Use of Bernstein Polynomials in Numerical Solutions of Volterra Integral Equations

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Abstract

In this paper the Bernstein polynomials are used to approximate the solutions of linear Volterra integral equations. Both second and first kind integral equations, with regular, as well as weakly singular kernels, have been considered.

Keywords: Volterra integral equation; Abel's integral equation; Bernstein polynomials

1.   Introduction

Bernstein polynomials have been recently used for the solution of some linear and non-linear differential equations, both partial and ordinary, by Bhatta and Bhatti [1] and Bhatti and Bracken [2]. Also these have been used to solve some classes of inegral equations of both first and second kinds, by Mandal and Bhattacharya [3]. These were further used to solve a Cauchy singular integro-differential equation by Bhattacharya and Mandal [4].

In this paper we have developed a very simple method to solve Volterra integral equations of both first and second kinds and having regular as well as weakly singular kernels, using Bernstein polynomials. Bernstein polynomials can be defined on some interval $[a, b]$ by,

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \quad i = 0, 1, 2, \ldots, n. \qquad (1.1)$$

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These polynomials form a partition of unity, that is $\sum_{i=0}^{n} B_{i,n}(x) = 1$, and can be used for approximating any function continuous in $[a, b]$.

Volterra integral equations arise in many problems pertaining to mathematical physics like heat conduction problems. Various methods are available in the literature concerning their numerical solutions. Recently chebychev polynomials were used by Maleknejad *et.al* [5], to solve certain Volterra integral equations with regular kernel, other methods include the Taylor series expansion (cf. Maleknejad and Aghazadeh [6]) and the theory of wavelets (cf. Maleknejad *et.al* [7]).

In this paper we have developed a simple method, based on approximation of the unknown function on the Bernstein polynomial basis, for the solution of Volterra integral equations with regular kernels, as well as weakly singular kernels, that is Abel's integral equation.

Abel's integral equations possess weakly singular kernels of the type $(x - t)^{-\alpha}$, $0 < \alpha < 1$. Although analytical solution of Abel's integral is very well known, yet the numerical solution is not so well pronounced due to some computational difficulties which arise due to the presence of the differential operator in the solution (cf. Golberg and Chen [8], p.27). However, the present method avoids any such computational difficulty, and uses a very direct algorithm for computation of the unknown function.

2.   The General Method

A.   Volterra integral equations with regular kernels

We consider the integral equation of the first kind given by,

$$\int_{a}^{x} k(x,t)\phi(t)dt = f(x), \quad a < x < b, \tag{2.1}$$

where $\phi(t)$ is the unknown function to be determined, $k(x, t)$, the kernel, is a continuous and square integrable function, $f(x)$ being the known function satisfying $f(a) = 0$.

To determine an approximate solution of (2.1), $\phi(t)$ is approximated in the Bernstein polynomial basis on $[a, b]$ as

$$\phi(t) = \sum_{i=0}^{n} a_i B_{i,n}(t) \tag{2.2}$$

where $a_i(i = 0, 1, \ldots, n)$ are unknown constants to be determined. Substituting (2.2) in (2.1) we obtain,

$$\sum_{i=0}^{n} a_i \alpha_i(x) = f(x), \quad a < x < b \tag{2.3}$$

where,

$$\alpha_i(x) = \int_a^x k(x,t) B_{i,n}(t) dt. \tag{2.4}$$

We now put $x = x_j$, $j = 0, 1, \ldots, n$ in (2.3), x_j's being chosen as suitable distinct points in (a, b), and x_0 is taken near a and x_n near b such that $a < x_0 < x_n < b$. Putting $x = x_j$ we obtain the linear system

$$\sum_{i=0}^n a_i \alpha_{ij} = f_j \qquad j = 0, 1, \ldots, n \tag{2.5}$$

where

$$\alpha_{ij} = \alpha_i(x_j) \qquad i, j = 0, 1, \ldots, n \tag{2.6}$$

and

$$f_j = f(x_j) \qquad j = 0, 1, \ldots, n. \tag{2.7}$$

The linear system (2.5) can be easily solved by standard methods for the unknown constants a_i's. These $a_i (i = 0, 1, \ldots, n)$ are then used in (2.2) to obtain the unknown function $\phi(t)$ approximately.

