\theta}^{(1)}$ for $\theta \in \Theta_1 \bigcap \Theta_*^{\perp}$ and $\tilde{\theta}^{(2)}$ for $\theta \in \Theta_2 \bigcap \Theta_*^{\perp}$ are optimal in the sense of being minimum variance unbiased. Hence every $\theta \in \Theta \bigcap \Theta_*^{\perp}$ admits optimal estimation and the problem of recovery of inter-block information is essentially concerned with that of estimation of $\theta \in \Theta_*$ for which no optimum solution is known. According to Yates' approach (crudely interpreted), $\theta \in \Theta_*$ would be estimated by a linear combination of $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$. The resulting estimator would agree with $\tilde{\theta}$ given by Rao's approach, only for special cases of θ which, for obvious reasons, are of interest to us. The following lemma clarifies the matter and motivates the definition that follows.

LEMMA 2.1. *Let $\theta \in \Theta_*$. Then $\tilde{\theta} = (1 - \alpha)\tilde{\theta}^{(1)} + \alpha\tilde{\theta}^{(2)}$ for some α if and only if $\theta = \boldsymbol{p}'\boldsymbol{r}^{\delta/2}\boldsymbol{\tau}$ for an eigenvector $\boldsymbol{p}$ of the matrix $\boldsymbol{A} = \boldsymbol{r}^{-\delta/2}\boldsymbol{NN}'\boldsymbol{r}^{-\delta/2}$ and then, $\alpha = \lambda/[\lambda + (1 - \lambda)\rho]$, where λ is the eigenvalue of $\boldsymbol{A}$ corresponding to $\boldsymbol{p}$ and $\rho = \sigma_2^2/\sigma_1^2$.*

DEFINITION 2.1. A treatment contrast $\theta = \boldsymbol{q}'\boldsymbol{\tau}$ is called a basic contrast if $\boldsymbol{q} = \boldsymbol{r}^{\delta/2}\boldsymbol{p}$ for some eigenvector $\boldsymbol{p}$ of $\boldsymbol{A}$.

It can be seen that canonical contrasts defined in Roy and Shah (1962) as well as their extension in Bhattacharya (1981) are basic. It is also interesting to observe that in all instances of combining $\tilde{\theta}^{(1)}$ and $\tilde{\theta}^{(2)}$ for $\theta \in \Theta_*$ by Yates, θ was a basic contrast. The term 'basic contrast' was introduced by Pearce *et al.* (1974) where it refers to a contrast vector $\boldsymbol{q} \in \boldsymbol{R}^v$ rather than the corresponding treatment contrast $\boldsymbol{q}'\boldsymbol{\tau}$ and is used to clarify the computational methods of Tocher (1952), Kuiper and Corsten [Kuiper(1952) and Corsten (1958)] and, Wilkinson (1970) for intra-block and inter-block analyses in the general case of a block design. It can be seen that all eigenvalues of $\boldsymbol{A}$ lie in the closed interval [0,1] and that $\boldsymbol{r}^{\delta/2}\mathbf{1}_v$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue 1.

Let $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_v\}$ with $\boldsymbol{p}_v = n^{-1/2}\boldsymbol{r}^{\delta/2}\mathbf{1}_v$ be a complete set of orthonormal eigenvectors of $\boldsymbol{A}$ and let λ_i be the eigenvalue corresponding to the eigenvector $\boldsymbol{p}_i$. Let $\theta_i = \boldsymbol{p}_i'\boldsymbol{r}^{\delta/2}\boldsymbol{\tau},\ i = 1, \ldots, v-1$. It can be seen that if θ is a basic contrast belonging to the eigenspace of λ then, it is estimable from intra-block analysis if and only if $\lambda \neq 1$ and from inter-block analysis if and only if $\lambda \neq 0$. Let $I = \{i : \lambda_i \neq 1\}$, $J = \{i : \lambda_i \neq 0\}$ and $K = I \cap J$. Then it can be seen that

