Higher order intertwining approach to quasinormal modes

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Abstract

Using higher order intertwining operators we obtain new exactly solvable potentials admitting quasinormal mode (QNM) solutions of the Klein-Gordon equation. It is also shown that different potentials exhibiting QNM's can be related through nonlinear supersymmetry.

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1 Introduction

Quasinormal modes (QNM) are basically discrete complex freequency solutions of real potentials. They appear in the study of black holes and in recent years they have been widely studied [1]. Interestingly QNM's have also been found in nonrelativistic systems [2]. However as in the case of bound states or normal modes (NM) there are not many exactly solvable potentials admitting QNM solutions and often QNM frequencies have to be determined numerically or other approximating techniques such as the WKB method, phase integral method etc. Consequently it is of interest to obtain new exactly solvable potentials admitting such solutions.

In the case of NM's or scattering problems a number of methods based on intertwining technique e.g, Darboux algorithm [3], supersymmetric quantum mechanics (SUSYQM) [4] etc. have been used successfully to construct new solvable potentials. Usually the intertwining operators are constructed using first order differential operators. However in recent years intertwining operators have been generalised to higher orders [5, 6, 7, 8, 9] and this has opened up new possibilities to construct a whole new class of potentials having nonlinear symmetry. In particular use of higher order intertwining operator or higher order Darboux algorithm leads to nonlinear supersymmetry.

QNM's are associated with outgoing wave like behaviour at spatial infinity and unlike normal modes (NM's), the QNM wave functions have rather unusual characteristic (for example, wave functions diverging at both or one infinity) [10]. Such open systems have been studied using (first order) intertwining technique [11]. Recently it has also been shown that open systems can be described within the framework of first order SUSY [10]. Here our objective is to examine whether or not intertwining method based on higher order differential operators can be applied to open systems. For the sake of simplicity we shall confine ourselves to second order intertwining operators (second order Darboux formalism) and it will be shown that the second order Darboux algorithm can indeed be applied to models admitting QNM's although not exactly in the same way as in the case of NM's. In particular we shall use the second order intertwining operator to the Pöschl-Teller potential to construct several new potentials admitting QNM solutions. It will also be shown that such potentials may be related to the Pöschl-Teller potential by second order SUSY. The organisation of the paper is as follows: in section 2, we present construction of new potentials using second order intertwining operators; in section 3, nonlinear SUSY underlying the potentials is shown and finally section 4 is devoted to a conclusion.

2 Second order intertwining approach to quasinormal modes

Two Hamiltonians H_0 and H_1 is said to be intertwined by an operator L if

$$LH_0 = H_1 L \tag{1}$$

Clearly if ψ is an eigenfunction H_0 with eigenvalue E then $L\psi$ is an eigenfunction of H_1 with the same eigenvalue provided $L\psi$ satisfies required boundary conditions. If L is constructed using first order differential operators then intertwining method is equivalent to Darboux formalism or SUSYQM. In particular if V_0 is the starting potential and $L = \frac{d}{dx} + W(x)$, then the isospectral potential is $V_1 = V_0 + 2\frac{dW}{dx}$ [4]. Similarly if L is generalised to higher orders then it is equivalent to higher order Darboux algorithm or higher order SUSY.

Let us now consider L to be a second order differential operator of the form [7]

$$\begin{aligned} L &= \frac{d^2}{dx^2} + \beta(x)\frac{d}{dx} + \gamma(\beta) \\ \beta(x) &= -\frac{d}{dx}\, log\; W_{i,j}(x) \\ \gamma(\beta) &= -\frac{\beta''}{2\beta} + \left(\frac{\beta'}{2\beta}\right)^2 + \frac{\beta'}{2} + \frac{\beta^2}{4} - \left(\frac{\omega_i^2 - \omega_j^2}{2\beta}\right)^2 \end{aligned} \tag{2}$$

where ψ_i and ψ_j are eigenfunctions of H_0 corresponding to the eigenvalues ω_i^2 and ω_j^2 and $W_{i,j} = (\psi_i\psi_j' - \psi_i'\psi_j)$ is the corresponding Wronskian. Then the isospectral partner potential $V_2(x)$ obtained via second order Darboux formalsim is given by

$$V_2(x) = V_0(x) - 2\frac{d^2}{dx^2}\log W_{i,j}(x) \tag{3}$$

The wave functions $\psi_i(x)$ and $\phi_i(x)$ corresponding to $V_0(x)$ and $V_2(x)$ are connected by

$$\phi_k(x) = L\psi_k(x) = \frac{1}{W_{i,j}(x)}\begin{vmatrix} \psi_i & \psi_j & \psi_k \\ \psi_i' & \psi_j' & \psi_k' \\ \psi_i'' & \psi_j'' & \psi_k'' \end{vmatrix}, \quad i,j \neq k \tag{4}$$