We now consider the second kind Volterra integral equation, given by,

$$c(x)\phi(x) + \int_a^x k(x,t)\phi(t) dt = f(x), \quad a < x < b, \tag{2.8}$$

where $k(x, t)$ is a regular kernel, $c(x)$, $f(x)$ are known functions, then applying the same procedure as described above, we obtain

$$\sum_{i=0}^n a_i \beta_i(x) = f(x) \tag{2.9}$$

where

$$\beta_i(x) = c(x) B_{i,n}(x) + \int_a^x k(x,t) B_{i,n}(t) dt. \tag{2.10}$$

Choosing x_j's $(j = 0, 1, \ldots, n)$ as described above we obtain the linear system

$$\sum_{i=0}^n a_i \beta_{i,j} = f_j, \quad j = 0, 1, \ldots, n \tag{2.11}$$

where

$$\beta_{ij} = \beta_i(x_j) \quad i, j = 0, 1, \ldots, n \tag{2.12}$$

and

$$f_j = f(x_j), \quad j = 0, 1, \ldots, n. \tag{2.13}$$

The system (2.11) is solved to obtain the unknown constants $a_i (i = 0, 1, \ldots, n)$ which are then used to obtain the unknown function $\phi(t)$.

B.   Volterra integral equations with weakly singular kernels

We consider the weakly singular integral equation of the first kind, that is the Abel's integral equation given by,

$$\int_a^x \frac{\phi(t)}{(x-t)^\alpha}dt = f(x), \quad a < x < b \tag{2.14}$$

with $0 < \alpha < 1$ and $f(a) = 0$. This is a Volterra integral equation with a weakly singular kernel. The analytical solution of (2.14) is well known (cf. Estrada and Kanwal [9])and is given by

$$\phi(t) = \frac{\sin\alpha\pi}{\pi} \frac{d}{dt}\left[\int_a^x \frac{f(x)}{(t-x)^{(\alpha-1)}}dx\right]. \tag{2.15}$$

Since $f(a) = 0$, (2.15) can be simplified to the form

$$\phi(t) = \frac{\sin\alpha\pi}{\pi} \int_a^x (t-x)^{(\alpha-1)} f'(x)dx. \tag{2.16}$$

In order to obtain a numerical solution of (2.14) we approximate $\phi(t)$ as in (2.2). Thus (2.14) reduces to

$$\sum_{i=0}^n a_i \binom{n}{i} \frac{1}{(b-a)^n} \sum_{l=0}^n d_l^{i,n} \int_a^x \frac{t^l}{(x-t)^\alpha}dt = f(x), \quad a < x < b \tag{2.17}$$

where,

$$d_l^{i,n} = \sum_s (-1)^{l-s} \binom{i}{s}\binom{n-i}{l-s} \quad l, i = 0, 1, \ldots, n, \tag{2.18}$$

the summation over s being taken as follows : for $i < n < n - i$, (i) $s = 0$ to l for $l \leq i$, (ii) $s = 0$ to i for $i < l \leq n - i$, (iii) $s = l - (n-i)$ to $n-i$ for $n - i < l \leq n$ while for $i = n - i$ (n being an even integer) (i) $s = 0$ to l for $k \leq i$, (ii) $s = l - i$ to i for $i < l \leq n$; for $i > n - i$, i and $n - i$ above are to be interchanged. For $\alpha = 1/2$, (2.17) can be written as

$$\sum_{i=0}^n a_i \gamma_i(x) = f(x) \tag{2.19}$$

where

$$\gamma_i(x) = \sum_{l=0}^n d_l^{i,n} \pi^{1/2} \frac{\Gamma(1+l)}{\Gamma(\frac{3}{2}+l)} x^{l+\frac{1}{2}}. \tag{2.20}$$

For proper choice of the distinct points $x = x_j (j = 0, 1, \ldots, n)$ in (a, b) we obtain the linear system,

$$\sum_{i=0}^n a_i \gamma_{ij} = f_j \quad j = 0, 1, \ldots, n \tag{2.21}$$

where

$$\gamma_{ij} = \gamma_i(x_j), \quad i, j = 0, 1, \ldots, n \tag{2.22}$$

$$f_j = f(x_j), \quad j = 0, 1, \ldots, n. \tag{2.23}$$

The linear system (2.22) can be solved to obtain the unknown constants $a_i (i = 0, 1, \ldots, n)$, which are then used to the approximate the unknown function $\phi(t)$.