$$(\tilde{\theta}_i^{(1)})_{i \in I} \cup (\tilde{\theta}_j^{(2)})_{j \in J} \cup (S_1, S_2) \text{ is an array of independent elements such that } \tilde{\theta}_i^{(1)} \sim N(\theta_i, \tfrac{\sigma_1^2}{1-\lambda_i}),\ \tilde{\theta}_j^{(2)} \sim N(\theta_j, \tfrac{\sigma_2^2}{\lambda_j}) \text{ and } S_l \sim \sigma_l^2 \chi_{e_l}^2, \quad \ldots (2.5)$$

and is a minimal sufficient (but not complete) statistic for M [Graybill and Weeks (1959), Roy and Shah (1962) and Bhattacharya (1981)]. Since $(\theta_1 \ldots \theta_{v-1})$ is a basis for the space of all treatment contrasts consisting of basic contrasts only, we can modify Yates' approach by restricting its application to θ_i's and generating estimate of any other contrast from its linear representation in terms of θ_i's. This leads to a flexible class of procedures including variants of Rao's as well as others not covered by Rao's approach, by removing the constraint on α of Lemma 2.1 for combining intra-block and inter-block estimates of θ's. Let $\boldsymbol{W} = (W_j)_{j \in K}$ where $W_j = \lambda_j(1-\lambda_j)(\tilde{\theta}_j^{(2)} - \tilde{\theta}_j^{(1)})^2$. Note that

$$(S_1, S_2) \bigcup \boldsymbol{W} \text{ is an array of independent elements such that } W_j \sim [\lambda_j\sigma_1^2 + (1-\lambda_j)\sigma_2^2]\chi_1^2. \quad \ldots (2.6)$$

Then it can be seen that every procedure (with appropriate modification in the case of Yates') of estimating treatment contrasts with recovery of inter-block information proposed in the literature can be represented in the following way:

(i) It is determined by an array $\boldsymbol{g} = (g_i)_{i \in K}$ of measurable functions from R_+^{s+2} to R.

(ii) Estimate θ_i's by $\hat{\theta}_i = \begin{cases} \tilde{\theta}_i^{(1)} & \text{if} \quad i \in I - K \\ \tilde{\theta}_i^{(2)} & \text{if} \quad i \in J - K \\ \tilde{\theta}_i^{(1)} + (\tilde{\theta}_i^{(2)} - \tilde{\theta}_i^{(1)})g_i(S_1, S_2, \boldsymbol{W}) & \text{if } i \in K \end{cases}$.

(iii) Estimate an arbitrary treatment contrast $\theta = \boldsymbol{q}'\boldsymbol{\tau}$ by $\hat{\theta} = \boldsymbol{q}'\boldsymbol{r}^{-\delta/2}\sum_{i=1}^{v-1}\boldsymbol{p}_i\hat{\theta}_i$.

In fact most procedures (including Yates-Rao and others using Rao's approach but differing in the method of estimation for ρ but excluding that in the last paragraph of Stein (1966) and some in Brown and Cohen (1974)) proposed in the literature use

$$g_i = \lambda_i/[\lambda_i + (1-\lambda_i)g],\ i \in K \quad \ldots (2.7)$$

where g is a measurable function from R_+^{s+2} to R such that $g(S_1, S_2, \boldsymbol{W})$ is the prescribed estimator of ρ. For $i \in K$, let $\boldsymbol{T}_i = (T_{i,j})_{j \in K},\ i \in K$ where

$$T_{ij} = W_j + \delta_{ij}[\lambda_j\sigma_1^2 + (1-\lambda_j)\sigma_2^2]V_* \text{ with } V_* \parallel (S_1, S_2, \boldsymbol{W}) \text{ and } V_* \sim \chi_2^2, \quad \ldots (2.8)$$

$$G_i = g_i(S_1, S_2, \boldsymbol{T}_i) \text{ and } \gamma_i = \lambda_i/[\lambda_i + (1-\lambda_i)\rho]. \qquad \ldots (2.9)$$

Then the following is a basic result which is applicable to all procedures proposed in the literature.

