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CONCOMITANT OF ORDER STATISTICS IN GUMBEL'S BIVARIATE EXPONENTIAL DISTRIBUTION

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SUMMARY. For the Gumbel's bivariate exponential distribution, we find the distribution of the concomitant of the r-th order statistic of one of the components. Properties that concomitants get from the corresponding order statistics are used to derive a number of results. Recurrence relations between moments of concomitants are also obtained. Finally, we obtain the expression for the joint distribution of concomitants of two order statistics.

1. **Introduction**

We consider Gumbel's bivariate exponential distribution (see Johnson and Kotz (1972), p.261), given by the cumulative distribution function (cdf)

$$F(x,y) = 1 - exp\{-x\} - exp\{-y\} + exp\{-(x+y+\theta xy)\},\ x,y > 0,\ 0 \leq \theta \leq 1, \quad \ldots (1)$$

probability density function (pdf)

$$f(x,y) = exp\{-x-y-\theta xy\}\{(1+\theta x)(1+\theta y) - \theta\},\ x,y > 0 \qquad \ldots (2)$$

and the marginal distributions of X and Y each standard exponential. The conditional density function of Y, given X, is

$$f(y|x) = exp\{-y(1+\theta x)\}\{(1+\theta x)(1+\theta y) - \theta\},\ y \geq 0. \qquad \ldots (3)$$

Consider a random sample $(X_i, Y_i),\ i = 1, 2, ..., n$ from a bivariate distribution. If the pairs are ordered by their X variates, then the Y variate associated

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with the r-th order statistic $X_{r:n}$ of X will be denoted by $Y_{[r:n]}$, $1 \le r \le n$ and called the concomitant of the r-th order statistic. The pdf of $Y_{[r:n]}$, denoted by $g_{[r:n]}$, is given by (see David (1981), p.110)

$$g_{[r:n]}(y) = \int f(y|x) f_{r:n}(x) dx, \qquad \ldots (4)$$

where $f_{r:n}(x)$ is the pdf of $X_{r:n}$.

The most important use of concomitants arises in selection procedures when k $(< n)$ individuals are chosen on the basis of their X-values. Then the corresponding Y-values represent performance on an associated characteristic. For example, X might be the score of a candidate on a screening test and Y the score on a later test.

Statements are made such as "if a student is good in mathematics then he will be poor in languages." If the average score in language of students good in mathematics is lower than the average score in language of ALL students, then the statement may be justified. To test hypotheses of this kind we need the distribution of the concomitants of order statistics. Thus to study a variable associated with another, distribution of concomitants of order statistics are usually crucial.

In this paper we study the properties of $Y_{[r:n]}$ for the distribution given by (2) and obtain recurrence relations between moments of concomitants. Finally, we obtain the expression for the joint distribution of concomitants of two order statistics.

2. Probability Density Function of $Y_{[r:n]}$

We first derive the pdf of $Y_{[1:n]}$,

$$\begin{aligned} g_{[1:n]}(y) &= \int_0^\infty exp\{-y(1+\theta x)\}\{(1+\theta x)(1+\theta y)-\theta\} n\ exp\{-nx\} dx \\ &= n\ exp\{-y\} \int_0^\infty exp\{-(\theta y + n)x\}\{\theta x(1+\theta y) + (1+\theta y) - \theta\} dx \\ &= \left[n - \tfrac{n(n-1)}{\theta(y+n/\theta)} - \tfrac{n(n-1)}{\theta(y+n/\theta)^2}\right] exp\{-y\},\ y \ge 0. \end{aligned} \qquad \ldots (5)$$

It is well known that cdf's of order statistics are connected by the relation

$$F_{r:n}(x) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} F_{1:i}(x),\ 1 \le r \le n \qquad \ldots (6)$$

(see David (1981), p. 48). The relation in (6) is also clearly true in terms of pdf's of order statistics. Thus the pdf of $Y_{[r:n]}$ readily follows from (4) and (6),

$$g_{[r:n]}(y) = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} g_{[1:i]}(y) \qquad \ldots(7)$$

$$= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \times \Big[i - \tfrac{i(i-1)}{\theta(y+i/\theta)} - \tfrac{i(i-1)}{\theta(y+i/\theta)^2}\Big] exp\{-y\},\ y \geq 0. \qquad \ldots(8)$$

In subsequent derivation we need the function

$$\int_0^\infty \frac{exp\{-at\}}{(b+t)} dt = exp\{ab\} E_1(ab),$$

where

$$E_1(c) = \int_c^\infty \frac{exp\{-t\}}{t} dt,\ c > 0$$

(see Abramowitz and Stegun, 1970, p. 230).