We next consider a Volterra integral equation of the second kind with weakly singular kernel given by

$$\phi(x) + \lambda \int_a^x \frac{\phi(t)}{(t-x)^\alpha} dt = f(x), \quad 0 < \alpha < 1, \quad a < x < b, \tag{2.24}$$

λ being a constant.

Approximating $\phi(t)$ as before we obtain

$$\sum_{i=0}^n a_i \mu_i(x) = f(x) \tag{2.25}$$

where for $\alpha = 1/2$,

$$\mu_i(x) = B_{i,n}(x) + \lambda \sum_{l=0}^n d_l^{i,n} \pi^{1/2} \frac{\Gamma(1+l)}{\Gamma(\frac{3}{2}+l)} x^{l+\frac{1}{2}}. \tag{2.26}$$

For suitable choice of $x = x_j$, we get the linear system

$$\sum_{i=0}^n a_i \mu_{ij} = f_j \quad j = 0, 1, \ldots, n \tag{2.27}$$

where

$$\mu_{ij} = \mu_i(x_j) \quad i, j = 0, 1, \ldots, n \tag{2.28}$$

and

$$f_j = f(x_j), \quad j = 0, 1, \ldots, n. \tag{2.29}$$

Solving the linear system (2.27) for $a_i (i = 0, 1, \ldots, n)$ we obtain the unknown function $\phi(t)$ ultimately.

3.   Illustrative Examples

Here we illustrate the above mentioned methods with the help of eight illustrative examples, which include two first kind and two second kind Volterra integral equations with regular kernels and three first kind and one second kind Volterra integral equation with weakly singular kernels.

Example 1

We consider the Volterra integral equation of the first kind given by,

$$\int_0^x \frac{\phi(t)}{x^2 + t^2} dt = x, \qquad 0 < x < 1 \tag{3.1}$$

which has the exact solution

$$\phi(x) = \frac{4}{4 - \pi} x^2. \tag{3.2}$$

Using the method illustrated in the section (2A) we solve the linear system

$$\sum_{i=0}^n a_i \alpha_{ij} = f_j, \qquad j = 0, 1, \ldots, n \tag{3.3}$$

where now

$$\alpha_{ij} = \int_0^{x_j} \frac{B_{i,n}(t)}{x_j^2 + t^2} dt$$

and

$$f_j = x_j.$$

We choose

$$x_0 = 10^{-10}, \quad x_i = x_0 + \frac{i}{n+1}, \qquad i = 0, 1, \ldots, n, \tag{3.4}$$

so that $0 < x_0 < x_1 < \cdots < x_n < 1$.

For $n = 7$, we solve the linear system (3.3) and obtain $a_i (i = 0, 1, \ldots, 7)$. Using these in the expansion of $\phi(t)$ given by (2.2), and choosing $n = 7$ the approximate solution for $\phi(t)$ is obtained. Other values of n can also be chosen. A plot of the absolute error between the exact solution and the approximate solution for different values of x is depicted in Figure 1. This plot shows that the error is of the order of 10^{-12}.
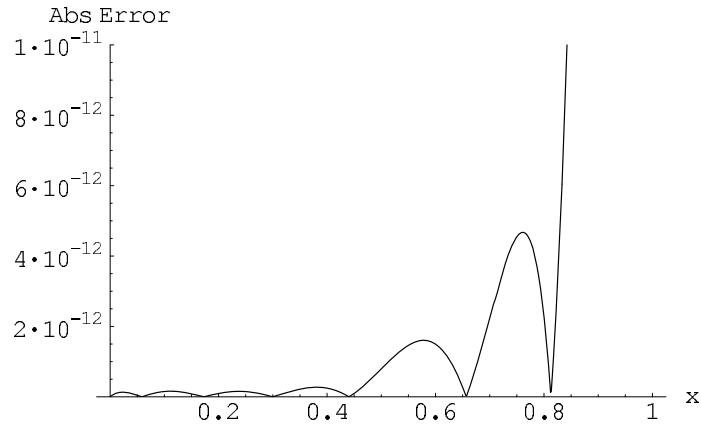
Fig 1 Absolute error between exact and approximate solutions of Eq.(3.1)