The eigenfunctions obtained from ψ_i and ψ_j are given by

$$f(x) \propto \frac{\psi_i(x)}{W_{i,j}(x)} \quad , \quad g(x) \propto \frac{\psi_j(x)}{W_{i,j}(x)} \tag{5}$$

It may be noted that in the case of normal modes, the new potential would be free of any new singularities if the Wronskian $W_{i,j}(x)$ is nodeless. This in turn requires that the Wronskian be constructed with the help of consecutive eigenfunctions (i.e, $j = i + 1$). Also, the eigenfunctions $f(x), g(x)$ in (5) are not acceptable because they do not satisfy the boundary conditions for the normal modes and in any case they are not SUSY partners of the corresponding states in the original potential. Thus in the case of normal modes the spectrum of the new potential is exactly the same as the starting potential except for the levels used in the construction of the Wronskian. However we shall find later that not all of these results always hold in the case of QNM's.

Let us now consider one dimensional Klein-Gordon equation of the form [10]

$$[\partial_t^2 - \partial_x^2 + V(x)]\psi(x,t) = 0 \tag{6}$$

The corresponding eigenvalue equation reads

$$H\psi_n = \omega_n^2\psi_n(x), \qquad H = -\frac{d^2}{dx^2} + V(x) \tag{7}$$

The QNM solutions of the equation (7) are characterised by the fact that they are either (1) increasing at both ends (II) (2) increasing at one end and decreasing at the other (ID,DI). The wave functions decreasing at both ends (DD) correspond to bound states or NM's. In the case of QNM's the eigenvalues (ω_n^2) may be complex or real and negative. If $Re(\omega_n) \neq 0$ then the SUSY formalsim can not be applied since in that case the superpotential $W(x)$ becomes complex and consequently one of the partner potential becomes complex. So we shall confine ourselves to the case when $Re(\omega_n) = 0$ i.e, ω_n^2 are real and negative. We would also like to mention that in case Eq.(7) is to be interpreted as a Schrödinger equation one just has to consider the replacement $\omega_n^2 \rightarrow E_n$.

There are a number of potentials which exhibit QNM's. A potential in this category is the inverted Pöschl-Teller potential. This potential is used as a good approximation in the study of Schwarzschild black hole and it is given by

$$V_0(x) = \nu sech^2 x \tag{8}$$

Eq.(7) for the potential (8) can be solved in different ways. One of the simplest way is to apply the shape invariance criteria [4] and the solutions are found to be [12, 13, 14]

$$\omega_n^{\pm} = -i(n - A^{\pm}),\ A^{\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \nu} = -\frac{1}{2} \pm q \tag{9}$$

$$\psi_n^{\pm}(x) = (sechx)^{(A^{\pm}-n)}{}_2F_1(\frac{1}{2} + q - i\omega_n^{\pm}, \frac{1}{2} - q - i\omega_n^{\pm}, 1 - i\omega_n^{\pm}, \frac{1 + tanhx}{2}) \tag{10}$$

We note that the behaviour of the wave functions (10) i.e, whether they represent a NM or QNM depends on the value of the parameter ν. For $\nu \in (0, 1/4)$ i.e, $A^+ \in (-1/2, 0), A^- \in (-1, -1/2)$, the wave functions represent outgoing waves and are of the type (II). For $\nu < 0$ (i.e, $A^+ > 0, A^- < 0$) then the wave functions represent NM's when $n < A^+$ while for $n > A^+$ they are QNM's. On the other hand the wave functions are always QNM's corresponding to ω_n^-. It may be noted that for the QNM's the wave functions (10) for even n are nodeless while those for odd n have exactly one node at the orign. This behaviour of the wave functions is quite different from those occuring in the case of NM's. Using the procedure mentioned above we shall now construct new exactly solvable potentials admitting QNM solutions.

2.1 Construction of isospectral partner potential using NM's

Case 1. V_0 with two NM's: In order to apply the second order intertwining approach one may start with a potential $V_0(x)$ admitting (1) at least two NM's and the rest QNM's or (2) only QNM's. We begin with the first possibility. Thus we consider $\nu = -5.04$ so that there are two NM's. In this case we obtain from (9) $A^{\pm} = 1.8, -2.8$. Thus the NM's correspond to $\omega_0^+ = 1.8i, \omega_1^+ = 0.8i$ and are given by

$$\psi_0^+(x) = (sech\ x)^{A^+} \quad , \quad \psi_1^+(x) = (sech\ x)^{(A^+-1)}{}_2F_1(2A^+, -1, A^+, \frac{1 + tanhx}{2}) \tag{11}$$

The QNM's in this case correspond to the frequencies $\omega_n^+, n = 2, 3, ...$ and are given by $\psi_n^+(x)$. Also for $\omega_n^-, n = 0, 1, 2, ...$ there is another set of QNM's and the corresponding wave functions are given by $\psi_n^-(x)$. In this case $A^- < 0$ and consequently there is no NM.