Differentiating $E_1(c)$, we get

$$E_1'(c) = -\frac{1}{c} exp\{-c\},\ c > 0 \qquad \ldots(10)$$

and hence

$$E_1'\Big(\frac{n}{\theta}\Big) = -\frac{\theta}{n} exp\{-n/\theta\}. \qquad \ldots(11)$$

3. Moments of $Y_{[r:n]}$

For convenience, we first obtain the k-th moment of $\Big(Y_{[1:n]} + \frac{n}{\theta}\Big)$, $k = 0, 1, 2, ...$ and denote it by $\nu_{1:n}^{(k)}$

$$\nu_{1:n}^{(k)} = E\Big[Y_{[1:n]} + \tfrac{n}{\theta}\Big]^k = \int_0^\infty \Big(y + \frac{n}{\theta}\Big)^k \Big[n - \frac{n(n-1)}{\theta(y+n/\theta)} - \frac{n(n-1)}{\theta(y+n/\theta)^2}\Big] exp\{-y\} dy. \qquad \ldots(12)$$

Suppose we write

$$\int_0^\infty \Big(y + \frac{n}{\theta}\Big)^k exp\{-y\} dy = \alpha_k(\frac{n}{\theta}),\ k = 0, 1, 2, ..., \qquad \ldots(13)$$

then (12) gives

$$\nu_{1:n}^{(k)} = n\alpha_k\Big(\frac{n}{\theta}\Big) - \frac{n(n-1)}{\theta}\alpha_{k-1}\Big(\frac{n}{\theta}\Big) - \frac{n(n-1)}{\theta}\alpha_{k-2}\Big(\frac{n}{\theta}\Big),\ k=0,1,2,..., \qquad \ldots(14)$$

where $\alpha_0(c)=1,\ \alpha_{-1}(c)=exp\{c\}E_1(c)$ and $\alpha_{-2}(c)=c^{-1}-exp\{c\}E_1(c)$. For $k=1$,

$$\nu_{1:n}^{(1)} = n + \Big(\frac{n}{\theta}\Big) - \frac{n(n-1)}{\theta}exp\{n/\theta\}E_1\Big(\frac{n}{\theta}\Big) \qquad \ldots(15)$$

and hence the mean of $Y_{[1:n]}$ is

$$\mu_{[1:n]} = n - \frac{n(n-1)}{\theta}exp\{n/\theta\}E_1\Big(\frac{n}{\theta}\Big). \qquad \ldots(16)$$

It is clear that

$$\alpha_k(c) = c^k + k\alpha_{k-1}(c),\ k=-1,0,1,2,... \qquad \ldots(17)$$

Even otherwise, directly

$$\alpha_k(c) = k!\sum_{j=0}^{k}\frac{c^{k-j}}{(k-j)!},\ k=0,1,2,... \qquad \ldots(18)$$

From (7),

$$\begin{aligned} \nu_{r:n}^{(k)} &= \int_0^\infty \Big(y+\frac{n}{\theta}\Big)^k g_{[r:n]}(y)dy \\ &= \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i} \\ &\times \int_0^\infty \Big(y+\frac{n}{\theta}\Big)^k g_{[1:i]}(y)dy. \end{aligned}$$

