Bifurcation analysis of first and second order Benney equations for viscoelastic fluid flowing down a vertical plane

B S Dandapat[1] **and A Samanta**

Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Calcutta–700 108, India

Abstract
Using the method of multiple scale we have studied the nonlinear stability of a travelling wave solution of an evolution equation for the viscoelastic fluid flowing down a vertical plane. Bifurcation analysis of first and second order Benney equations (BEs) for the viscoelastic fluid shows that the first order BE gives both subcritical unstable and supercritical stable zones depending on the Reynolds number greater or smaller than its critical value and the supercritical stable/subcritical unstable region decreases/increases as the viscoelastic parameter increases. However, the second order BE exhibits only supercritical bifurcation and this stable region increases with the increase in either the Reynolds number or the viscoelastic parameter. The spatially uniform solution of the complex Ginzburg–Landau equation for sideband disturbances is also investigated.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Liquid films are encountered both in nature and in many technological applications such as the flow of molten lava/metal, in the cooling system and coating process. It is well known that the waves that develop on the surface of a thin film enhance the transport of heat, mass and momentum across the liquid–gas and liquid–solid interfaces. So, for a better modelling of devices such as distillation, adsorption columns, evaporators, condensers and nuclear emergency cooling system, one needs a clear understanding of the flow development and its finite amplitude behaviour. A great deal has been learned about the onset of the film waves and their weakly nonlinear evolution, but often their observed strong nonlinear character remains to be understood quantitatively. The viscous film flow on an inclined plane has been widely studied since the pioneering experimental observation of the development of waves on the surface by Kapitza and Kapitza [1].

Benjamin [2] and Yih [3] first investigated the linear stability analysis of the liquid film flowing down an inclined plane. They determined the critical Reynolds number as a function of the angle of inclination. Finite amplitude stability analysis was initiated by Benney [4] and he derived an evolution equation referred to as the Benney equation (BE) in terms of the flow depth h, by using a perturbation expansion technique in terms of the long wave parameter α. Later nonlinear analysis was performed by Lin [5], Gjevik [6], Nakaya [7] and others. A comprehensive review on the works on Newtonian fluid can be found in [8]. Pumir *et al* [9], Oron and Gottlieb [10] and many others have pointed out that in spite of a successful description of the dynamics of falling liquid films the BE has a serious drawback. It turns out that there exists a subdomain in parameter space in which the BE manifests solutions that may have growing unbounded amplitude. In this case the BE loses its physical relevance. To overcome this blow-up property of the BE, several attempts have been made [11–14] by devising alternative approaches and finally deriving an appropriate evolution equation for the same physical problem. Mainly these alternative approaches are nothing but the refinements of the integral methods presented in [1, 15].

Recently, substantial effort has been devoted towards analysing the nonlinear solutions of the evolution equation governing non-Newtonian fluids and their stability

[1] Author to whom any correspondence should be addressed.

0022-3727/08/095501+09$30.00

characteristics [16–19]. The nonlinear stability of the thin micropolar liquid film flowing down a vertical wall and a vertical cylinder has been analysed by Hung *et al* [20] and Cheng *et al* [21] and their results show that the micropolar parameter plays an important role in stabilizing the film flow. The viscoelastic fluid exhibits certain viscoelastic effects on normal and shear stresses and several investigations have been reported on the flow and stability of a falling film of viscoelastic fluids [22–24]. For these classes of fluids, (i) the current state of stress is a function of the history of past motion; (ii) the phenomena of elastic recoil creep and stress relaxation occur and (iii) the relation between the stress and velocity field is highly nonlinear, even in situations where the history of the strain is highly repetitive. The stability of a thin viscoelastic film flow travelling down a vertical wall has been examined by Cheng *et al* [24] and the influence of the viscoelastic parameter on the equilibrium finite amplitude has been obtained. The weakly nonlinear stability analysis of a thin viscoelastic liquid film flowing down the outside surface of a vertical cylinder has been investigated by Cheng *et al* [21]. The results show that both subcritical instability and supercritical stability conditions are possible in a viscoelastic film flow system and that the degree of instability in the film flow is further intensified by the lateral curvature of the cylinder. The stability characteristics of the system have been shown to be influenced by the viscoelastic parameter and the radius of the curvature of the cylinder. The study of Cheng *et al* [24] was based on the first order BE in which they have shown that there exists an explosive region in which both linear and nonlinear analyses give instability. As mentioned above in the case of the Newtonian liquid film this explosive zone leads to a numerical inconsistency and one may expect the same type of numerical inconsistency that may occur in the viscoelastic film flowing down a vertical plane also. Further the nonlinear stability analysis of the thin viscoelastic film has not received proper attention. Here, we are interested in studying the bifurcation analysis of the second order BE that arises due to the viscoelastic liquid film.

2. Mathematical formulation

Consider the steady flow of an incompressible second order fluid flowing down a vertical plane. Using the postulate of gradually fading memory, Coleman and Noll [25] derived the constitutive equation for the fluid as

$$\tau_{ij} = -P\delta_{ij} + \mu_0 A_{(1)_{ij}} + \beta_0 A_{(1)_{ik}} A_{(1)_{kj}} + m_0 A_{(2)_{ij}}, \qquad (1)$$

where τ_{ij} is the stress tensor, P is an indeterminate pressure and μ_0, β_0 and m_0 are material constants. The rate of strain tensor $A_{(1)_{ij}}$ and the acceleration tensor $A_{(2)_{kj}}$ are defined by

$$A_{(1)_{ij}} = v_{i,j} + v_{j,i},$$

$$A_{(2)_{ij}} = a_{i,j} + a_{j,i} + 2v_{m,i}v_{m,j},$$

where v_is are the velocity components, a_is are the acceleration components given by $v_{i,T} + v_j v_{i,j}$ and T is the time. It is worth pointing out that such a fluid shows normal stress differences in the shear flow (which is a characteristic property of a viscoelastic fluid) and (1) is applicable to the flow of some dilute polymer solutions (e.g. polyethylene oxide in aqueous solution (POLYOX), polyisobutylene in cetane), which are only slightly viscoelastic, and may have a bearing on the study of fluids with drag-reducing properties (Toms phenomenon), m_0 being negative from thermodynamic considerations. The governing equations can be expressed in terms of Cartesian coordinates (X_1, X_2) as

$$\frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} = 0,$$

$$\rho\left[\frac{\partial u_1}{\partial T} + u_1\frac{\partial u_1}{\partial X_1} + u_2\frac{\partial u_1}{\partial X_2}\right] = \rho g + \frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2}, \qquad (2)$$

$$\rho\left[\frac{\partial u_2}{\partial T} + u_1\frac{\partial u_2}{\partial X_1} + u_2\frac{\partial u_2}{\partial X_2}\right] = \frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2},$$

where (u_1, u_2) are the velocity components of the viscoelastic fluid following vertically downwards and normal to the vertical directions, respectively, and the disturbances are assumed to be two-dimensional. The corresponding boundary conditions are as follows: for the no-slip condition at the wall at $X_2 = 0$ and at the free surface $X_2 = H(X_1, T)$, the shear stress vanishes and the normal stress balances with surface tension times curvature, where $H(X_1, T)$ is the deflection from the mean depth of the fluid film. Further the kinematic condition at the free surface is

$$\frac{\partial H}{\partial T} + u_1\frac{\partial H}{\partial X_1} = u_2, \qquad \text{at } X_2 = H(X_1, T).$$