Example 2

We consider another first kind integral equation with a regular kernel given by,

$$\int_0^x \frac{\phi(t)}{(x^2 + t^2)^{\frac{1}{2}}} dt = x, \quad 0 < x < 1 \tag{3.5}$$

whose exact solution is

$$\phi(x) = \frac{x}{2^{1/2} - 1}. \tag{3.6}$$

Following the procedure described above we obtain the linear system

$$\sum_{i=0}^n a_i \alpha_{ij} = f_j, \quad j = 0, 1, \dots, n \tag{3.7}$$

where now,

$$\alpha_{ij} = \int_0^{x_j} \frac{B_{i,n}(t)}{(x_j^2 + t^2)^{\frac{1}{2}}} dt, \quad i, j = 0, 1, \dots, n$$

and

$$f_j = x_j, \quad j = 0, 1, \dots, n.$$

We choose x_j's to be the same as given by (3.4).

For $n = 7$ we solve the linear system (3.7) and obtain $a_0, \dots, a_7$. These when substituted in the expansion (2.2) for $n = 7$ produce $\phi(t)$ approximately.

In Figure 2, the plot of the absolute error between the exact solution and the approximate solutions shows that accuracy is of the order of 10^{-14}.
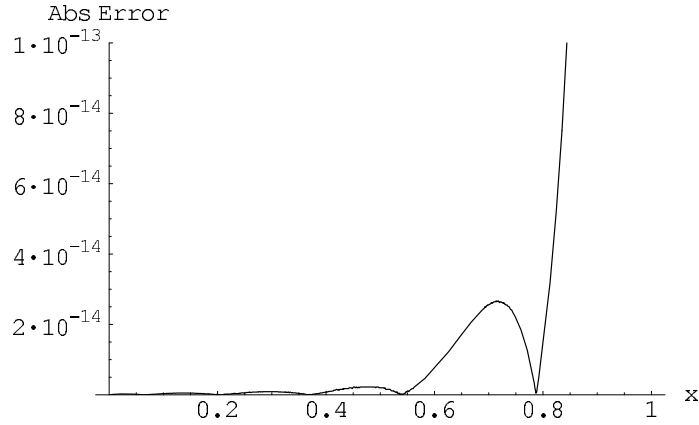
Fig 2 Absolute error between exact and approximate solutions of Eq.(3.5)

Example 3

We consider the second kind Volterra integral equation,

$$\phi(x)+\int_0^x (t^2-3x^3)\phi(t)dt = \frac{1}{4}[4x^3+x-1]e^{-2x}+\frac{3}{8}[1-2x^2], \quad 0<x<1 \quad (3.8)$$

whose exact solution is

$$\phi(x) = xe^{-2x} \quad (3.9)$$

(cf. Polyanin and Manzhirov [10]).

Using the method described in section (2A) we obtain the linear system

$$\sum_{i=0}^n a_i\beta_{ij} = f_j, \quad j=0,1,\ldots,n \quad (3.10)$$

where

$$\beta_{ij} = B_{i,n}(x_j) + \int_0^{x_j}(t^2-3x_j^2)B_{i,n}(t)dt, \quad i,j=0,1,\ldots,n$$

and

$$f_j = \frac{1}{4}[4x_j^3+x_j-1]e^{-2x_j}+\frac{3}{8}[1-2x_j^2], \quad j=0,1,\ldots,n.$$

The linear system (3.10) when solved gives the unknown constants $a_i(i=0,1,\ldots,n)$ from which approximate value of $\phi(t)$ is obtained by using (2.2).

For $n=13$ Figure 3 gives a plot of the absolute error between the exact and approximate solutions. This figure shows that error is of the order of 10^{-13}.
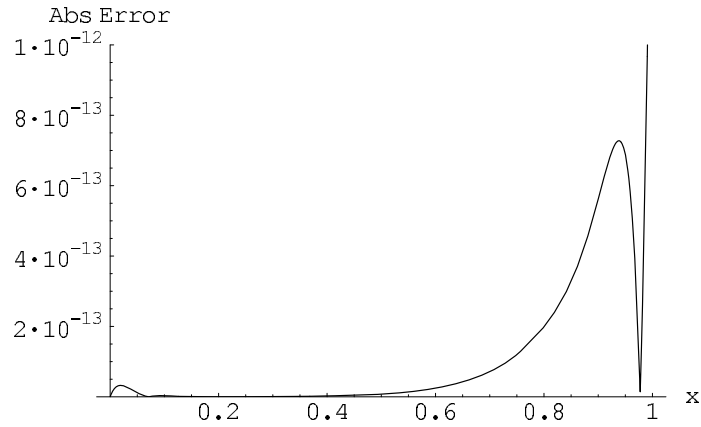
Fig 3 Absolute error between exact and approximate solutions of Eq.(3.8)