THEOREM 2.1. *Let* $\{i, j\} \subseteq K$. *Then*

(i) $E(\hat{\theta}_i - \theta_i)^2 < \infty \Leftrightarrow EG_i^2 < \infty \Rightarrow E\hat{\theta}_i = \theta_i$.

(ii) $\max(EG_i^2, EG_j^2) < \infty \Rightarrow \mathrm{cov}(\hat{\theta}_i, \hat{\theta}_j) = \delta_{ij} V(\tilde{\theta}_i^{(1)})[1 + E(G_i^2/\gamma_i - 2G_i)]$.

PROOF. Easy noting that $E[f(S_1, S_2, \boldsymbol{W})W_i] = EW_i E[f(S_1, S_2, \boldsymbol{T}_i)]$ for any f [as in Lemma 2.2 of Brown and Cohen (1974) or statement (2.5) of Khatri and Shah (1974)] by the identity "$xh_1(x) = h_3(x)$" where h_m stands for the chi-square density with m degrees of freedom and using Stein's theorem [Stein (1950)] [as in Lemma 6.1 of Roy and Shah (1962)].

In the following we shall refer to the procedure determined by $\boldsymbol{g}$ as $\boldsymbol{g}$. The procedure $\boldsymbol{g} = \boldsymbol{0}$ will be referred to as $\boldsymbol{g}_0$. $G_i^{(j)}$ will denote the value of G_i when $\boldsymbol{g} = \boldsymbol{g}_j$. We then have, from Theorem 2.1, the following general theorem for comparing two procedures.

THEOREM 2.2. $\boldsymbol{g}_1$ *is better than* $\boldsymbol{g}_2$ *for estimating* $\boldsymbol{\theta} = \boldsymbol{q}'\boldsymbol{\tau} \in \Theta$ *iff* (A_i): $E(G_i^{(1)} - \gamma_i)^2 \leq E(G_i^{(2)} - \gamma_i)^2, \forall i \in K_q = \{i : \boldsymbol{q}'\boldsymbol{r}^{-\delta/2}\boldsymbol{p}_i \neq 0\}$. $\boldsymbol{g}_1$ *is better than* $\boldsymbol{g}_2$ *iff* (A_i) *holds* $\forall i \in K$.

We now introduce :

DEFINITION 2.2. The procedure $\boldsymbol{g}$ is good for $\theta \in \Theta$ if $\boldsymbol{g}$ is better than $\boldsymbol{g}_0$ for estimating θ. $\boldsymbol{g}$ is good if $\boldsymbol{g}$ is good $\forall \theta \in \Theta$.
Let

$$H_i = G_i/\gamma_i \text{ and } \nu_i = \inf_{\gamma_i \in (0,1)} EH_i/EH_i^2, \; i \in K. \qquad \ldots (2.10)$$

Then, Theorem 2.2 implies

THEOREM 2.3. $\boldsymbol{g}$ *is good for* $\theta = \boldsymbol{q}'\boldsymbol{\tau} \in \Theta$ *iff* (B_i): $\sup_{\gamma_i \in (0,1)} EH_i^2 < \infty$ *and* $\nu_i \geq \frac{1}{2}$ $\forall i \in K_q$. $\boldsymbol{g}$ *is good iff* (B_i) *holds* $\forall i \in K$.

We conclude this section by quoting an inequality from Bhattacharya (1984).