Introducing the stream function

$$u_1 = \frac{\partial \Phi}{\partial X_2}, \qquad u_2 = -\frac{\partial \Phi}{\partial X_1}$$

into the above equations and their corresponding boundary conditions, we then express these sets of equations into their dimensionless form by using

$$x = \frac{\alpha X_1}{h_0} \quad y = \frac{X_2}{h_0} \quad t = \frac{\alpha u_0 T}{h_0} \quad h = \frac{H}{h_0} \quad R = \frac{u_0 h_0}{\nu}$$

$$\phi = \frac{\Phi}{u_0 h_0} \quad p = \frac{P - p_a}{\rho u_0^2} \quad \alpha = \frac{2\pi h_0}{\lambda} \quad M = \frac{m_0}{\varrho h_0^2},$$

where $u_0 = gh_0^2/2\nu$ is the basic Nusselt velocity and λ and h_0 represent the perturbed wavelength and the mean undisturbed film thickness, respectively. The parameters R and M denote the Reynolds number and the viscoelastic parameter, respectively. The dimensionless equations and the boundary conditions can be written as

$$\begin{aligned}\phi_{yyy} = {} & -2 + \alpha R[p_x + \phi_{ty} + \phi_y\phi_{xy} - \phi_x\phi_{yy} \\ & - M(\phi_{yy}\phi_{xyy} - \phi_{yyy}\phi_{xy} + \phi_x\phi_{yyyy} - \phi_y\phi_{xyyy})] \\ & - \alpha^2\phi_{xxy} + \alpha^3 RM(\phi_y\phi_{xxxy} + \phi_{yy}\phi_{xxx} - 3\phi_{xy}\phi_{xxy} \\ & - \phi_x\phi_{xxyy} + 2\phi_{xx}\phi_{xyy}),\end{aligned} \qquad (3)$$

$$\begin{aligned}p_y = {} & -\alpha R^{-1}\phi_{xyy} + \alpha^2[\phi_{tx} + \phi_y\phi_{xx} - \phi_x\phi_{xy} \\ & + M(3\phi_{xyy}\phi_{xy} + \phi_y\phi_{xxyy} - \phi_{xx}\phi_{yyy} \\ & - \phi_x\phi_{xyyy} - 2\phi_{xxy}\phi_{yy})] - \alpha^3 R^{-1}\phi_{xxx} \\ & + \alpha^4 M(-\phi_{xy}\phi_{xxx} + \phi_{xx}\phi_{xxy} + \phi_y\phi_{xxxx} - \phi_x\phi_{xxxy}).\end{aligned} \qquad (4)$$

For the boundary conditions on the wall at $y = 0$ we have

$$\phi = \phi_x = \phi_y = 0. \tag{5}$$

For the boundary conditions on the free surface at $y = h(x, t)$, we have

$$\begin{aligned}\phi_{yy} &= \alpha RM[2(1-\alpha^2 h_x^2)^{-1}\phi_{yy}^2 h_x + \phi_y\phi_{xyy} \\ &\quad - \phi_x\phi_{yyy} + 2\phi_{yy}\phi_{xy}] + \alpha^2[\phi_{xx} + 4(1-\alpha^2h_x^2)^{-1}\phi_{xy}h_x] \\ &\quad + \alpha^3 RM[4(1-\alpha^2h_x^2)^{-1}(-\phi_y\phi_{xxy} + \phi_x\phi_{xyy})h_x \\ &\quad - \phi_y\phi_{xxx} + \phi_x\phi_{xxy} + 2\phi_{xx}\phi_x] \\ &\quad - \alpha^5[2RM(1-\alpha^2h_x^2)^{-1}\phi_{xx}^2 h_x],\end{aligned} \tag{6}$$

$$\begin{aligned}p &= -2\alpha^2 S_1 R^{-5/3} h_{xx}(1+\alpha^2h_x^2)^{-3/2} - \alpha[2(1+\alpha^2h_x^2)^{-1} \\ &\quad \times R^{-1}(\phi_{yy}h_x + \phi_{xy})] - \alpha^2\{2M(1+\alpha^2h_x^2)^{-1} \\ &\quad \times[(\phi_y\phi_{xxy} - \phi_x\phi_{xyy})(\alpha^2h_x^2 - 1) \\ &\quad - (\phi_y\phi_{xyy} - \phi_x\phi_{yyy} + 2\phi_{yy}\phi_{xy})h_x] - \phi_{yy}^2 h_x^2 \\ &\quad + \phi_{xx}\phi_{yy} - 4M\phi_{xy}^2\} + \alpha^3[2(1+\alpha^2h_x^2)^{-1} \\ &\quad \times R^{-1}(\phi_{xx}h_x + \phi_{xy}h_x^2)] - 2M\alpha^4(1+\alpha^2h_x^2)^{-1} \\ &\quad \times[\phi_{xx}\phi_{yy}h_x^2 - \phi_{xx}^2 + (\phi_y\phi_{xxx} - \phi_x\phi_{xxy} \\ &\quad - 2\phi_{xx}\phi_{xy})h_x],\end{aligned} \tag{7}$$

$$h_t + \phi_y h_x + \phi_x = 0. \tag{8}$$

For long wave stability analysis, we expand the dependent variables ϕ and the pressure p in terms of the wave number α as

$$\begin{aligned}\phi &= \phi_0 + \alpha\phi_1 + \alpha^2\phi_2 + O(\alpha^3), \\ p &= p_0 + \alpha p_1 + \alpha^2 p_2 + O(\alpha^3).\end{aligned} \tag{9}$$

Substituting (9) into the governing equations (3)–(7), and collecting the terms of different orders in α and finally solving these equations up to $O(\alpha^2)$, we can obtain the solution of the above system of equations. Using the solutions so obtained, in the kinematic equation (8) one can obtain the BE for the viscoelastic liquid film falling down a vertical plane as

$$\begin{aligned}&h_t + A(h)h_x + \alpha[B(h)h_x + C(h)h_{xxx}]_x + \alpha^2[D(h)h_x^2 \\ &\quad + E(h)h_{xx} + F(h)h_{xxxx} + G(h)h_xh_{xxx} + H(h)h_{xx}^2 \\ &\quad + I(h)h_x^2h_{xx}]_x + O(\alpha^3) = 0,\end{aligned} \tag{10}$$

where

$$\begin{cases} A(h) = 2h^2, \qquad B(h) = R\left(\frac{8}{15}h^6 + \frac{10}{3}Mh^4\right), \\ C(h) = \frac{2}{3}S_1\alpha^2h^3R^{-2/3}, \\ D(h) = \frac{596}{315}R^2h^9 + \frac{5876}{315}Mh^7 + \frac{136}{3}M^2h^5 + \frac{14}{3}h^3, \\ E(h) = \frac{2}{7}R^2h^{10} + \frac{1081}{315}Mh^8 + \frac{32}{3}M^2h^6 + 2h^4, \\ F(h) = \left(\frac{5}{14}h^7 + 2Mh^5\right)R^{1/3}S_1\alpha^2, \\ G(h) = \left(\frac{9}{2}h^6 + \frac{20}{3}Mh^4\right)R^{1/3}S_1\alpha^2, \\ H(h) = \frac{16}{5}h^6R^{1/3}S_1\alpha^2, \qquad I(h) = \frac{32}{5}h^5R^{1/3}S_1\alpha^2. \end{cases} \tag{11}$$

