Nonlocal $\mathcal{PT}$-symmetric potentials

B Roy and R Roychoudhury

Physics and Applied Mathematics Unit, Indian Statistical Institute, Calcutta 700 035, India

Abstract

The factorization approach for complex Hamiltonians has been used to obtain the exactly solvable nonlocal variant of $\mathcal{PT}$-symmetric local potentials. The formalism is used to obtain exact eigenvalues and eigenfunctions of the nonlocal $\mathcal{PT}$-symmetric Scarf potential.

PACS number: 03.65.−w

1. Introduction

Recently there has been a growing interest in the study of non-Hermitian Hamiltonians which appear in different branches of physics [1]. The main reason for this is that the energy spectrum of a number of complex potentials turned out to be real (at least partly). Bender and his collaborators [2] attributed this unusual behaviour of the energy spectrum to the so-called $\mathcal{PT}$ symmetry, i.e. the invariance of the Hamiltonian with respect to the simultaneous space $\mathcal{P}$ and time $\mathcal{T}$ reflection. However, it is now known that for the real spectrum of a $\mathcal{PT}$-symmetric Hamiltonian the energy eigenfunctions have to be necessarily simultaneous eigenstates of the combined operator $\mathcal{PT}$ [2]. Otherwise, $\mathcal{PT}$ is spontaneously broken and the eigenvalues are arranged in complex conjugate pairs. In recent times, it has been stressed [3] that a quantum Hamiltonian H having a complete set of eigenvectors will have a real spectrum if and only if there exists a positive definite operator η such that

$$H^{\dagger} = \eta H \eta^{-1}, \tag{1}$$

i.e. H is η-pseudo-Hermitian. The other equivalent conditions are discussed in [3].

On the other hand, complex nonlocal potentials [5–9], in particular the $\mathcal{PT}$-symmetric ones, have attracted a lot of interest in recent years. The scattering by $\mathcal{PT}$-symmetric nonlocal potential was studied in [10]. In [11, 15], $\mathcal{PT}$-symmetric point (nonlocal) interactions were used to clarify certain properties of $\mathcal{PT}$-symmetric quantum-mechanical Hamiltonians. The symmetries and general characteristics of $\mathcal{PT}$ symmetrical point interactions were discussed in [12]. Eigenvalues of $\mathcal{PT}$ symmetrical Hamiltonians were calculated in [13]. Integrability and $\mathcal{PT}$ symmetry of many-body systems with pseudo-Hermitian point interactions were studied in [14]. The application of the ideas of supersymmetric quantum mechanics [16] in constructing non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians has been considered in [17] and a formulation of $\mathcal{PT}$-symmetric supersymmetry has been outlined in [18]. In this paper, our aim is to extend the idea of nonlocal potentials from complex point interactions to other types of interactions. To this end we shall use the factorization approach, which was extended to the case of complex potentials in [19] to obtain exact solutions of a nonlocal deformation of exactly solvable complex $\mathcal{PT}$ invariant local potentials. We shall find the energy eigenfunctions and eigenvalues and shall show that in some cases the partner Hamiltonians have normalizable ground states while in other cases they have not. The organization of the paper is as follows. In Section 2, we give the details of factorization approach to treat the nonlocal deformation of $\mathcal{PT}$-symmetric potentials. Application of the formalism to a specific potential is given in section 3 and section 4 is devoted to a conclusion.

2. Complex factorization approach

The time-independent Schrödinger equation in the position representation is given by

$$\tilde{H}\psi(x) = E\psi(x), \tag{2}$$

where the Hamiltonian $\tilde{H}$ is given by

$$\tilde{H}\psi(x) = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + V(x)\psi(x) + \int_{-\infty}^{\infty}\mathrm{d}y\, v(x, y)\psi(y) = E\psi(x), \tag{3}$$

with $V(x)$ and $v(x, y)$ being the complex local and nonlocal potentials, respectively.