Example 4

We consider another Volterra integral equation of the second kind given by,

$$\phi(x) - \int_0^x \frac{1+x}{1+t}\phi(t)dt = 1 - x - \frac{3}{2}x^2 + \frac{x^3}{2}, \quad 0 < x < 1 \tag{3.11}$$

having the exact solution

$$\phi(x) = 1 - x^2 \tag{3.12}$$

(cf. Polyanin and Manzhirov [10]).

Here,

$$\beta_{ij} = B_{i,n}(x_j) + \int_0^{x_j} \frac{1+x_j}{1+t} B_{i,n}(t)dt, \quad i,j = 0, 1, \ldots, n$$

and

$$f_j = 1 - x_j - \frac{3}{2}x_j^2 + \frac{x_j^3}{2}, \quad j = 0, 1, \ldots, n.$$

We solve the system $\sum_{i=0}^{n} a_i\beta_{ij} = f_j$ for the unknown constants a_i, which produce an approximate solution.

For $n = 7$, Figure 4 gives a plot of the absolute error between the exact and approximate solutions for various values of x, and this plot shows that the error is of the order of 10^{-11}.
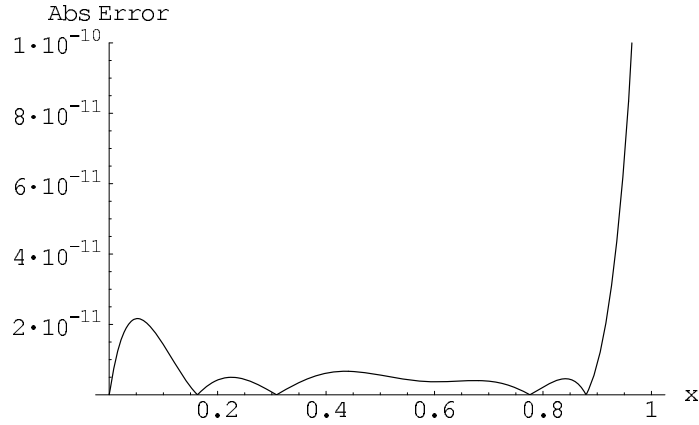
Fig 4 Absolute error between exact and approximate solutions of Eq.(3.11)

Example 5

Here we consider the Abel integral equation given by,

$$\int_0^x \frac{\phi(t)}{(x-t)^{1/2}} dt = x^r, \quad 0 < x < 1 \tag{3.13}$$

where r is any positive number. This is a first kind Volterra integral equation with weak singularity.

The exact solution of the integral equation (3.13) is given by,

$$\phi(x) = \frac{2^{2r-1}}{\pi} r \frac{(\Gamma(r))^2}{\Gamma(2r)} x^{r-\frac{1}{2}}. \tag{3.14}$$

In one numerical example r is chosen as $r = 5$ (integral value) while in another it is chosen as $3/2$ (non-integral value).

a. $r = 5$

For $r = 5$ the exact solution is

$$\phi(x) = \frac{1280}{315\pi} x^{9/2}. \tag{3.15}$$

Performing approximations as given in (2B) we get the linear system

$$\sum_{i=0}^n a_i \gamma_{ij} = f_j, \quad j = 0, 1, \dots, n$$

where,

$$\gamma_{ij} = \binom{n}{i} \sum_{s=0}^{n-i} \binom{n-i}{s} (-1)^s \pi^{1/2} \frac{\Gamma(s+i+1)}{\Gamma(s+1+\frac{3}{2})} x^{s+i+\frac{1}{2}}, \quad i,j = 0,1,\ldots,n$$

(3.16)

and

$$f_j = x_j^5, \quad j = 0,1,\ldots,n$$

x_j's being chosen to be the same as in (3.4).

Solving the linear system (3.23) we get the unknown constants a_i, which give the approximate value of the unknown function $\phi(t)$ by using (2.2).