The above procedure is lengthy but straightforward and for details one can refer to either Dandapat and Gupta [16] or Cheng *et al* [24]. Here the parameter α denotes the aspect ratio and $S_1 = (S_0^3/2^2 g\varrho^3\nu^4)^{1/3} = (\mathrm{Fi}/4)^{1/3} = WR^{2/3}$, where $\mathrm{Fi} = (S_0^3/g\varrho^3\nu^4)$ is the film number and $W = S_0/\varrho g h_0^2$ is the Weber number. S_0 and g denote the surface tension and gravity, respectively. $h = h(x, t)$ represents the nondimensional film thickness depending on the dimensionless variables x and t, denoted as the spatial and temporal variables, respectively. Equation (10) is derived under long wave approximation, with the assumption $S_1\alpha^2 \approx O(1)$ and we denote $S_1\alpha^2 = S$ as the modified surface tension parameter.

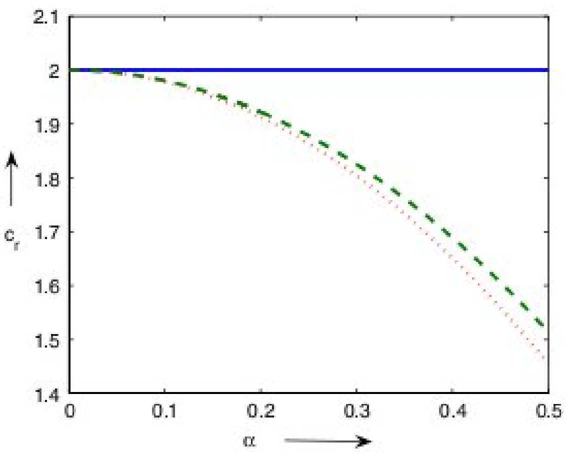

Figure 1. Variation of c_r with respect to the aspect ratio α. The solid curve represents the first order BE and dashed and dotted curves represent that second order BE with $M = 0.01$ and $M = 0.1$, respectively, when $S = 1$, $R = 1$.

3. Linear stability analysis

The linearized stability of equation (10) is studied by assuming a small disturbance at the interface as $h = 1 + \eta(x, t)$, where $\eta(x, t)$ is the deviation from its flat film solution $h_0 = 1$. The linearized evolution equation becomes

$$\begin{aligned}&\eta_t + A_1\eta_x + \alpha(B_1\eta_{xx} + C_1\eta_{xxxx}) \\ &\quad + \alpha^2(E_1\eta_{xxx} + F_1\eta_{xxxxx}) + O(\alpha^3),\end{aligned} \tag{12}$$

where

$A_1 = A(h = 1)$, $B_1 = B(h = 1)$,

$C_1 = C(h = 1)$, $E_1 = E(h = 1)$, $F_1 = F(h = 1)$.

Equation (12) has a travelling wave solution of the form

$$\eta(x, t) = \Gamma\exp[\mathrm{i}(x - ct)] + \bar{\Gamma}\exp[-\mathrm{i}(x - \bar{c}t)], \tag{13}$$

where bars denote the complex conjugates. Γ is a complex wave amplitude and the complex wave celerity $c = c_r + \mathrm{i}c_i$. The dispersion relation is given by

$$\mathrm{i}[A_1 - c + \alpha^2(F_1 - E_1)] - \alpha(B_1 - C_1) = 0,$$

which yields the long wave linear stability result as

$$c_r = A_1 + \alpha^2(F_1 - E_1), \qquad c_i = \alpha(B_1 - C_1). \tag{14}$$

Here, c_r and c_i are regarded as the linear wave speed and the linear growth rate of the amplitude, respectively. It is to be

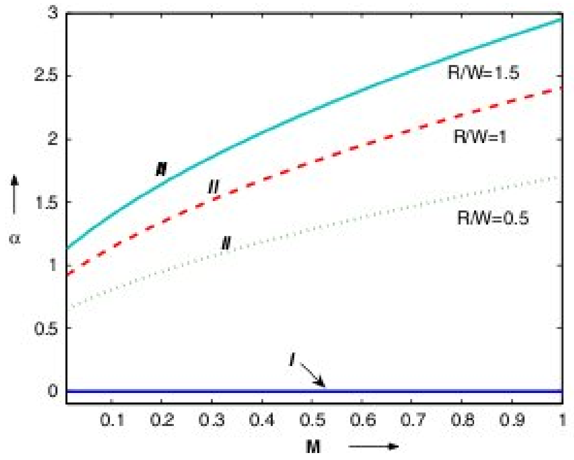

Figure 2. Neutral curves I and II represent the variation of α with respect to the viscoelastic parameter M for different values of R/W.

noted here that there is no difference in the expression of c_i between the present result and that of Cheng *et al* [24]. But according to them the wave speed is constant and fixed at 2, while in the present case c_r contains an extra term $\alpha^2(F_1 - E_1)$ which yields dispersion of waves. This difference is due to the consideration of the second order correction of the evolution equation. Figure 1 depicts the variation of the two results. The solution $h(x, t) = 1$ is asymptotically stable or unstable if $c_i < 0$ or $c_i > 0$. This is equivalent to $B_1 < C_1$ or $B_1 > C_1$. For neutral perturbations, assuming $c_i = 0$ we obtain two relations:

$$\begin{cases} \alpha = 0, & \text{I} \\ \alpha = \alpha_H = \sqrt{\left(\dfrac{4}{5} + 5M\right)\dfrac{R}{W}}, & \text{II} \end{cases} \tag{15}$$

which yield two branches of a neutral curve. The flow instability takes place in the region between the two branches I and II of the neutral curves. It is evident from figure 2 that as M increases the flow instability increases. The Reynolds number at the neutral curve becomes

$$R = \left(\frac{S}{4/5 + 5M}\right)^{3/5}. \tag{16}$$

It is clear from equation (16) that for the viscoelastic parameter $M = 0$, the present result coincides with the results for the Newtonian fluid given by Lin [5].