Let us construct a potential isospectral to (8) using the NM frequencies ω_0^+ and ω_1^+. Then from (10) the Wronskian $W_{0,1}^+$ is found to be

$$W_{0,1}^+ = -(sechx)^{2A^+-1} \tag{12}$$

Clearly $W_{0,1}^+$ does not have a zero. In this case the new potential $V_2^+(x)$ is free of singularities and is given by

$$V_2^+(x) = V_0(x) - 2\frac{d^2}{dx^2} \log W_{0,1}^+(x) = -(1 - A^+)(2 - A^+)sech^2 x \tag{13}$$

Using the value of A^+ it is easy to see that the new potential $V_2^+(x)$ in (13) does not support any bound state but only QNM's. This is also reflected by the explicit expressions for the wave functions. Using (5) we find

$$f^+(x) = (sechx)^{(1-A^+)} \quad , \quad g^+(x) = -(sechx)^{-A^+} tanhx \tag{14}$$

From (14) it follows that the above wave functions are QNM's corresponding to $-\omega_1^+$ and $-\omega_0^+$ respectively. Note that these two QNM's are new and were not present in the original potential. This in fact is where the behaviour of the new potential is different from the usual case. In the case of potentials supporting only NM's the wave functions $f^+(x), g^+(x)$ obtained through (5) do not have acceptable behaviour. However in the present case both these wave functions become QNM's instead of NM's and they have acceptable behaviour at $\pm\infty$ as can be seen from (14) as well as from figure 1. The other wave functions $\phi_n^+(x), n = 2, 3, ...$ corresponding to QNM frequencies $\omega_n^+ = -i(n - A^+)$ can be obtained using (4) and are given by

$$\begin{aligned} &\phi_n^+(x) = (sechx)^{(A^+-n)} \left[(n-1)n\ F_n\ tanh^2x \right. \\ &+c_1(2n-3)\ F_{n+1}\ sech^2x\ tanhx \\ &\left. + c_1c_2\ F_{n+2}\ sech^4x\right], \quad n = 2, 3, ... \end{aligned} \tag{15}$$

where

$$\begin{aligned} c_1 &= -\frac{n(2A^+ - n + 1)}{2(A^+ - n + 1)} \quad , \quad c_2 = \frac{(-n+1)(2A^+ - n + 2)}{2(A^+ - n + 2)} \\ F_n &= {}_2F_1(-n, 2A^+ - n + 1, A^+ - n + 1, \tfrac{1+tanhx}{2}) \\ F_{n+1} &= {}_2F_1(-n+1, 2A^+ - n + 2, A^+ - n + 2, \tfrac{1+tanhx}{2}) \\ F_{n+2} &= {}_2F_1(-n+2, 2A^+ - n + 3, A^+ - n + 3, \tfrac{1+tanhx}{2}) \end{aligned} \tag{16}$$

To see the nature of the wave functions (15) we have plotted $\phi_2^+(x)$ and $\phi_3^+(x)$ in figure 1. From the figure it can be seen that these wave functions are indeed QNM's and for even n they do not have nodes while for odd n they have one node at the origin. We would like to point out that the new potential $V_2^+(x)$ has two more QNM's than $V_0(x)$. Thus except for two additional QNM's, the QNM frequencies ω_n^+ is common to both $V_0(x)$ and $V_2^+(x)$. We now examine the second set of solutions corresponding to ω_n^-. It can be shown by direct calculation that the new potential (13) also possess this set of solutions.