For $n = 10$, Figure 5a gives a plot of absolute error between the exact and approximate solutions against x, and this plot shows that error is of the order of 10^{-7}.
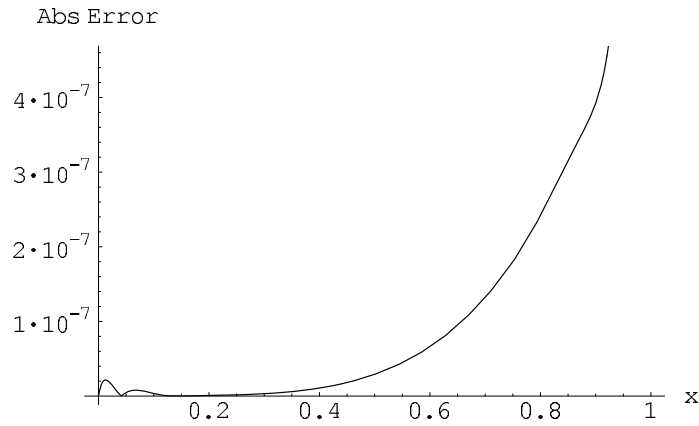

Fig 5a Absolute error between exact and approximate solutions of Eq.(3.13) for $r = 5$

b. $r = 3/2$

For $r = 3/2$ exact solution for $\phi(x)$ is obtained as

$$\phi(x) = \frac{3}{4}x.$$

(3.17)

Solving the system $\sum_{i=0}^{n} a_i \gamma_{ij} = f_j$, $j = 0,1,\ldots,n$ with γ_{ij} being same as in example 5a above and $f_j = x_j^{3/2}$, we get the approximate value of the unknown function, x_j's being chosen to be the same as above. For $n = 5$, a plot of the absolute error between the exact and approximate solution against x shows that the error is of the order of 10^{-15}.
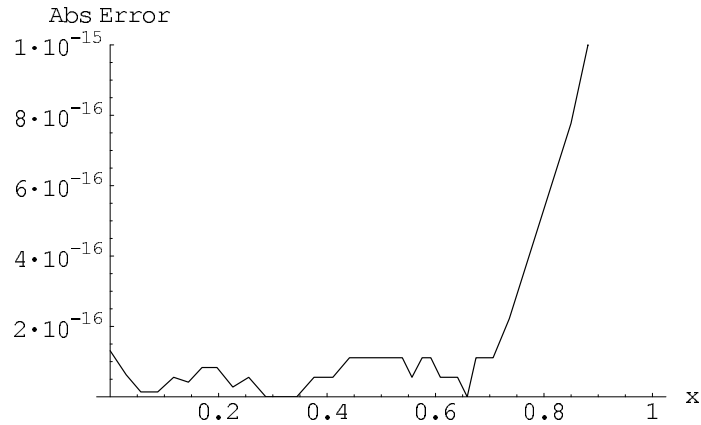
Fig 5b Absolute error between exact and approximate solutions of Eq.(3.13) for $r = \frac{3}{2}$

Example 6

Here we consider the integral equation given by

$$\int_0^x \frac{\phi(t)}{(x-t)^{1/2}}dt = \frac{x^{1/2}}{2}\left[{}_1F_1(1,\frac{3}{2};ix) +_1 F_1(1,\frac{3}{2};-ix)\right], \quad 0 < x < 1 \quad (3.18)$$

'i' here is imaginary unit, and $_1F_1(a,b;z)$ is the hypergeometric function.

Then (3.18) has the exact solution given by $\phi(x) = \cos x$. (cf. Gradshteyn and Ryzhik [11],pp.424)

Applying the method illustrated in (2B) we get the linear system

$$\sum_{m=0}^n a_m\gamma_{mj} = f_j, \quad j = 0,1,\ldots,n \quad (3.19)$$

where γ_{mj} is same as in (3.16) with i replaced by m and

$$f_j = \frac{x_j^{1/2}}{2}\left[{}_1F_1(1,\frac{3}{2};ix_j) +_1 F_1(1,\frac{3}{2};-ix_j)\right].$$

Solving the linear system (3.19)for $n = 5$ we get $a_0, a_1, \ldots, a_5$ which give an approximation to the unknown function $\phi(x)$.