4. Nonlinear stability analysis

In this section we are interested in studying those small-amplitude waves which develop immediately just after the breakdown of the flat film solution ($\eta = 0$) by assuming $c_i = O(\varepsilon^2)$, which is equivalent to $(B_1 - C_1) = O(\varepsilon^2)$. To study the growth of the weakly nonlinear evolution of waves, we need to consider two or more characteristic time scales of different orders of magnitude. As a rule, problems of this kind cannot be solved by means of classical perturbation methods and the method of matched asymptotic expansions is also insufficient. This gap is filled up by introducing the method of multiple scales considering small variables as

$$X = \varepsilon x, \qquad T_1 = \varepsilon t, \qquad T_2 = \varepsilon^2 t,$$

where ε is the measure of the difference in R from its criticality. Under this scale we have

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} \tag{17}$$

and

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X}. \tag{18}$$

Expanding the solution

$$\begin{aligned} h(\alpha, x, t, X, T_1, T_2) &\equiv 1 + \varepsilon\eta(\alpha, x, t, X, T_1, T_2) \\ &= 1 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 + \cdots \end{aligned} \tag{19}$$

and using (17)–(19) in (10), we get

$$\begin{aligned} &(L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \cdots)(\varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots) \\ &\quad = \text{nonlinear terms}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} L_0 &= \frac{\partial}{\partial t} + A_1\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial x}\left[B_1\frac{\partial}{\partial x} + C_1\frac{\partial^3}{\partial x^3}\right] \\ &\quad + \alpha^2\frac{\partial}{\partial x}\left[E_1\frac{\partial^2}{\partial x^2} + F_1\frac{\partial^4}{\partial x^4}\right], \\ L_1 &= \frac{\partial}{\partial T_1} + A_1\frac{\partial}{\partial X} + \alpha\frac{\partial}{\partial X}\left[2B_1\frac{\partial}{\partial x} + 4C_1\frac{\partial^3}{\partial x^3}\right] \\ &\quad + \alpha^2\frac{\partial}{\partial X}\left[3E_1\frac{\partial^2}{\partial x^2} + 5F_1\frac{\partial^4}{\partial x^4}\right], \\ L_2 &= \frac{\partial}{\partial T_2} + \alpha\frac{\partial^2}{\partial X^2}\left[B_1 + 6C_1\frac{\partial^2}{\partial x^2}\right] \\ &\quad + \alpha^2\frac{\partial^2}{\partial X^2}\left[3E_1\frac{\partial}{\partial x} + 10F_1\frac{\partial^3}{\partial x^3}\right]. \end{aligned} \tag{21}$$

The terms of order ε in (20), give

$$L_0\eta_1 = 0. \tag{22}$$

The solution of equation (22) has the form

$$\eta_1 = \Gamma\exp[\mathrm{i}(x - c_r t)] + \bar{\Gamma}\exp[-\mathrm{i}(x - c_r t)],$$

where $\Gamma(X, T_1, T_2)$ is the complex amplitude and $\bar{\Gamma}$ is the complex conjugate of Γ. The next higher order equation in ε^2 gives

$$\begin{aligned} L_0\eta_2 &= -\left(\frac{\partial}{\partial T_1} + H_2\frac{\partial}{\partial X}\right)\Gamma\exp[\mathrm{i}(x - c_r t)] \\ &\quad + Q_1\Gamma^2\exp[2\mathrm{i}(x - c_r t)] + \text{c.c.}, \end{aligned} \tag{23}$$

where c.c. denotes complex conjugate and the notations H_2 and Q_1 are defined as

$$\begin{cases} H_2 = H_{2r} + \mathrm{i}H_{2i} = A_1 + \alpha^2(5F_1 - 3E_1) + \mathrm{i}\alpha(2B_1 - 4C_1), \\ Q_1 = -\mathrm{i}A_1' + 2[\alpha(B_1' - C_1') \\ \quad + \mathrm{i}\alpha^2(D_1 + E_1' - F_1' - G_1 - H_1)]. \end{cases} \tag{24}$$

We get the solution of (23) after elimination of the secular term:

$$\eta_2 = \tilde{H}_1 \Gamma^2 \exp[2\mathrm{i}(x - c_r t)] + \mathrm{c.c.}, \tag{25}$$

where

$$\tilde{H}_1 = \tilde{H}_{1r} + \mathrm{i}\tilde{H}_{1i} = \frac{Q_1}{2\alpha[2(4C_1 - B_1) + 3\mathrm{i}\alpha(5F_1 - E_1)]}.$$

Substituting η_1 and η_2 in the equation of third order in ε^3 and elimination of its secular term gives

$$\frac{\partial \Gamma}{\partial T_2} + \mathrm{i}V\frac{\partial \Gamma}{\partial X} - c_i'\Gamma + (J_{1r} + \mathrm{i}J_{1i})\frac{\partial^2 \Gamma}{\partial X^2} + (J_2 + \mathrm{i}J_4)|\Gamma|^2\Gamma = 0, \tag{26}$$

where

$$\begin{cases} c_i' = \varepsilon^{-2}c_i, \qquad V = 2\alpha(B_1 - 2C_1)\varepsilon^{-1} < 0, \\ J_{1r} = \alpha(B_1 - 6C_1), \qquad J_{1i} = \alpha^2(3E_1 - 10F_1), \\ J_2 = -A_1'\tilde{H}_{1i} + \alpha\left[\frac{1}{2}(C_1'' - B_1'') + (7C_1' - B_1')\tilde{H}_{1r}\right] \\ \quad + \alpha^2\tilde{H}_{1i}Z, \\ J_4 = A_1'\tilde{H}_{1r} + \frac{1}{2}A_1'' + \alpha\tilde{H}_{1i}(7C_1' - B_1') \\ \quad + \alpha^2\left[D_1' - \frac{3}{2}(E_1'' - F_1'') - G_1' + 3H_1' - I_1 - Z\tilde{H}_{1r}\right], \end{cases} \tag{27}$$

and

$$Z = -4D_1 + 5E_1' - 17F_1' + 10G_1 - 8H_1.$$

It is to be noted here that the second term in equation (26) is absent in the analysis of Cheng *et al* [24]. Moreover the coefficient of the diffusion term is not real as given in equation (59) of Cheng *et al* [24]. The present analysis has a correction in the imaginary component at $O(\alpha^2)$ of this term. The coefficients J_2 and J_4 also have corrections at $O(\alpha^2)$. Equation (26) can be transformed into the standard complex Ginzburg–Landau equation (CGLE) by using the transformation

$$\Gamma(X, T_2) = \exp(\mathrm{i}qX)\Upsilon(X, T_2), \qquad q = -\frac{V}{2(J_{1r} + \mathrm{i}J_{1i})}, \tag{28}$$

in (26) and, finally, we obtain

$$\frac{\partial \Upsilon}{\partial T_2} + \Upsilon\left[\frac{V^2}{4(J_{1r} + \mathrm{i}J_{1i})} - c_i'\right] + (J_{1r} + \mathrm{i}J_{1i})\frac{\partial^2 \Upsilon}{\partial X^2} + (J_2 + \mathrm{i}J_4)|\Upsilon|^2\Upsilon = 0. \tag{29}$$

It is to be noted here that the coefficients of linear and spatially homogeneous terms are complex. To find the linear stability of the amplitude in equation (26), we first linearized the equation at about $\Gamma = 0$ and then followed the normal mode analysis. Substituting $\Gamma = \mathrm{const} \times \exp(\omega T_2 + \mathrm{i}\kappa X)$ in the linearized equation, we finally arrived at the dispersion relation

$$\omega = c_i' + \kappa V + (J_{1r} + \mathrm{i}J_{1i})\kappa^2.$$

Here, ω is the complex growth rate and κ is the wave number. The disturbance will grow with time of about $\Gamma = 0$ if

$$\omega_r = c_i' + \kappa V + J_{1r}\kappa^2 > 0. \tag{30}$$

where $\omega = \omega_r + \mathrm{i}\omega_i$.