To apply the factorization technique we would assume that $\tilde{H}$ in equation (3) can be generalized to

$$\tilde{H} = \begin{pmatrix} \tilde{H}_+ & 0 \\ 0 & \tilde{H}_- \end{pmatrix}, \tag{4}$$

where $\tilde{H}_+ = \hat{A}\hat{B}$ and $\tilde{H}_- = \hat{B}\hat{A}$ are isospectral partners and $\hat{A}, \hat{B}$ are linear first-order differential operators. Details of $\hat{A}, \hat{B}$ will be discussed later on. Also let $V_\pm$ and $v_\pm$ be, respectively, the local and nonlocal potentials for $H_\pm$ and $E_\pm$ be the corresponding energies.

Writing

$$\begin{aligned} \langle x|V_\pm(x)|\psi_\pm\rangle &= V_\pm(x)\psi_\pm(x), \\ \langle x|v_\pm(x, y)|\psi_\pm\rangle &= \int_{-\infty}^{\infty}\mathrm{d}y\, v_\pm(x, y)\psi_\pm(y), \end{aligned} \tag{5}$$

the Schrödinger equations corresponding to $H_\pm$ can be written as

$$\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_\pm(x) + \langle x|V_\pm(x)|\psi_\pm\rangle + \langle x|v_\pm(x, y)|\psi_\pm\rangle = E_\pm\psi_\pm(x). \tag{6}$$

Let us write the potentials in an operator form as

$$\hat{V}_\pm = \int_{-\infty}^{\infty}\mathrm{d}x|x\rangle V_\pm(x)\langle x|, \qquad \hat{v}_\pm = \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y|x\rangle v_\pm(x, y)\langle y|. \tag{7}$$

The partner Hamiltonians $\tilde{H}_+$ and $\tilde{H}_-$ can be factorized, respectively, as $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$, where

$$\hat{B} = -\frac{\mathrm{i}\hat{p}}{\sqrt{2m}} + \hat{W} + \hat{w} \qquad \hat{A} = \frac{\mathrm{i}\hat{p}}{\sqrt{2m}} + \hat{W} + \hat{w} \tag{8}$$

and $\hat{p} = -\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}$. $\hat{W}$ and $\hat{w}$ are defined as in equation (7), namely,

$$\hat{W} = \int_{-\infty}^{\infty} \mathrm{d}x|x\rangle W(x)\langle x|, \qquad \hat{w} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y|x\rangle w(x, y)\langle y|. \tag{9}$$

The potentials $V_\pm$ and $v_\pm$ are written in terms of the factorization potentials $W(x)$ and $w(x, y)$ (superpotentials in the supersymmetric quantum mechanics) as

$$\begin{aligned} V_\pm(x) &= [W(x)]^2 \pm \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W(x)}{\mathrm{d}x}, \\ v_\pm(x, y) &= \int_{-\infty}^{\infty} \mathrm{d}u\, w(x, u)w(u, y) + [W(x) + W(y)]w(x, y) \\ &\quad \pm \frac{\hbar}{\sqrt{2m}}\left[\frac{\partial w(x, y)}{\partial x} + \frac{\partial w(x, y)}{\partial y}\right]. \end{aligned} \tag{10}$$

It is to be mentioned here [20] that in models with local potentials, there is a one-to-one relationship between the ground state and the factorization potential $W(x)$, but it is not so with nonlocal potentials. In nonlocal models, given the factorization potentials $W(x)$ and $w(x, y)$, the zero-energy ground state $\psi_0(x)$ of $\hat{H}_-$ say is obtained from the integro-differential equation

$$\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}\psi_0(x)}{\mathrm{d}x} + W(x)\psi_0(x) + \int_{-\infty}^{\infty} \mathrm{d}y\, w(x, y)\psi_0(y) = 0. \tag{11}$$

We shall now construct a class of exactly solvable models with both complex ($\mathcal{PT}$) invariant local and nonlocal potentials starting from any exactly solvable local model with the factorization potential $W_0(x)$ and to this end we choose

$$W(x) = (1 - c)W_0(x), \qquad w(x, y) = \frac{C_1\hbar}{\sqrt{2m}}\frac{\partial}{\partial x}\delta(x - y), \tag{12}$$

where C_1 is a parameter of nonlocality and c is a constant.