On the right hand side, writing

$$\Big(y+\frac{n}{\theta}\Big)^k = \Big(y+\frac{i}{\theta}+\frac{n-i}{\theta}\Big)^k = \sum_{j=0}^{k}\binom{k}{j}\Big(\frac{n-i}{\theta}\Big)^{k-j}\Big(y+\frac{i}{\theta}\Big)^j,$$

and using (14), we get

$$\begin{aligned} \nu_{r:n}^{(k)} &= \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}\binom{i-1}{n-r}\binom{n}{i}\sum_{j=0}^{k}\binom{k}{j}\Big(\frac{n-i}{\theta}\Big)^{k-j} \\ &\times \Big[i\alpha_j\Big(\tfrac{i}{\theta}\Big) - \tfrac{i(i-1)}{\theta}\alpha_{j-1}\Big(\tfrac{i}{\theta}\Big) - \tfrac{i(i-1)}{\theta}\alpha_{j-2}\Big(\tfrac{i}{\theta}\Big)\Big]. \end{aligned} \qquad \ldots(19)$$

Now, from (19), we can easily obtain $\mu_{[r:n]}^{(k)} = E\Big[Y_{[r:n]}^k\Big]$ by writing

$$E\Big[Y_{[r:n]}^k\Big] = E\Big[Y_{[r:n]} + \frac{n}{\theta} - \frac{n}{\theta}\Big]^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \Big(\frac{n}{\theta}\Big)^{k-j} \nu_{r:n}^{(j)}. \tag{20}$$

4. Moment Generating Function of $Y_{[r:n]}$

The moment generating function (mgf) of $Y_{[1:n]}$ is given by

$$\begin{aligned} M_{[1:n]}(t) &= E\{exp\{tY_{[1:n]}\}\} \\ &= \int_0^\infty exp\{ty\}\Big[n - \frac{n(n-1)}{\theta(y+n/\theta)} - \frac{n(n-1)}{\theta(y+n/\theta)^2}\Big] exp\{-y\}dy \\ &= \int_0^\infty exp\{-y(1-t)\}\Big[n - \frac{n(n-1)}{\theta(y+n/\theta)} - \frac{n(n-1)}{\theta(y+n/\theta)^2}\Big] dy \\ &= \tfrac{n}{1-t} - (n-1) - t\ exp\{(1-t)n/\theta\} \times \tfrac{n(n-1)}{\theta} E_1\Big(\tfrac{n(1-t)}{\theta}\Big) \end{aligned} \qquad \ldots(21)$$

and hence, from (7), the mgf of $Y_{[r:n]}$ is given by

$$\begin{aligned} M_{[r:n]}(t) &= E\{exp\{tY_{[r:n]}\}\} \\ &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} M_{[1:i]}(t) \\ &= \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \\ &\times \Big[\tfrac{i}{1-t} - (i-1) - t\ exp\{(1-t)i/\theta\} \times \tfrac{i(i-1)}{\theta} E_1\Big(\tfrac{i(1-t)}{\theta}\Big)\Big]. \end{aligned} \qquad \ldots(22)$$

5. Recurrence Relation Between Moments of Concomitants

From (21), we simply obtain

$$\frac{\theta}{n(n-1)}\Big[\frac{n}{1-t} - (n-1) - M\Big] exp\{-(1-t)n/\theta\} = t\ E_1\Big(\frac{n(1-t)}{\theta}\Big), \qquad \ldots(23)$$

where $M = M_{[1:n]}(t)$. Differentiating (23) w.r.t. t, we get

$$\begin{aligned} &\frac{1}{(n-1)}\Big[\frac{n}{1-t} - (n-1) - M\Big] exp\{-(1-t)n/\theta\} \\ +\ &\frac{\theta}{n(n-1)}\Big[\frac{n}{(1-t)^2} - M'\Big] exp\{-(1-t)n/\theta\} \\ =\ &E_1((1-t)n/\theta) + t(-n/\theta)E_1'((1-t)n/\theta) \end{aligned}$$

or,

$$
\begin{aligned}
& \frac{1}{(n-1)}\Big[\frac{n}{(1-t)}-(n-1)-M\Big] \\
+ \quad & \frac{\theta}{n(n-1)}\Big[\frac{n}{(1-t)^2}-M'\Big] \\
= \quad & exp\{(1-t)n/\theta\}E_1((1-t)n/\theta)+\frac{t}{(1-t)}
\end{aligned}
$$

or,

$$
\begin{aligned}
& \tfrac{1}{(n-1)(1-t)}-\tfrac{M}{n-1}+\tfrac{\theta}{n(n-1)}\Big[\tfrac{n}{(1-t)^2}-M'\Big] \\
= \quad & exp\{(1-t)n/\theta\}E_1((1-t)n/\theta) \\
= \quad & \int_0^\infty \frac{exp\{-(1-t)y}{(y+n/\theta)}dy.
\end{aligned}
\qquad \ldots(24)
$$