Case 2. $V_0(x)$ with three NM's: Let us now consider the potential (8) supporting three NM's. A convenient choice of the parameter is $\nu = -6.2$ so that $A^+ = 2.04, A^- = -3.04$. We shall now construct the new potential using the NM frequencies $\omega_1^+ = 1.04i$ and $\omega_2^+ = 0.04i$. The Wronskian $W_{1,2}^+$ is found to be

$$W_{1,2}^+(x) = \frac{(sechx)^{2A^+-1}}{2(A^+-1)}[A^+ - 2 - (A^+ - 1)cosh2x] \tag{17}$$

Now using (3) we obtain

$$V_2^+(x) = -(A^+-1)(A^+-2)sech^2x + 8(A^+-1)\frac{(A^+-2)cosh2x - (A^+-1)}{[(A^+-1)cosh2x - (A^+-2)]^2} \tag{18}$$

To get an idea of the potential, we have plotted $V_2^+(x)$ in figure 2. From figure 2, it is clear that $V_2^+(x)$ supports at least one NM. Next to examine the wave functions we first consider $f^+(x)$ and $g^+(x)$. ¿From the relation (5) we obtain

$$f^+(x) = \frac{2(A^+ - 1)(sechx)^{-A^+} tanhx}{(A^+ - 1)cosh2x - (A^+ - 2)} \quad , \quad g^+(x) = \frac{(sechx)^{-(A^+ + 1)}[1 - (2A^+ - 1)tanh^2 x]}{(A^+ - 1)cosh2x - (A^+ - 2)} \tag{19}$$

Also from (4) it follows that

$$\phi_0^+(x) = \frac{4(A^+ - 1)(sechx)^{(A^+ - 2)}}{(A^+ - 1)cosh2x + A^+ - 2} \tag{20}$$

¿From (19) it follows that $f^+(x)$ and $g^+(x)$ are new QNM's corresponding to frequencies $-\omega_2^+ = -0.04i$ and $-\omega_1^+ = -1.04i$ respectively. The former has one node the later has two nodes. The nodal structure of the QNM wave functions are different from those obtained earlier. The reason for this is that since we started with the first and second excited state NM's and the Wronskian $W_{1,2}^+$ is nodeless, the behaviour of the original wave functions $\psi_{1,2}^+(x)$ are retained by $f^+(x)$ and $g^+(x)$. However, $\phi_0^+(x)$ is a NM at $\omega_0^+ = 2.04i$ and it does not have a node because $\psi_0^+(x)$ does not have one. Also other QNM wave functions $\phi_n^+(x), n = 3, 4, ...$ have either no node or one node. In figure 3 we have plotted the some of the wave functions. We also note that although the potential in (13) is of a similar nature as (8), the potential (18) is of a completely different type. In particular it is a non shape invariant potential. Finally we discuss the possibility of a second set of solutions for the potential (18). We recall that the existence of two sets of solutions for the potential (8) (or (13)) was due to the fact that the parameter ν could be expressed as a product of two different parameters $A^\pm$. However in the case of (18) the entire potential can not be expressed in terms of two distinct parameters because of the presence of the second term. Consequently the potential (18) has only one set of solution mentioned above.

2.2 Construction of isospectral partner potential using QNM's

Case 1. Potential based on consecutive QNM's: Here we shall construct isospectral partner of a potential which has only QNM's. Thus we consider $\nu = 0.24$ and in this case $A^\pm = -0.4, -0.6$. We consider the A^+ sector and begin with the freequencies ω_0^+ and ω_1^+. In this case the expression for the Wronskian $W_{0,1}^+$, the new potential $V_2^+(x)$ and the QNM wave functions can be derived from the expressions (12), (13) and (15) respectively except that we now have to use a different parameter value. Thus the new potential is given by

$$V_2^+(x) = -3.36 \ sech^2 x \tag{21}$$

For this potential the NM's corresponding to $-\omega_0^+ = 0.4i$ and $-\omega_1^+ = 1.4i$ are given respectively by

$$f^+ = (sechx)^{1.4} \quad , \quad g^+ = (sechx)^{0.4} tanhx \tag{22}$$

Clearly these NM's are not SUSY partner of any levels in H_0. The QNM's are correspond to $\omega_n^+ = -i(n + 0.4), n = 2, 3, ...$ and are given by (15) with $A^+ = -0.4$. We have plotted some of the wave functions in fig 4. From the figure we find that the wave functions $f^+(x)$ and $g^+(x)$ correspond to NM's and the other wave functions represent QNM's which are the SUSY partners of the QNM's in H_0. We note that as in (13) the potential (21) has two sets of QNM's, the second of which corresponds to ω_n^-.