Substituting $W(x)$ and $w(x, y)$ from equation (12) into equation (10), we obtain

$$\begin{aligned} V_\pm(x) &= (1 - c)^2[W_0(x)]^2 \pm (1 - c)\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W_0(x)}{\mathrm{d}x}, \\ \int_{-\infty}^{\infty} \mathrm{d}y\, v_\pm(x, y)\psi_\pm(y) &= \frac{\hbar^2}{2m}C_1^2\frac{\mathrm{d}^2\psi_\pm(x)}{\mathrm{d}x^2} + 2\frac{\hbar}{\sqrt{2m}}C_1(1 - c)W_0(x)\frac{\mathrm{d}\psi_\pm(x)}{\mathrm{d}x} \\ &\quad \pm (1 - c)\frac{\hbar}{\sqrt{2m}}C_1\frac{\mathrm{d}W_0(x)}{\mathrm{d}x}\psi_\pm(x). \end{aligned} \tag{13}$$

Thus, the contribution of nonlocal potential to the Hamiltonian is given by

$$\hat{v}_\pm = -C_1^2\frac{\hat{p}^2}{2m} + \frac{2\mathrm{i}(1 - c)C_1}{\sqrt{2m}}W_0(x)\hat{p} \pm \frac{\hbar(1 - c)C_1}{\sqrt{2m}}W_0'(x). \tag{14}$$

Consequently, the eigenvalue equations for the Hamiltonians $\tilde{H}_\pm$ are written as

$$\tilde{H}_\pm\psi_\pm = E_\pm\psi_\pm, \tag{15}$$

where

$$\begin{aligned} \tilde{H}_\pm &= -\frac{\hbar^2}{2m}\left(1 - C_1^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\hbar C_1(1 - c)}{\sqrt{2m}}W_0(x)\frac{\mathrm{d}}{\mathrm{d}x} \\ &\quad + \frac{\hbar}{\sqrt{2m}}(1 - c)(C_1 \pm 1)\frac{\mathrm{d}W_0(x)}{\mathrm{d}x} + (1 - c)^2W_0^2(x). \end{aligned} \tag{16}$$

To solve the eigenvalue problem in equation (15), our strategy would be to find a similarity transformation mapping the Hamiltonians in equation (16) into a standard form.

To this end, we make the transformation

$$\psi_\pm(x) = \mathrm{e}^{-\frac{1}{2}\int f(x)\,\mathrm{d}x}\phi_\pm(x) = \eta\phi_\pm(x), \tag{17}$$

where

$$\eta = \mathrm{e}^{-\frac{1}{2}\int f(x)\,\mathrm{d}x}, \qquad f(x) = -\frac{2\sqrt{2m}C_1(1-c)}{\hbar\left(1-C_1^2\right)}W_0(x), \tag{18}$$

so that the transformed Hamiltonians $\bar{H}_\pm = \eta^{-1}\tilde{H}_\pm\eta$ are given by

$$\begin{aligned}\bar{H}_\pm\phi_\pm(x) = &-\frac{\hbar^2}{2m}\left(1-C_1^2\right)\frac{\mathrm{d}^2\phi_\pm(x)}{\mathrm{d}x^2} + \frac{(1-c)^2}{\left(1-C_1^2\right)}W_0^2(x)\phi_\pm(x)\\ &\pm\frac{\hbar}{\sqrt{2m}}(1-c)W_0'(x)\phi_\pm(x) = E_+\phi_\pm(x).\end{aligned} \tag{19}$$

It is to be noted that the factorization can again be applied to $\bar{H}_\pm$. In fact

$$\bar{H}_+ = \hat{C}\hat{D}, \qquad \bar{H}_- = \hat{D}\hat{C}, \tag{20}$$

where

$$\begin{aligned}\hat{C} &= \frac{\hbar}{\sqrt{2m}}(1+C_1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(1-c)}{(1-C_1)}W_0(x),\\ \hat{D} &= -\frac{\hbar}{\sqrt{2m}}(1-C_1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(1-c)}{(1+C_1)}W_0(x).\end{aligned} \tag{21}$$

The factorization (20) indicates that $\bar{H}_\pm$ are isospectral depending on the vanishing of the operator D and/or normalizability of the eigenfunctions.