THEOREM 2.4. *Suppose* u, v, w *are functions of random variables* $x_1, \cdots, x_n$ *such that* v *is positive with a finite expectation and* $E(wv) > 0$. *Let '*$E_r f$*'* ($r = 1 \ldots n$)*, denote the conditional expectation of* f *given* $(x_1, \cdots, x_r)$. *Let '*f *SD* h $|x_r$*' mean that either both* f *and* g *are non-decreasing in* x_r *or both are non-increasing in* x_r. *Let* $f_r = E_r(wv)/E_r v$ *and* $h_r = E_r u/E_r v$. *Then*

(i) f_r *SD* h_r $|x_r$ $\forall r \leq n \Rightarrow E(wu)/E(wv) \geq Eu/Ev$

(ii) f_r *OD* h_r $|x_r$ $\forall r \leq n \Rightarrow E(wu)/E(wv) \leq Eu/Ev$

3. Results

For the Yates-Rao procedure the prescribed estimator of ρ is that obtained by equating the error SS and adjusted block SS in the ANOVA for (2.1) to their respective expectations under (2.3) or 1, whichever is larger [since $\rho \geq 1$ under (2.3) assumed by them]. The two SS involved can be easily seen to be same as S_1 and $S_2 + \sum_{i\in K} W_i$ respectively. Hence using (2.5) and (2.6) the prescribed estimator of ρ is

$$g = \max(1, g_*) \text{ with } g_* = \frac{-\lambda_0 S_1 + e_1(S_2 + \sum_{i\in K} W_i)}{(e_2 + s - \lambda_0)S_1} \qquad \ldots(3.1)$$

where $\lambda_0 = \sum_{i\in K} \lambda_i$ and $s = \#[\mathrm{K}]$. From (2.7) along with (3.1) g_i's for the Yates-Rao procedure are given by

$$g_i = 1/\max(\lambda_i^{-1}, b_i + c_i(S_2 + \textstyle\sum_{j\in K} W_j)/S_1) \qquad \ldots(3.2)$$

where

$$b_i = 1 - (\lambda_i^{-1} - 1)\lambda_0/(e_2 + s - \lambda_0) \text{ and } c_i = (\lambda_i^{-1} - 1)e_1/(e_2 + s - \lambda_0) \quad \ldots(3.3)$$

Let $V_1 = S_1/\sigma_1^2,\ V_2 = S_2/\sigma_2^2,$ and $V_{ij} = T_{ij}/[\lambda_j\sigma_1^2 + (1-\lambda_j)\sigma_2^2],\ j \in K$. From (2.8) note that $\sum_{j\in K} V_{ij}$ does not depend on i and let

$$V_0 = V_2 + \textstyle\sum_{j\in K} V_{ij},\ V = V_0/V_1,\ U_{ij} = V_{ij}/V_0 \text{ and } U_{i*} = \textstyle\sum_{j\in K} \lambda_j U_{ij} \quad \ldots(3.4)$$

Then from (2.10) along with (2.9) and (3.2), we have

$$H_i = 1/\max(\alpha_i, R_i^{-1}) \qquad \ldots(3.5)$$

where

$$R_i = 1/[dV(1-U_{i*})(1-\alpha_i) + \{b_i\lambda_i + dV(1-\lambda_i)\}\alpha_i] \qquad \ldots(3.6)$$

with

$$d = e_1/(e_2 + s - \lambda_0) \text{ and } \alpha_i = \gamma_i/\lambda_i. \qquad \ldots(3.7)$$

From (3.4) along with (2.5), (2.6) and (2.8) note that

$$V \parallel (U_{ij})_{j\in K},\ V \sim f_{e_2+s+2,e_1} \text{ and } (U_{ij})_{j\in K} \sim d_s(\frac{1}{2}\mathbf{1'} + \mathbf{e'}_i, \frac{1}{2}e_2) \quad \ldots(3.8)$$

where $\mathbf{e}_i$ stands for the i-th column of $\mathbf{I}_s$. We shall use the following conditions:

(C): $e_2 + s \geq 3$ and (D$_i$): $\gamma_i \geq \gamma_0/(e_2 + s)$ for $i \in K$.
We first prove two lemmas which are applicable to any proper block design.

LEMMA 3.1. $Sup_{\gamma_i \in (0,1)} EH_i^2 < \infty \Rightarrow (C)$.