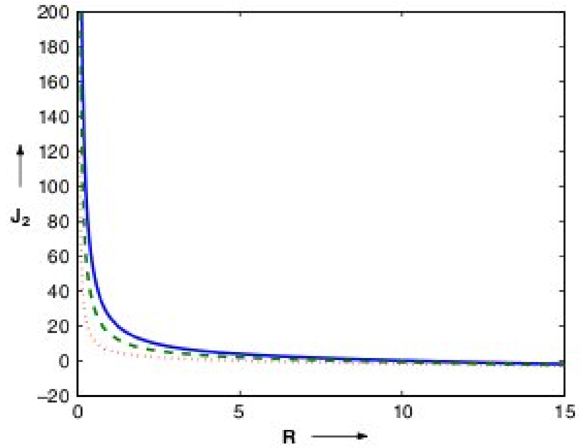

Figure 3. Variation of J_2 obtained from the first order BE with respect to the Reynolds number R for different values of viscoelastic parameter when $\alpha = 0.1$, solid line for $M = 0$, dashed line for $M = 0.1$, dotted line for $M = 0.5$.

Case I: If $c_i' > 0$. Equation (30) has two roots κ_1 and κ_2 where

$$\kappa_1 = \frac{2c_i'}{\sqrt{V^2 - 4J_{1r}c_i'} - V}, \qquad \kappa_2 = \frac{2c_i'}{-\sqrt{V^2 - 4J_{1r}c_i'} - V}. \tag{31}$$

It is clear from (31) that $\kappa_2 < 0 < \kappa_1$. Therefore the unstable zone lies in the regions $\kappa > \kappa_1$ while in the other region $0 < \kappa \leqslant \kappa_1$ the disturbance is stable.

Case II: If $c_i' < 0$. In this case the two roots of equation (30) are

$$\kappa_1 = \frac{-2c_i'}{-\sqrt{V^2 - 4J_{1r}c_i'} + V}, \qquad \kappa_2 = \frac{-2c_i'}{\sqrt{V^2 - 4J_{1r}c_i'} + V}. \tag{32}$$

Both roots are negative, implying the system is stable in this case. The nonlinear stability of the perturbation dynamics depends solely on the sign of J_2. If $J_2 > 0$ then the bifurcation is supercritical while for subcritical bifurcation $J_2 < 0$. Bifurcation analysis of the first order BE for the viscoelastic fluid is depicted in figure 3. This figure shows that the supercritical stable zone at a small Reynolds number changes to the subcritical unstable zone by increasing either the Reynolds number or the viscoelastic parameter.

5. Stability analysis of the first order BE

In the case of the first order BE we consider the terms up to $O(\alpha)$. Then equation (10) is reduced to

$$h_t + A(h)h_x + \alpha[B(h)h_x + C(h)h_{xxx}]_x + O(\alpha^2) = 0. \tag{33}$$

The corresponding equation (26) reduces to

$$\frac{\partial \Gamma}{\partial T_2} + \mathrm{i}V\frac{\partial \Gamma}{\partial X} - c_i'\Gamma + J_{1r}\frac{\partial^2\Gamma}{\partial X^2} + (J_2 + \mathrm{i}J_4)|\Gamma|^2\Gamma = 0, \qquad (34)$$

where

$$\begin{cases} c_i' = \varepsilon^{-2}c_i, \qquad V = 2\alpha(B_1 - 2C_1)\varepsilon^{-1}, \\ J_{1r} = \alpha(B_1 - 6C_1), \\ J_2 = -A_1'\tilde{H}_{1i} + \alpha\left[\frac{1}{2}(C_1'' - B_1'') + (7C_1' - B_1')\tilde{H}_{1r}\right], \\ J_4 = A_1'\tilde{H}_{1r} + \frac{1}{2}A_1'' + \alpha\tilde{H}_{1i}(7C_1' - B_1'), \\ \tilde{H}_{1r} = \dfrac{B_1' - C_1'}{2(4C_1 - B_1)}, \quad \tilde{H}_{1i} = \dfrac{-A_1'}{4\alpha(4C_1 - B_1)}. \end{cases} \qquad (35)$$

It is to be noted here that an imaginary, apparently 'convective', V-term appears in equations (26) and (34) which are different from equation (59) of Cheng *et al* [24]. This extra term appears due to the fact that $(B_1 - C_1) \sim O(\varepsilon^2)$ was not properly taken into account while deducing their equation (59). Substituting the corresponding values of the coefficients given in equation (11) and using relation (16) to replace the surface tension S in the above equation (35), we get

$$J_2 = \frac{4}{3\alpha R(8/15 + 10M/3)} - \alpha R\frac{(192/25 + 128M/3 + 100M^2/9)}{6(8/15 + 10M/3)}, \qquad (36)$$

$$J_4 = \frac{-8/5 - 90M/3}{8/5 + 10M}.$$

Now, $J_2 = 0$ gives

$$R = \frac{\sqrt{2}}{\alpha\sqrt{\dfrac{48}{25} + \dfrac{32M}{3} + \dfrac{25M^2}{9}}} = R_c. \qquad (37)$$

It is evident from the above equation (37) that as M increases R_c decreases, implying the destabilizing effect of the viscoelastic parameter M. This result was reported earlier by Gupta [22], Dandapat and Gupta [16] and Cheng *et al* [24]. It is clear from (36) that bifurcation is possible for the first order BE and it will be supercritical if $J_2 > 0$; in other words for

$$R < R_c = \frac{\sqrt{2}}{\alpha\sqrt{\dfrac{48}{25} + \dfrac{32M}{3} + \dfrac{25M^2}{9}}} \qquad (38)$$

and for

$$R > R_c = \frac{\sqrt{2}}{\alpha\sqrt{\dfrac{48}{25} + \dfrac{32M}{3} + \dfrac{25M^2}{9}}} \qquad (39)$$

bifurcation is subcritical. It is also clear from (38) and (39) that the supercritical stable/subcritical unstable region decreases/increases as the viscoelastic parameter increases. By setting $M = 0$ in (37), (38) and (39) one can get the results obtained by Oron and Gottlieb [10].