For the special case $C_1^2 = c$, the Hamiltonians $\bar{H}_\pm$ can be written in terms of the local Hamiltonians $\bar{H}_{\pm,\mathrm{local}}$ as

$$\bar{H}_\pm = (1-c)\bar{H}_{\pm,\mathrm{local}}. \tag{22}$$

From the above equation it follows that

$$\phi_\pm = \chi_\pm, \qquad E_\pm = (1-c)E_{\pm,\mathrm{local}}, \tag{23}$$

where $\chi_\pm$ are the eigenfunctions of the local Hamiltonian.

It may be noted that in the case of $\mathcal{PT}$-symmetric systems, neither the standard definition of the inner product in Hilbert space $\mathcal{H}$ nor the straightforward generalization would work, because the norm becomes negative for some of the states. For details about the positive definite scalar products in $\mathcal{PT}$-symmetric systems, we refer the reader to [3, 23, 25].

3. Example

Here we shall apply the formalism of section 2 to obtain exact solutions of a nonlocal variant of the $\mathcal{PT}$-symmetric Scarf potential [21]. In this case, the factorization potential is taken as

$$W_0(x) = \lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x. \tag{24}$$

Then from equation (14), $\hat{v}_\pm$ are found to be

$$\begin{aligned}\hat{v}_\pm = &-C_1^2\frac{\hat{p}^2}{2m} + \frac{2\mathrm{i}(1-c)C_1}{\sqrt{2m}}(\lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)\hat{p}\\ &\pm\frac{\hbar(1-c)C_1}{\sqrt{2m}}(\lambda\,\mathrm{sech}^2 x - \mathrm{i}\mu\,\mathrm{sech}\,x\tanh x),\end{aligned} \tag{25}$$

and from equation (16), $\tilde{H}_\pm$ are given by

$$\tilde{H}_\pm = -\frac{\hbar^2}{2m}\left(1 - C_1^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\hbar C_1(1-c)}{\sqrt{2m}}(\lambda \tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\hbar(1-c)(C_1 \pm 1)}{\sqrt{2m}}(\lambda\,\mathrm{sech}^2 x - \mathrm{i}\mu\,\mathrm{sech}\,x\tanh x) + (1-c)^2(\lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)^2. \tag{26}$$

With $f(x) = -\frac{2\sqrt{2m}C_1(1-c)}{\hbar\left(1-C_1^2\right)}(\lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)$, the transformed Hamiltonians $\bar{H}_\pm$ are given by

$$\bar{H}_\pm = -\frac{\hbar^2}{2m}\left(1 - C_1^2\right)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{(1-c^2)}{\left(1-C_1^2\right)}(\lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)^2 \pm \frac{\hbar}{\sqrt{2m}}(1-c)(\lambda\,\mathrm{sech}^2 x - \mathrm{i}\mu\,\mathrm{sech}\,x\tanh x)\ . \tag{27}$$

These two Hamiltonians admit the factorizations

$$\bar{H}_\pm = \left[\pm\frac{\hbar}{\sqrt{2m}}(1 \pm C_1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(1-c)}{(1 \mp C_1)}(\lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)\right] \times \left[\left(\mp\frac{\hbar}{\sqrt{2m}}\right)(1 \mp C_1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(1-c)}{(1 \pm C_1)}(\lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)\right]. \tag{28}$$

