

Construction of \mathcal{C} operator for a \mathcal{PT} symmetric model

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abstract

We obtain a closed form expression of the $\mathcal{C}(x, y)$ operator for the \mathcal{PT} symmetric Scarf I potential. It is also shown that the eigenfunctions form a complete set.

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In recent years non Hermitian systems, in particular the \mathcal{PT} symmetric ones [1] have been studied widely. Many of these systems are characterised by the fact that they possess real eigenvalues. However for non Hermitian systems the concept of a scalar product is a non trivial one. In fact a straight forward \mathcal{PT} symmetric generalisation of the usual scalar product for Hermitian systems produces a norm which alternates in sign i.e,

$$\langle \psi_m | \psi_n \rangle_{\mathcal{PT}} = (-1)^n \delta_{mn} \quad (1)$$

With a view to circumvent this difficulty an operator $\mathcal{C}(x, y)$ was introduced [2]. This operator is defined as [2]

$$\mathcal{C}(x, y) = \sum_{n=0}^{\infty} \psi_n(x) \psi_n(y) \quad (2)$$

where $\psi_n(x)$ are eigenfunctions of the Hamiltonian H :

$$H\psi_n(x) = \lambda_n \psi_n(x) \quad (3)$$

However, it is not always easy to obtain a closed form expression of the $\mathcal{C}(x, y)$ operator and often one has to construct it using various approximating techniques [3]. Our purpose here is to obtain a closed form expression of the $\mathcal{C}(x, y)$ operator for a \mathcal{PT} symmetric Scarf I potential.

We consider the Scarf I potential defined by

$$V(x) = \left(\frac{\alpha^2 + \beta^2}{2} - \frac{1}{4} \right) \frac{1}{\cos^2 x} + \frac{\alpha^2 - \beta^2}{2} \frac{\sin x}{\cos^2 x}, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \quad (4)$$

where α and β are complex parameters such that $\beta^* = \alpha$ and $\alpha_R > \frac{1}{2}$. In this case the (real) eigenvalues and the corresponding eigenfunctions are given by [5]

$$\begin{aligned} E_n &= \left(n + \frac{\alpha + \beta + 1}{2} \right)^2 \\ \psi_n(x) &= D_n (1 - \sin x)^{\frac{\alpha}{2} + \frac{1}{4}} (1 + \sin x)^{\frac{\alpha^*}{2} + \frac{1}{4}} P_n^{(\alpha, \alpha^*)}(\sin x), \quad n = 0, 1, 2, \dots \end{aligned} \quad (5)$$

where $P_n^{(a, b)}(x)$ denotes the Jacobi polynomial and D_n is a normalisation constant given by

$$D_n = i^n \sqrt{\frac{(2n + 2\alpha_R + 1)n! \Gamma(n + 2\alpha_R + 1)}{2^{2\alpha_R + 1} \Gamma(n + \alpha + 1) \Gamma(n + \alpha^* + 1)}} \quad (6)$$

Using the orthogonality properties of Jacobi polynomials [4] it can be shown [5] that the wave functions in (5) satisfy the relation

$$\int_{-\pi/2}^{\pi/2} (\mathcal{PT} \psi_m(x)) \psi_n(x) dx = (-1)^n \delta_{mn} \quad (7)$$

We now turn to the evaluation of the $\mathcal{C}(x, y)$ operator. Using (5) we obtain from (2)

$$\mathcal{C}(x, y) = \prod_{z=x, y} (1 - \sin z)^{\frac{\alpha}{2} + \frac{1}{4}} (1 + \sin z)^{\frac{\alpha^*}{2} + \frac{1}{4}} \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 2\alpha_R + 1) n! \Gamma(n + 2\alpha_R + 1)}{2^{2\alpha_R + 1} \Gamma(n + \alpha + 1) \Gamma(n + \alpha^* + 1)} P_n^{(\alpha, \alpha^*)}(\sin x) P_n^{(\alpha, \alpha^*)}(\sin y) \quad (8)$$