6. Stability analysis for the second order BE

Following the above procedure of nonlinear stability analysis in equation (26) we get the corresponding value of J_2 as

$$\begin{aligned} J_2 = {} & -\frac{\alpha R}{2}\left(\frac{64}{5} + 20M\right) + \alpha R\left(8 + \frac{170M}{3}\right)\left[6R^2\left(\frac{8}{15} + \frac{10M}{3}\right)\left(\frac{8}{5} + \frac{10M}{3}\right)\right. \\ & \left. + \left\{\frac{8R^2}{7} + M\left(\frac{237R^2}{14} - \frac{1081}{315}\right) + \left(50R^2M^2 - \frac{32M^2}{3} - 2\right)\right\}\right] \\ & \times \left[\frac{\left\{-6 + 3\alpha^2\left[\frac{38}{3} - \frac{5372R^2}{1575} + M^2\left(\frac{328}{3} - \frac{250R^2}{3}\right) + M\left(\frac{14524}{315} - \frac{965R^2}{15}\right)\right]\right\}}{\left\{6R\left(\frac{8}{15} + \frac{10M}{3}\right)\right\}^2 + \left\{3\alpha\left[\frac{8R^2}{7} + M\left(\frac{237R^2}{14} - \frac{1081}{315}\right) + \left(50R^2M^2 - \frac{32M^2}{3} - 2\right)\right]\right\}^2}\right] \\ & + \left\{-4 + \alpha^2\left[\frac{64}{3} - \frac{18526R^2}{1575} + \left(\frac{416}{3} - \frac{1550R^2}{3}\right)M^2 + \left(\frac{19736}{315} - \frac{1189R^2}{6}\right)M\right]\right\} \\ & \times \left\{\frac{\left[-3\alpha R\left(\frac{8}{5} + \frac{10M}{3}\right)\left\{\frac{8R^2}{7} + M\left(\frac{237R^2}{14} - \frac{1081}{315}\right) + \left(50R^2M^2 - \frac{32M^2}{3} - 2\right)\right\}\right]}{\left\{6R\left(\frac{8}{15} + \frac{10M}{3}\right)\right\}^2 + \left\{3\alpha\left[\frac{8R^2}{7} + M\left(\frac{237R^2}{14} - \frac{1081}{315}\right) + \left(50R^2M^2 - \frac{32M^2}{3} - 2\right)\right]\right\}^2}\right. \\ & \left. + \frac{\left(\frac{8}{15} + \frac{10M}{3}\right)\left[-\frac{12R}{\alpha} + 6\alpha R\left\{\frac{38}{3} - \frac{5372R^2}{1575} + M^2\left(\frac{328}{3} - \frac{250R^2}{3}\right) + M\left(\frac{14524}{315} - \frac{965R^2}{15}\right)\right\}\right]}{\left\{6R\left(\frac{8}{15} + \frac{10M}{3}\right)\right\}^2 + \left\{3\alpha\left[\frac{8R^2}{7} + M\left(\frac{237R^2}{14} - \frac{1081}{315}\right) + \left(50R^2M^2 - \frac{32M^2}{3} - 2\right)\right]\right\}^2}\right\}. \end{aligned} \qquad (40)$$

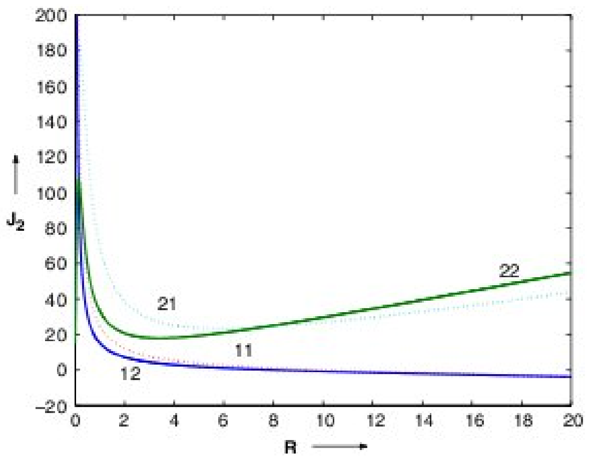

Figure 4. Variation of J_2 with respect to the Reynolds number R. The curves (11, 21) and (12, 22) represent first and second order BEs when ($M = 0$, $\alpha = 0.1$) and ($M = 0.1$, $\alpha = 0.1$), respectively.

Figure 4 represents a comparison of the behaviour of J_2 that was obtained separately from the analysis of the first and second order BEs. It is evident from figure 4 that the second order BE predicts a supercritical stable region irrespective of the increase in either the Reynolds number R or the viscoelastic parameter M, while the first order BE gives a supercritical stable zone at small values of R and M and crosses to the subcritical unstable region when the Reynolds number R or the viscoelastic parameter M increases.

7. Sideband stability analysis

In this section we shall study the sideband instability of the spatially uniform solution of the CGLE (26) with respect to the infinitesimal sideband disturbances. The spatial uniformity of solutions implies that for a filtered wave, there is no spatial modulation and the second and fourth terms of equation (26) vanish. Following Lin [5] we take

$$\Gamma_\infty(T_2) = |\Gamma_\infty| \exp(-\mathrm{i}QT_2),$$

where $Q = J_4 c_i'/J_2$, $|\Gamma_\infty|^2 = c_i'/J_2$ and $\Gamma_\infty(T_2)$ is the limiting solution of equation (26) as $T_2 \to \infty$ for the filtered wave. This solution is perturbed by small spatial sideband disturbances in the form

$$\Gamma = \Gamma_\infty(T_2) + [\delta\Gamma_+(T_2)\exp(\mathrm{i}KX) + \delta\Gamma_-(T_2)\exp(-\mathrm{i}KX)]\exp(-\mathrm{i}QT_2), \quad (41)$$

where K is the modulation wave number. This is substituted in (26), and neglecting the terms containing nonlinearities of $\delta\Gamma_+$, $\delta\Gamma_-$, we obtain

$$\frac{\partial\delta\Gamma_+}{\partial T_2} = \delta\Gamma_+\{\mathrm{i}Q + VK + c_i' - 2(J_2 + \mathrm{i}J_4)|\Gamma_\infty|^2 + (J_{1r} + \mathrm{i}J_{1i})K^2\} - \overline{\delta\Gamma}_-\{(J_2 + \mathrm{i}J_4)|\Gamma_\infty|^2\}, \quad (42)$$

$$\frac{\partial\overline{\delta\Gamma}_-}{\partial T_2} = \overline{\delta\Gamma}_-\{-\mathrm{i}Q - VK + c_i' - 2(J_2 - \mathrm{i}J_4)|\Gamma_\infty|^2 + (J_{1r} - \mathrm{i}J_{1i})K^2\} - \delta\Gamma_+\{(J_2 - \mathrm{i}J_4)|\Gamma_\infty|^2\}. \quad (43)$$