PROOF. From (3.5) and (3.6) note that

$$H_i \to F_i \text{ a.s. where } F_i = [dV(1 - U_{i*})]^{-1}. \qquad \ldots (3.9)$$

Hence, if (C) does not hold then using (3.8) and Fatou's lemma $\sup_{\gamma_i \in (0,1)} EH_i^2 \geq \liminf_{\gamma_i \to 0} EH_i^2 \geq EF_i^2 = \infty$.

LEMMA 3.2. $EH_i/EH_i^2 \geq 1 \ \forall \gamma_i \in [\lambda_i, 1)$.

PROOF. Follows from $0 < H_i \leq 1 \ \forall \alpha_i \geq 1$.

From now on, unless otherwise stated, we assume that both (C) and (D$_i$) hold and consider values of γ_i in $(0, \lambda_i)$ so that $\alpha_i \in (0, 1)$. From (3.3) note that

$$(\mathrm{D}_i) \Leftrightarrow b_i \geq 0 \Rightarrow R_i \geq 0 \Rightarrow H_i = \min(\alpha_i^{-1}, R_i). \qquad \ldots (3.10)$$

LEMMA 3.3. $\{H_i^2 : \alpha_i \in (0, 1)\}$ *is uniformly integrable.*

PROOF. From (3.10) and (3.6) note that (D$_i$) $\Rightarrow H_i \leq H_{i*} \ \forall \alpha_i \in (0, 1)$ where $H_{i*} = [dV \min(1 - U_{i*}, 1 - \lambda_i)]^{-1}$ is integrable by (3.8) along with (C).

LEMMA 3.4. $EF_i > ER_i \ \forall \alpha_i \in (0, 1)$.

PROOF. We shall prove that *(i)* $E(F_i/R_i) > 1$ and *(ii)* $EF_i/ER_i > E(F_i/R_i)$ from which the desired result is obvious. To prove (i) note that $F_i/R_i = 1 + (D_i - 1)\alpha_i$ where $D_i = [1 - \lambda_i + b_i\lambda_i/(dV)]/(1 - U_{i*})$ and then using (3.3) and (3.8) note also that $ED_i = [1 - \lambda_0/(e_2 + s)]E(1 - U_{i*})^{-1} > 1$ since $E(1 - U_{i*})^{-1} \geq (1 - EU_{i*})^{-1}$ and by (3.8) and (D$_i$), $EU_{i*} = (\lambda_0 + 2\lambda_i)/(e_2 + s + 2) \geq \lambda_0/(e_2 + s)$. To prove (ii) apply Theorem 2.4 with $n = 2$, $x_1 = U_{i*}$, $x_2 = V$, $u = F_i/R_i$, $v = 1$, $w = R_i$. We have $f_1 = E(R_i|U_{i*}) \uparrow U_{i*}$, $h_1 = E(F_i/R_i|U_{i*}) \uparrow U_{i*}$, $f_2 = R_i \downarrow V$ and $h_2 = F_i/R_i \downarrow V$ which, by that theorem, imply (ii).

LEMMA 3.5. $\inf_{\alpha_i \in (0,1)} [EH_i/EH_i^2] = \inf_{\alpha_i \in (0,1)} ER_i/ER_i^2 = EF_i/EF_i^2$.