With $C_1^2 = c$, the eigenvalue equations for $\bar{H}_\pm$ can be written as

$$\bar{H}_\pm\phi_\pm(x) = (1-c)\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\phi_\pm(x)}{\mathrm{d}x^2} + (\lambda\tanh x + \mathrm{i}\mu\,\mathrm{sech}\,x)^2\phi_\pm(x) \pm \frac{\hbar}{\sqrt{2m}}(\lambda\,\mathrm{sech}^2 x - \mathrm{i}\mu\,\mathrm{sech}\,x\tanh x)\phi_\pm(x)\right] = E_\pm\phi_\pm(x). \tag{29}$$

The eigenvalues and eigenfunctions can be written from equation (29) using the results of the corresponding local model [21]. In the following we shall consider the case with an unbroken $\mathcal{PT}$ symmetry, i.e. the energies are real. If the potential is written as

$$V(x) = V_1\,\mathrm{sech}^2 x + \mathrm{i}V_2\,\mathrm{sech}\,x\tanh x,$$

this will be true if $|V_2| \leqslant V_1 + \frac{1}{4}$, V_1, V_2 being given in the following equations. Three cases will arise [21]:

Case 1. Positive square roots taken in both t and s (given in the following equations). In this case, the eigenvalues are given by

$$E_{\pm,n^+} = (1-c)\left[\lambda^2 - \frac{\hbar^2}{2m}\left\{n^+ + \frac{1}{2} - \frac{1}{2}(t+s)\right\}^2\right], \tag{30}$$

where

$$n^+ = 0, 1, 2 \cdots < \frac{s+t-1}{2}$$

$$V_1 = \frac{2m}{\hbar}(\lambda^2 + \mu^2) \mp \frac{\sqrt{2m}}{\hbar}\lambda$$

$$V_2 = \pm\frac{\sqrt{2m}}{\hbar}\mu - \frac{2m}{\hbar^2}2\lambda\mu$$

$$t = \sqrt{\frac{1}{4} + V_1 - V_2}$$
$$s = \sqrt{\frac{1}{4} + V_1 + V_2} \tag{31}$$

and the eigenfunctions corresponding to these real eigenvalues are

$$\psi_{\pm,n^+}(x) \approx N_n^+ (\operatorname{sech} x)^{\frac{(1-\sqrt{c})(s+t)-(1\pm\sqrt{c})}{2}}$$
$$\exp\left[\frac{\mathrm{i}}{2}(1+\sqrt{c})(t-s)\tan^{-1}(\sinh x)\right] P_n^{(-t,-s)}(\mathrm{i}\sinh x), \tag{32}$$

where $P_n^{(-t,-s)}$ denotes the Jacobi polynomial and N_n^+ is the normalization constant.

It is to be noted that the nonlocality of the original Hamiltonian is reflected in the spectrum, equation (30) (as well as in the eigenfunctions, equation (32)), through c which is related to the nonlocal parameter C_1.

Case 2. $V_2 > 0$, positive square root in s and negative square root in t.
In this case

$$E_{\pm,n^-} = (1-c)\left[\lambda^2 - \frac{\hbar^2}{2m}\left\{n^- + \frac{1}{2} - \frac{1}{2}(s-t)\right\}^2\right], \tag{33}$$

where

$$n^- = 0, 1, 2 \cdots < \frac{s-t-1}{2}$$
$$V_1 = \frac{2m}{\hbar}(\lambda^2 + \mu^2) \mp \frac{\sqrt{2m}}{\hbar}\lambda$$
$$V_2 = \pm\frac{\sqrt{2m}}{\hbar}\mu - \frac{2m}{\hbar^2}2\lambda\mu \tag{34}$$
$$t = -\sqrt{\frac{1}{4} + V_1 - V_2}$$
$$s = \sqrt{\frac{1}{4} + V_1 + V_2}$$

and

$$\psi_{\pm,n^-}(x) \approx N_n^- (\operatorname{sech} x)^{\frac{(1-\sqrt{c})(s-t)-(1\pm\sqrt{c})}{2}}$$
$$\exp\left[-\frac{\mathrm{i}}{2}(1+\sqrt{c})(t+s)\tan^{-1}(\sinh x)\right] P_n^{(-t,s)}(\mathrm{i}\sinh x). \tag{35}$$