Equation (42) and (43) can be written in the matrix form as

$$\begin{pmatrix} \dfrac{\partial\delta\Gamma_+}{\partial T_2} \\ \dfrac{\partial\overline{\delta\Gamma}_-}{\partial T_2} \end{pmatrix} = A\begin{pmatrix} \delta\Gamma_+ \\ \overline{\delta\Gamma}_- \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \delta\Gamma_+ \\ \overline{\delta\Gamma}_- \end{pmatrix},$$

where

$$\begin{cases} a_{11} = \mathrm{i}Q + VK + c_i' + K^2(J_{1r} + \mathrm{i}J_{1i}) - 2(J_2 + \mathrm{i}J_4)|\Gamma_\infty|^2, \\ a_{12} = -(J_2 + \mathrm{i}J_4)|\Gamma_\infty|^2, \qquad a_{21} = -(J_2 - \mathrm{i}J_4)|\Gamma_\infty|^2, \\ a_{22} = -\mathrm{i}Q - VK + c_i' + K^2(J_{1r} - \mathrm{i}J_{1i}) \\ \quad - 2(J_2 - \mathrm{i}J_4)|\Gamma_\infty|^2. \end{cases} \quad (44)$$

We seek the solution in the form

$$\begin{pmatrix} \delta\Gamma_+ \\ \overline{\delta\Gamma}_- \end{pmatrix} = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}\exp(\lambda T_2),$$

where λ is the eigenvalue. The eigenvalues are given by

$$\lambda_1 = \tfrac{1}{2}[(a_{11} + a_{22}) + \{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{21})\}^{1/2}],$$
$$\lambda_2 = \tfrac{1}{2}[(a_{11} + a_{22}) - \{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{21}a_{21})\}^{1/2}]. \quad (45)$$

The condition for stability of the sideband disturbances is that $\lambda < 0$. It can be readily seen that

$$\mathrm{tr}(\boldsymbol{A}) = a_{11} + a_{22} = -2c_i' + 2K^2 J_{1r}$$

is real and negative provided $c_i' > 0$. This shows that at least one of the eigenvalues' real part is negative implying the linear stable sideband mode. The other eigenvalue may be either positive or negative real part depending on the values of the parameters responsible for the sideband stability. For a particular case $V = 0$, it can be seen from (44) that $a_{22} = \overline{a}_{11}$ and this will reduce the

$$\det(\boldsymbol{A}) = (K^2 J_{1r} - 2c_i')K^2 J_{1r} + (K^2 J_{1i} - 2Q)K^2 J_{1i} \quad (46)$$

to be real. The first term on the right-hand side in (46) is positive as long as the sideband is supercritical ($c_i' > 0$) and the wave number is not zero. But the second term on the right-hand side may take any sign. Thus the spatially uniform solution of (CGLE) leading to (26) may become sideband stable or unstable for positive and negative signs of $\det(\boldsymbol{A})$, respectively. It is to be recalled that J_{1i} is nonzero in the case of the second order BE, and assuming $V \sim o(\epsilon)$ one can show that $\det(\boldsymbol{A}) < 0$ provided

$$K^2 \leqslant K_c^2 \equiv \frac{2c_i'}{|J_1|^2}\left(J_{1r} + \frac{J_4 J_{1i}}{J_2}\right). \quad (47)$$

This shows that in the supercritical case sideband instability may be possible if $J_{1r} + (J_4 J_{1i}/J_2) > 0$. This is nothing but the well-known condition for Benjamin–Feir instability [26].

8. Stability analysis for truncated bimodal dynamical system

In this section we are interested in showing the coincidence of the Hopf bifurcation curve for bimodal and unimodal systems.

Consider the solution of equation (33) in a truncated Fourier series:

$$h(x,t) = 1 + \sum_{n=1}^{N} [Z_n(t)\exp(\mathrm{i}nx) + \overline{Z}_n \exp(-\mathrm{i}nx)], \tag{48}$$

where $\overline{Z}_n$ is complex conjugate of Z_n. Now substituting (48) in (33) and keeping terms up to $N = 2$ the coefficients of $\exp(\mathrm{i}x)$ and $\exp(-\mathrm{i}x)$, give

$$\begin{aligned}\dot{Z}_1 &= [\mu_{111}Z_1 + \mu_{121}\overline{Z}_1 Z_2 + Z_1(\mu_{131}|Z_1|^2 + \mu_{132}|Z_2|^2)],\\ \dot{Z}_2 &= [\mu_{211}Z_2 + \mu_{221}Z_1^2 + Z_2(\mu_{231}|Z_1|^2 + \mu_{232}|Z_2|^2)],\end{aligned} \tag{49}$$

where the coefficients μ_{ijk} are given by

$$\begin{cases} \mu_{111} = \alpha(B_1 - C_1) - 2\mathrm{i}, \\ \mu_{211} = \alpha(4B_1 - 16C_1) - 4\mathrm{i}, \\ \mu_{121} = \alpha\left(6B_1 - \dfrac{20MR}{3} - 21C_1\right) - 4\mathrm{i}, \\ \mu_{221} = \alpha\left(12B_1 - \dfrac{40MR}{3} - 6C_1\right) - 4\mathrm{i}, \\ \mu_{131} = \alpha(15B_1 - 30MR - 3C_1) - 2\mathrm{i}, \\ \mu_{231} = \alpha(120B_1 - 240MR - 96C_1) - 8\mathrm{i}, \\ \mu_{132} = \alpha(30B_1 - 60MR - 6C_1) - 4\mathrm{i}, \\ \mu_{232} = \alpha(60B_1 - 140MR - 48C_1) - 4\mathrm{i}. \end{cases} \tag{50}$$

For convenience we write (49) in polar notation using $Z_n = a_n \exp(\mathrm{i}\theta_n)$, where both a_n and θ_n are functions of time. Assuming the phase relation $\phi = 2\theta_1 - \theta_2$, we get

$$\begin{aligned}\dot{a_1} &= \beta_{111}a_1 + (\beta_{121}\cos\phi - 4\sin\phi)a_1a_2\\ &\quad + a_1(\beta_{131}a_1^2 + \beta_{132}a_2^2),\end{aligned} \tag{51}$$

$$\begin{aligned}\dot{a_2} &= \beta_{211}a_2 + (\beta_{221}\cos\phi + 4\sin\phi)a_1^2\\ &\quad + a_2(\beta_{231}a_1^2 + \beta_{232}a_2^2),\end{aligned} \tag{52}$$

$$\begin{aligned}\dot{\phi} &= (-2\beta_{121}\sin\phi - 8\cos\phi)a_2 + (-\beta_{221}\sin\phi + 4\cos\phi)\frac{a_1^2}{a_2}\\ &\quad -4(a_2^2 - a_1^2),\end{aligned} \tag{53}$$

where $\beta_{ijk} = \Re e(\mu_{ijk})$.