PROOF. By (3.9) and Lemma 3.3, we have $\inf_{\alpha_i \in (0,1)} [EH_i/EH_i^2] \leq \lim_{\alpha_i \to 0} [EH_i/EH_i^2] = EF_i/EF_i^2$. To complete the proof, we shall prove that $EH_i/EH_i^2 \geq ER_i/ER_i^2 \geq EF_i/EF_i^2$ which implies the reverse of the preceeding inequality. To prove the first inequality in the above statement apply Theorem 2.4 with $n = 1$, $x_1 = R_i$, $u = R_i$, $v = R_i^2$, $w = H_i/R_i$. Note that $f_1 = H_i/R_i = \min(R_i^{-1}\alpha_i^{-1}, 1) \downarrow R_i$ and $h_1 = R_i/R_i^2 = R_i^{-1} \downarrow R_i$. Hence by that theorem, $EH_i/E(H_iR_i) \geq ER_i/ER_i^2$ which implies the desired inequality since $R_i \geq H_i$ by (3.10). To prove the second inequality in that statement, we shall prove *(i)* $ER_i^2/E(F_iR_i) \leq ER_i/EF_i$ and *(ii)* $EF_i^2/E(F_iR_i) \geq EF_i/ER_i$ which imply the desired inequality in view of Lemma 3.4. To prove (i) and (ii) apply Theorem 2.4 with $n = 2$, $x_1 = U_{i*}$, $x_2 = V$, $u = R_i$, $v = F_i$, $w = R_i$ in the case of (i), and $n=2,\ x_1=U_{i*},\ x_2=V,\ u=F_i,\ v=R_i,\ w=F_i$ in the case of (ii). In the first case, $f_1=E(R_iF_i|U_{i*})/(F_i|U_{i*}) = E[(dV)^{-1}R_i|U_{i*}]/E(dV)^{-1} \uparrow U_{i*}$, $h_1=E(R_i|U_{i*})/E(F_i|U_{i*}) = E[(dV)^{-1}R_i/F_i|U_{i*}]/E(dV)^{-1} \downarrow U_{i*}$, $f_2=R_i \downarrow V$ and $h_2 = R_i/F_i \uparrow V$. In the second case, $f_1 = E(F_iR_i|U_{i*})/E(R_i|U_{i*}) = E[(dV)^{-1}R_i|U_{i*}]/E[(dV)^{-1}R_i/F_i|U_{i*}] \uparrow U_{i*}$, $h_1=E(F_i|U_{i*})/E(R_i|U_{i*}) \uparrow U_{i*}$, $f_2=F_i \downarrow V$ and $h_2=F_i/R_i \downarrow V$. Hence, (i) and (ii) hold by that theorem.

REMARKS 3.1. It was proved in Bhattacharya (1984) that if $f_\alpha=1/[\alpha x_1+(1-\alpha x_2)]$ where x_1 and x_2 are independent random variables such that $\max(Ex_1^{-2}, Ex_2^{-2}) < \infty$ and $g(\alpha) = Ef_\alpha/Ef_\alpha^2$, then $\inf_{\alpha\in(0,1)} g(\alpha) = \min[g(0), g(1)]$. Although the conjecture that the above result holds without the assumption of independence still appears to be hard, the preceding Lemma is a nontrivial example of the validity of the above result without the assumption of independence.

Let $A_0=E(dV)^{-1}/E(dV)^{-2}$, $\psi_i=E(1-U_{i*})^{-1}/E(1-U_{i*})^{-2}$ and note that

$$A_0=\begin{cases} [e_1/(e_2+s-\lambda_0)][(e_2+s-2)/(e_1+2)) & \text{if } e_2+s\geq 3 \\ 0 & \text{otherwise} \end{cases}$$

Combining Lemmas 3.2 and 3.5, we have $\min(A_0\psi_i, 1) \leq \nu_i \leq A_0\psi_i$. Hence by Theorem 2.3 along with Lemma 3.1. we have

THEOREM 3.1. (i) *For any proper block design and any $\theta \in \Theta$ a necessary condition for Yates-Rao procedure to be good for θ is that (C) holds.*

(ii) *If $\theta=\boldsymbol{q}'\boldsymbol{\tau} \in \Theta$ and* (D_i) *holds $\forall i \in K_q$, then Yates-Rao procedure is good for θ iff* (E_i) : $A_0\psi_i \geq \frac{1}{2}\ \forall i \in K_q$

(iii) *If* (D_i) *holds $\forall i \in K$, then Yates-Rao procedure is good iff* (E_i) *holds $\forall i \in K$*

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