Case 3. $V_2 < 0$, positive square root taken in t and negative square root in s.
In this case

$$E_{\pm,n^-} = (1-c)\left[\lambda^2 - \frac{\hbar^2}{2m}\left\{n^- + \frac{1}{2} - \frac{1}{2}(t-s)\right\}^2\right], \tag{36}$$

where

$$n^- = 0, 1, 2 \cdots < \frac{t-s-1}{2}$$
$$V_1 = \frac{2m}{\hbar}(\lambda^2 + \mu^2) \mp \frac{\sqrt{2m}}{\hbar}\lambda$$
$$V_2 = \pm\frac{\sqrt{2m}}{\hbar}\mu - \frac{2m}{\hbar^2}2\lambda\mu$$

$$
\begin{aligned}
t &= \sqrt{\frac{1}{4} + V_1 - V_2} \\
s &= -\sqrt{\frac{1}{4} + V_1 + V_2}
\end{aligned}
\tag{37}
$$

and

$$
\begin{aligned}
\psi_{\pm,n^-}(x) \approx N_n^-(\operatorname{sech} x)^{\frac{(1-\sqrt{c})(t-s)-(1\pm\sqrt{c})}{2}} \exp\left[\frac{\mathrm{i}}{2}(1+\sqrt{c})(t+s)\tan^{-1}(\sinh x)\right] \\
\times P_n^{(-t,s)}(\mathrm{i}\sinh x).
\end{aligned}
\tag{38}
$$

Now for case 1 and case 3, it is found that the ground state is shared by both the partner Hamiltonians only if $\lambda > \frac{\hbar}{\sqrt{2m}(1-\sqrt{c})} > 0$. For case 2, the ground state will not be shared by the partner Hamiltonians irrespective of the value λ takes.

4. Discussions

The complex factorization approach of quantum mechanics is formally extended in this paper to complex ($\mathcal{PT}$ invariant) nonlocal Hamiltonians. The formalism is applied to obtain exact eigenvalues and eigenfunctions of nonlocal deformation of the $\mathcal{PT}$-invariant Scarf potential. It is seen that in some cases the ground state is shared by partner Hamiltonians for some particular values of the parameters involved and in some other cases this does not appear. This is a typical feature of nonlocality. It should be mentioned that, by similarity transformation equation (17), the nonlocal Hamiltonian is related to local Hamiltonian thereby making it possible to talk about the normalization of wavefunctions of the original nonlocal Hamiltonian which otherwise would not be possible because as far as we know, the inner product for $\mathcal{PT}$-symmetric nonlocal systems is yet to be defined. Here it must be admitted that some theoretical questions for nonlocal potentials are yet to be resolved. We hope to take up this problem in near future. However, the present framework provides a way to link, albeit for special value of the nonlocal parameter, the nonlocal potential to a corresponding local potential. Finally, we feel it would be interesting to study other interactions, especially the shape invariant ones [26] within the present framework.

Acknowledgments

Helpful correspondences with Professor Je-Young Choi are highly appreciated. The authors are grateful to the referees for their constructive suggestions. One of us (RR) is grateful to Council of Scientific and Industrial Research (CSIR) for a grant (project no 21/(0659)/06/EMR-II).

References

[1] Ramirez A and Mielnik B 2003 *Rev. Mex. Fis.* **49S2** 130
[2] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M and Boettcher S 1998 *J. Phys. A: Math. Gen.* **31** L273
Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2201
[3] Mostafazadeh A 2002 *J. Math. Phys.* **43** 205
Mostafazadeh A 2002 *J. Math. Phys.* **43** 2814
Mostafazadeh A 2002 *J. Math. Phys.* **43** 5439
[4] Scholtz F G, Geyer H B and Hahne F J W 1992 *Ann. Phys.* **213** 74
Kretschmer R and Szymanowski L 2004 *Phys. Lett.* A **325** 112
[5] Potthoff M, Lachnitt J, Nolting W and Braun J 1997 *Phys. Status Solidi* b **203** 441