To evaluate the summation in (8) we now use the result [6]

$$\sum_{n=0}^{\infty} n! \frac{(2\alpha_R + 1)_n}{(\alpha + 1)_n (\beta + 1)_n} (2n + 2\alpha_R + 1) P_n^{(\alpha, \alpha^*)}(\sin x) P_n^{(\alpha, \alpha^*)}(\sin y) t^n = \frac{(2\alpha_R + 1)(1 - t)}{(1 + t)^{2\alpha_R + 1}} F_4(a, b, c, d, U, V) \quad (9)$$

where

$$F_4(a, b, c, d, U, V) = \sum_{r,s=0}^{\infty} \frac{(a)_s (b)_s}{s! (d)_s} \frac{(a+s)_r (b+s)_r}{r! (c)_r} U^r V^s = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s V^s}{s! (d)_s} {}_2F_1(a+s, b+s, c, U) \quad (10)$$

$$a = \alpha_R + 1, \quad b = \alpha_R + 3/2, \quad c = 1 + \alpha, \quad d = \beta + 1, \quad U = \frac{(1 - \sin x)(1 - \sin y)t}{(1+t)^2}, \quad V = \frac{(1 + \sin x)(1 + \sin y)t}{(1+t)^2} \quad (11)$$

and ${}_2F_1(a, b, c, z)$ is the standard hypergeometric function.

Now taking the limit $t \rightarrow -1$, we obtain

$$F_4(a, b, c, d, U, V) = (-U)^a \sum_{s=0}^{\infty} \frac{\Gamma(c)\Gamma(1/2)}{\Gamma(b+s)\Gamma(c-a-s)} \frac{(-V/U)^s (a)_s (b)_s}{s! (d)_s} \quad (12)$$

Then using (12) we obtain from (9) and (10)

$$\mathcal{C}(x, y) = \mathcal{N} \frac{[(1 + \sin x)(1 + \sin y)]^{(\alpha^*/2+1/4)}}{[(1 - \sin x)(1 - \sin y)]^{(\alpha^*/2+3/4)}} {}_2F_1(a, 1 - c + b, d, z), \quad z = \frac{(1 + \sin x)(1 + \sin y)}{(1 - \sin x)(1 - \sin y)} \quad (13)$$

where \mathcal{N} is a constant given by

$$\mathcal{N} = \frac{2\Gamma(\alpha_R + 1)\sin(\pi(1 - c + a))\Gamma(1 - c + a)}{\pi \Gamma(\alpha^* + 1)} \quad (14)$$

It may be noted that (13) is an exact result.

Completeness of the eigenfunctions

The completeness property is very important feature of eigenfunctions. However, to the best of our knowledge for \mathcal{PT} symmetric systems this property has been verified numerically [7]. Here we shall show analytically that the eigenfunctions (5) form a complete set. To do this we note that in a \mathcal{PT} symmetric theory with unbroken \mathcal{PT} symmetry the completeness property can be expressed as [2, 3]

$$\sum_{n=0}^{\infty} (-1)^n \psi_n(x) \psi_n(y) = \delta(x - y) \quad (15)$$

To prove (15) we consider the result [8]

$$\sum_{n=0}^{\infty} \frac{n! \Gamma(a+b+2n+1) \Gamma(a+b+n+1)}{\Gamma(a+n+1) \Gamma(b+n+1)} P_n^{(a,b)}(x) P_n^{(a,b)}(y) = (1+x)^{-b/2} (1-x)^{-a/2} (1+y)^{-b/2} (1-y)^{-a/2} \delta(x-y) \quad (16)$$

where $-1 < x, y < 1$, $Re(a) > -1$, $Re(b) > -1$. Now putting $a = \alpha$, $b = \beta$ in (16) and using (5) we obtain

$$\sum_{n=0}^{\infty} (-1)^n \psi_n(x) \psi_n(y) = \sqrt{\cos x \cos y} \delta(\sin x - \sin y) = \delta(x - y) \quad (17)$$

Thus the eigenfunctions (5) form a complete set.

It is interesting to note that two important results can be derived using (16). First we recall that in Hermitian systems, the operator $\mathcal{C}(x, y)$ is just the parity operator i.e., $\mathcal{C}(x, y) = \delta(x + y)$. So for $\alpha = \alpha^*$, (13) should reduce to this limit. Now using the properties of Hypergeometric functions it can be shown that for real α, β , $\mathcal{C}(x, y) = \delta(x + y)$. The other properties of the \mathcal{C} operator viz, $\mathcal{C}\psi_n = (-1)^n \psi_n$ follow from the definition (2) and (1) while $\mathcal{C}^2 = 1$ can be derived using the results (7) and (16).

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