Case 2. Potential based on non consecutive QNM's: Here we shall consider the previous parameter values (i.e, $A^+ = -0.4$) and construct the new potential using the non consecutive levels ω_0^+ and ω_3^+. In this case the Wronskian is given by

$$W_{0,3}^+ = \frac{(sechx)^{(2A^+ - 3)}}{2(A^+ - 2)}[(9 - 6A^+)tanh^2 x + 3] \tag{23}$$

It can be shown that the Wronskian (23) is nodeless. Now using the (3) the new potential is found to be

$$V_2^+(x) = \frac{(A^+-2)[2(A^+(A^+-2)(3A^+-7) - 2A^+(A^+-1)(A^+-4)\ cosh2x - (3-2A^+)^2(A^+-1)\ sech^2x]}{[1-A^+ + (A^+-2)\ cosh2x]^2} \tag{24}$$

The potential (24) is free of any singularity and is plotted in fig 5. From figure 5 we find that it supports NM's. Also as explained earlier, this potential has also one set of solution. We now consider the wave functions corresponding to $\psi_0^+(x)$ and $\psi_3^+(x)$. These are obtained from (5) and are given by

$$f^+(x) = \frac{2(A^+-2)}{(9-6A^+)tanh^2x+3}\ (sechx)^{(3-A^+)} \quad , \quad g^+(x) = \frac{(1-2A^+)tanh^2x+3}{(9-6A^+)tanh^2x+3}\ sinhx\ (sechx)^{(1-A^+)} \tag{25}$$

The above wave functions (with zero and one node respectively) represent NM's corresponding to $-\omega_3^+ = 3.4i$ and $-\omega_0^+ = 0.4i$. The other wave functions can be obtained through (4). The two QNM wave functions lying between ω_0^+ and ω_3^+ are $\phi_{1,2}^+(x)$ corresponding to $\omega_1^+ = -1.4i$ and $\omega_2^+ = -2.4i$. We have plotted these wave functions in fig 6. From figure 6, it can be seen that $f^+(x)$ and $g^+(x)$ are NM's while $\phi_{1,2}^+(x)$ are QNM's, with the later having two nodes. The rest of the QNM wave functions corresponding to the freequencies $\omega_n^+ = -(n+0.4)i, n \neq 0, 3$ are given by $\phi_n^+(x)$ and they have either zero or one node.

3 Polynomial SUSY

In first order SUSY, the anticommutator $\{Q, Q^\dagger\}$ of the supercharges is a linear function of the Hamiltonian. On the other hand in higher order SUSY, $\{Q, Q^\dagger\}$ is a nonlinear function of the Hamiltonian. It will be shown here that the Hamiltonians H_0 and H_2 are related by second order SUSY. To this end we define the supercharges Q and $Q^\dagger$ as follows:

$$Q = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix} \quad , \quad Q^\dagger = \begin{pmatrix} 0 & L^\dagger \\ 0 & 0 \end{pmatrix} \tag{26}$$

where the operator L is given by (2).

Clearly the supercharges Q and $Q^\dagger$ are nilpotent. We now define a super Hamiltonian H of the form

$$H = \begin{pmatrix} H_0 & 0 \\ 0 & H_2 \end{pmatrix} \tag{27}$$

It can be easily verified that $Q, Q^\dagger$ and H satisfy the following relations :

$$[Q, H] = [Q^\dagger, H] = 0 \tag{28}$$

Then the anticommutator of the supercharges Q and $Q^\dagger$ is given by a second order polynomial in H :

$$H_{ss} = \{Q, Q^\dagger\} = \begin{pmatrix} L^\dagger L & 0 \\ 0 & LL^\dagger \end{pmatrix} = \left(H + \frac{\delta}{2}\right)^2 - c\mathcal{I} \tag{29}$$

where $\mathcal{I}$ is the 2×2 unit matrix and

$$\delta = -(\omega_i^2 + \omega_j^2) \quad , \quad c = \left(\frac{\omega_i^2 - \omega_j^2}{2}\right)^2 \tag{30}$$

Also we have

$$[Q, H_{ss}] = [Q^\dagger, H_{ss}] = 0 \tag{31}$$

The relations (29) and (31) constitute second order SUSY algebra.

As an example let us consider the potentials (8) and (13). The corresponding Hamiltonians H_0 and H_2 are obtained from (7). In this case $\delta = 0.5416$ and $c = 0.2916$ so that from (29) we get

$$H_{ss} = (H + 0.5416)^2 - 0.2916\mathcal{I} \tag{32}$$

In a similar fashion one may obtain H_{ss} for the other pair of potentials.

4 Conclusion

Here we applied the second order Darboux algorithm to the Pöschl-Teller potential and obtained new exactly solvable potentials admitting QNM solutions. We have considered a number of possibilties to construct the new potentials e.g, starting from NM's or starting from QNM's. It has also been shown that the new potentials are related to the original one by second order SUSY. We feel it would also be also useful to analyse the construction of potentials using various levels as well as for different values of the parameter ν (for example, $\nu =$ half-integer) [10]. Finally we belive it would be interesting to extend the present approach to other effective potentials appearing in the study of Reissner-Nordström, Kerr black hole etc.

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