[6] Horacek J, Cizek M and Domcke W 2004 *Theor. Chim. Acta* **100** 31
[7] Trevisan C S, Houfek K, Zhang Z, Orel A E, MaCurdy C W and Rescigno T N 2005 *Phys. Rev.* A **71** 052714
[8] Cederbaum L S and Domcke W 1981 *J. Phys. B: At. Mol. Phys.* **14** 4665
[9] Cassola R L and Koshel R D 1968 *J. Phys. A: Gen. Phys.* **1** 224
[10] Cannata F and Ventura A 2006 *Preprint* quant-ph/0606006 v1, June 2006
Weigert S 2004 *Czech. J. Phys.* **54** 1139
[11] Znojil M and Jakubsky V 2005 *J. Phys. A: Math. Gen.* **38** 5041
Jakubsky V and Znojil M 2005 *Czech. J. Phys.* **55** 1113
[12] Albeverio S *et al* 2002 *Lett. Math. Phys.* **59** 227
Albeverio S *et al* 1998 *Lett. Math. Phys.* **45** 33
Albeverio S *et al* 2005 *J. Phys. A: Math. Gen.* **38** 4975
[13] Cervero J M 2003 *Phys. Lett.* A **317** 26
Cervero J M and Rodriguez A 2004 *J. Phys. A: Math. Gen.* **37** 10167
Demiralp E 2005 *Czech. J. Phys.* **55** 1081
Uncu H and Demiralp E 2006 *Phys. Lett.* A **359** 190
Ahmed Z 2001 *Phys. Lett.* A **286** 231
[14] Fei S M 2003 *Czech. J. Phys.* **53** 1027
Fei S M 2004 *Czech. J. Phys.* **54** 43
Fei S M 2005 *Czech. J. Phys.* **55** 1085
[15] Coutinho F A B *et al* 2005 *J. Phys. A: Math. Gen.* **38** L519
[16] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **25** 268
Bagchi B K 2001 *Supersymmetry in Quantum and Classical Mechanics* (London: Chapman and Hall/CRC)
[17] Cannata F, Junker G and Trost J 1998 *Phys. Lett.* A **246** 219
Andrianov A A, Ioffe M V, Cannata F and Dedonder J P 1999 *Int. J. Mod. Phys.* A **14** 2675
Bagchi B, Cannata F and Quesne C 2000 *Phys. Lett.* A **269** 79
Cannata F, Ioffe M, Roychoudhury R and Roy P 2001 *Phys. Lett.* A **281** 305
Dorey P, Dunning C and Tateo R 2001 *Preprint* hep-th/0104119
Bagchi B, Mallik S and Quesne C 2001 *Int. J. Mod. Phys.* A **16** 2859
[18] Znojil M, Cannata F, Bagchi B and Roychoudhury R 2000 *Phys. Lett.* B **483** 284
Znojil M 2001 *Preprint* hep-th/0101038
[19] Baye D, Levai G and Sparenberg J M 1996 *Nucl. Phys.* A **599** 435
[20] Choi J Y and Hong S I 1999 *Phys. Rev.* A **60** 796
[21] Ahmed Z 2001 *Phys. Lett.* A **282** 343
Ahmed Z 2001 *Phys. Lett.* A **287** 295
[22] Samani K A and Mostafazadeh A 2001 *Nucl. Phys.* B **595** 467
Mostafazadeh A 2002 *Nucl. Phys.* B **640** 419
[23] Bender C M, Brody D C and Jones H F 2002 *Phys. Rev. Lett.* **89** 271401
[24] Mostafazadeh A 2005 *J. Phys. A: Math. Gen.* **38** 3213
Mostafazadeh A 2003 *J. Math. Phys.* **44** 974
[25] Japaridze G S 2002 *J. Phys. A: Math. Gen.* **35** 1709
Mostafazadeh A 2006 *Czech. J. Phys.* **56** 919
[26] Levai G and Znojil M 2000 *J. Phys. A: Math. Gen.* **33** 7165