Travelling waves of the modal system corresponding to a fixed point with a constant nonzero phase difference are obtained by considering $\dot{a_1} = \dot{a_2} = \dot{\phi} = 0$. If we assume modal amplitudes are small and are of order $a_1 \to \varepsilon a_1$, $a_2 \to \varepsilon^2 a_2$ then the phase evolution is governed by the second term on the right-hand side of (53). From (53) we get the phase component of the fixed point as

$$\tan\phi = \frac{4}{\beta_{221}} = \text{const} + O(\epsilon^2).$$

Using the above relation in (51) and (52) we obtain explicit relations for a_1^2 and a_2^2 as

$$\beta_{111} + \gamma a_2 + \beta_{131}a_1^2 = 0 \tag{54}$$

and

$$\gamma\beta_{231}a_2^2 + a_2(\beta_{111}\beta_{231} - \beta_{131}\beta_{211} + \gamma\delta) + \beta_{111}\delta = 0, \tag{55}$$

where

$$\gamma = \beta_{121}\cos\phi - 4\sin\phi, \qquad \delta = \beta_{221}\cos\phi + 4\sin\phi. \tag{56}$$

At neutral stability, we have $B_1 = C_1$, which gives $\beta_{111} = 0$. Therefore the amplitude of the nonzero travelling wave is given by

$$a_2 = \frac{\beta_{131}\beta_{211} - \gamma\delta}{\gamma\beta_{231}}. \tag{57}$$

It is to be noted here that the zero amplitude in (57) corresponds to the threshold for the subcritical travelling waves, which gives

$$\beta_{131}\beta_{211} = \beta_{121}\beta_{221} - 16. \tag{58}$$

Substituting the value of β_{ijk} in the above equation, we get

$$R = \frac{\sqrt{2}}{\alpha\sqrt{\dfrac{48}{25} + \dfrac{32M}{3} + \dfrac{25M^2}{9}}} = R_c. \tag{59}$$

This result coincides with the result for the unimodal system given in equation (37).

9. Conclusion

In this analysis we have carried out the bifurcation analysis of the first and second order BEs for the viscoelastic fluid and found that the first order BE gives both subcritical unstable and supercritical stable zones depending on the Reynolds number greater or smaller than its critical value. Further, it is observed that the supercritical stable/subcritical unstable region decreases/increases as the viscoelastic parameter increases. However, the second order BE exhibits only supercritical bifurcation and this stable region increases with the increase in either the Reynolds number or the viscoelastic parameter M. We have also shown that the critical Reynolds number of the truncated bimodal system coincides with the unimodal system of the analysis for the first order BE. It is also observed that the side band disturbance of the spatially uniform solution for the complex Ginburg–Landau equation may be either stable or unstable.

We would like to make a comment regarding the validity of the model of the second order fluid in our analysis. If the disturbance time scale is large compared with the characteristic time scale (relaxation time) of the fluid, then the second order fluid model is internally consistent with the stress-relaxing fluid due to Oldroyd [27]. As pointed out by Porteous and Denn [28], this would happen if $M \ll 1$.

Acknowledgment

The authors are thankful to the referees for their valuable comments for improving the text as well as the contents of the manuscript.

References

[1] Kapitza P L and Kapitza S P 1949 Wave flow of thin layers of a viscous fluid *Zh. Teor. Fiz.* **19** 105–20

[2] Benjamin T B 1957 Wave formation in laminar flow down an inclined plane *J. Fluid Mech.* **2** 554–74

[3] Yih C S 1963 Stability of liquid flow down an inclined plane *Phys. Fluids.* **6** 321–34

[4] Benney D J 1966 Long waves on liquid films *J. Math. Phys.* **45** 150–5

[5] Lin S P 1974 Finite amplitude side-band stability of a viscous film *J. Fluid. Mech.* **63** 417–29

[6] Gjevik B 1970 Occurrence of finite-amplitude surface waves on falling liquid films *Phys. Fluids.* **13** 1918–25

[7] Nakaya C 1975 Long waves on a thin fluid layer flowing down an inclined plane *Phys. Fluids.* **15** 1407–12

[8] Chang H C 1994 Wave evolution on a falling film *Annu. Rev. Fluid Mech.* **36** 103–36

[9] Pumir A, Manneville A and Pomeau Y J 1983 On solitary waves running down an inclined plane *J. Fluid Mech.* **135** 27–50

[10] Oron A and Gottlieb O 2004 Subcritical and supercritical bifurcations of the first- and second- order Benny equations *J. Eng. Math.* **50** 121–40

[11] Chang H C, Demekhin E A and Kopelevich D I 1993 Nonlinear evolution of waves on a vertically falling film *J. Fluid Mech.* **250** 433–80

[12] Nguyen L T and Balakotaiah V 2000 Modeling and experimental studies of wave evolution on free falling viscous films *Phys. Fluids.* **12** 2236–56

[13] Ruyer-Quil C and Manneville P 2000 Improved modeling of flows down inclined planes *Eur. Phys. J.* B **15** 357–69

[14] Ruyer-Quil C and Manneville P 2002 Further accuracy and convergence results on the modeling of flows down inclined planes by weighted-residual approximations *Phys. Fluids.* **14** 170–83

[15] Shkadov V Ya 1967 Wave regimes of thin layer flow of viscous fluid under the gravity effect *Izv. Akad. Nauk. SSR. Mekh. Zhidk. Gaza.* **1** 43–51

[16] Dandapat B S and Gupta A S 1978 Long waves on a layer of a viscoelastic fluid down an inclined plane *Rheol. Acta.* **17** 492–9

[17] Dandapat B S and Gupta A S 1997 Solitary waves on the surface of a viscoelastic fluid running down an inclined plane *Rheol. Acta.* **36** 135–43

[18] Shaqfeh E S G, Larson R G and Fredrickson G H 1989 The stability of gravity driven viscoelastic film-flow at low to moderate Reynolds number *J. Non-Newton. Fluid Mech.* **31** 87–113

[19] Lin J S and Hwang C C 2000 Finite amplitude long wave instability of power-law liquid films *Int. J. Non-lin. Mech.* **35** 769–77

[20] Hung C I, Tasi J S and Chen C K 1996 Nonlinear stability of the thin micropolar liquid film flowing down on a vertical plane *ASME Trans. J. Fluid Eng.* **118** 498–505

[21] Cheng P J, Chen C K and Lai H Y 2001 Nonlinear stability analysis of the thin micropolar liquid film flowing down on a vertical cylinder *ASME Trans. J. Fluid Eng.* **123** 411–21

[22] Gupta A S 1967 Stability of a viscoelastic liquid film flowing down an inclined plane *J. Fluid Mech.* **28** 17–28

[23] Gupta A S and Rai L 1968 Note on the stability of a viscoelastic liquid film flowing down an inclined plane *J. Fluid Mech.* **33** 87–91

[24] Cheng P J, Lai H Y and Chen C K 2000 Stability analysis of thin viscoelastic liquid film flowing down on a vertical wall *J. Phys. D: Appl. Phys.* **33** 1674–82

[25] Coleman B D and Noll W 1960 An approximation theorem of functionals with application in continuum mechanics *Arch. Ration. Mech. Anal.* **6** 355–70

[26] Aranson A and Karmer L 2002 The world of the complex Ginzburg–Landau equation *Rev. Mod. Phys.* **74** 99–143

[27] Oldroyd J G 1950 On the formulation of rheological equations of state *Proc. R. Soc.* A **200** 523–41

[28] Porteous K C and Denn M M 1972 Linear stability of plane Poiseuille flow of viscoelastic liquids *Trans. Soc. Rheol.* **16** 295–308