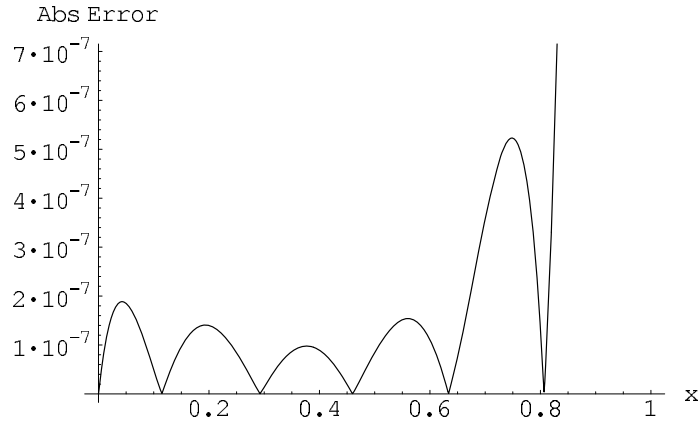Figure 6, which is a plot of the absolute error against x shows that error is of the order of 10^{-7}.

Fig 6 Absolute error between exact and approximate solutions of Eq.(3.18)

Example 7

In this example we consider a second kind weakly singular Volterra integral equation given by,

$$\phi(x) - \int_0^x \frac{\phi(t)}{(x-t)^{1/2}} dt = x^7 (1 - \frac{4096}{6435} x^{1/2}), \quad 0 < x < 1 \qquad (3.20)$$

which has the exact sloution

$$\phi(x) = x^7.$$

Then approximating the unknown function $\phi(x)$ using Bernstein polynomials and following a procedure similar to the one given in (2B) we get the linear system

$$\sum_{i=0}^{n} a_i \mu_{ij} = f_j, \quad j = 0, 1, \ldots, n \qquad (3.21)$$

where

$$\mu_{ij} = B_{i,n}(x_j) - \binom{n}{i} \sum_{s=0}^{n-i} \binom{n-i}{s} (-1)^s \pi^{1/2} \frac{\Gamma(s+i+1)}{\Gamma(s+1+\frac{3}{2})} x^{s+i+\frac{1}{2}}, \quad i, j = 0, 1, \ldots, n$$

and

$$f_j = x_j^7 (1 - \frac{4096}{6435} x_j^{1/2}), \quad j = 0, 1, \ldots, n.$$

For $n = 10$ the system (3.21) is solved for the unknowns $a_i (i = 0, 1, \ldots, n)$. These then give the approximate value of the unknown function $\phi(t)$. Figure 7 shows that the absolute error is of the order of 10^{-7}.
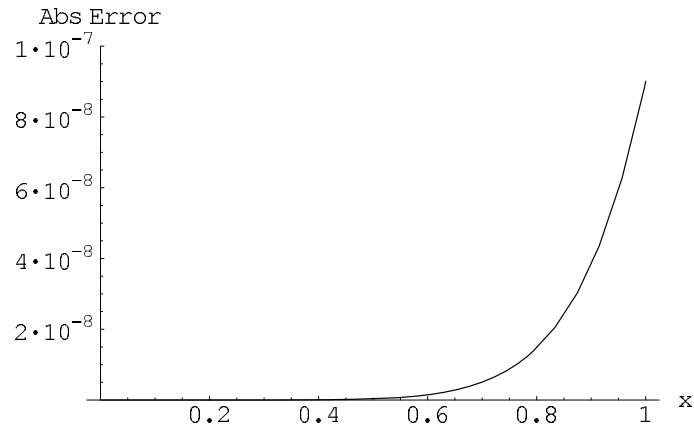
Fig 7 Absolute error between exact and approximate solutions of Eq.(3.20)

4.   Conclusion

Here a very simple and straight forward method, based on approximation of the unknown function of an integral equation on the Bernstein polynomial basis is developed. Use of this method produces very accurate results. It may be mentioned that the linear systems avoid appearance of any ill-conditioned matrix.

Further, using this method, numerical solutions of Abel integral equations are obtained quite correctly, which is otherwise somewhat difficult and tedious process due to the presence of the differential operator in the inverse problem, as has been pointed out in the literature.

Thus a simple method of approximation of the unknown function on the Bernstein polynomial basis for the solution of Volterra integral equations of first and second kinds with regular and weakly singular kernels produces very accurate numerical results.

Acknowledgement

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