Differentiating both sides of (24) m times, w.r.t. t, we get

$$
\begin{aligned}
& \tfrac{m!}{(n-1)(1-t)^{m+1}}-\tfrac{M^{(m)}}{(n-1)}+\tfrac{\theta}{n(n-1)}\Big[\tfrac{n(m+1)!}{(1-t)^{m+2}}-M^{(m+1)}\Big] \\
= \quad & \int_0^\infty y^m\frac{exp\{-(1-t)y\}}{(y+n/\theta)}dy.
\end{aligned}
\qquad \ldots(25)
$$

Putting $t=0$ in (25), we get

$$
\begin{aligned}
& \tfrac{m!}{(n-1)}-\tfrac{1}{(n-1)}\mu_{[1:n]}^{(m)}+\tfrac{\theta}{n(n-1)}\Big[n(m+1)!-\mu_{[1:n]}^{(m+1)}\Big] \\
= \quad & \int_0^\infty y^m\frac{exp\{-y\}}{(y+n/\theta)}dy \\
= \quad & \int_0^\infty \Big(y+\frac{n}{\theta}-\frac{n}{\theta}\Big)^m\frac{exp\{-y\}}{(y+n/\theta)}dy \\
= \quad & \int_0^\infty \sum_{j=0}^m \binom{m}{j}\Big(y+\frac{n}{\theta}\Big)^{j-1}\Big(-\frac{n}{\theta}\Big)^{m-j}exp\{-y\}dy \\
= \quad & \sum_{j=0}^m \binom{m}{j}(-1)^{m-j}\Big(\frac{n}{\theta}\Big)^{m-j}\alpha_{j-1}\Big(\frac{n}{\theta}\Big).
\end{aligned}
\qquad \ldots(26)
$$

Hence, we have the following recurrence relation

$$
\begin{aligned}
& \tfrac{\theta}{n(n-1)}\mu_{[1:n]}^{(m+1)}=\tfrac{m!}{(n-1)}-\tfrac{1}{(n-1)}\mu_{[1:n]}^{(m)}+\theta\tfrac{(m+1)!}{(n-1)} \\
- \quad & \sum_{j=0}^m \binom{m}{j}(-1)^{m-j}\Big(\frac{n}{\theta}\Big)^{m-j}\alpha_{j-1}\Big(\frac{n}{\theta}\Big),\ m=0,1,2,...,
\end{aligned}
\qquad \ldots(27)
$$

where $\alpha_k(c)=k!\sum_{j=0}^k\frac{c^j}{j!},\ k\geq 0$ and $\alpha_{-1}(c)=exp\{c\}E_1(c),\ c>0.$

When $m = 0$, (27) reduces to

$$\mu_{[1:n]} = n - \frac{n(n-1)}{\theta} exp\{n/\theta\} E_1\Big(\frac{n}{\theta}\Big), \qquad \ldots(28)$$

which agrees with (16). As $\theta \to 0$, $\mu_{[1:n]} \to 1$, for $\frac{1}{\theta} exp\{n/\theta\} E_1(n/\theta) = \int_0^\infty \frac{exp\{-y\}}{(\theta y + n)} dy \to \frac{1}{n}$ as $\theta \to 0$.

We can express $\nu_{1:n}^{(k)}$ in terms of $\mu_{[1:n]}^{(j)}$'s as follows

$$\nu_{1:n}^{(k)} = E\Big[Y_{[1:n]} + \frac{n}{\theta}\Big]^k = \sum_{j=0}^{k} \binom{k}{j} \Big(\frac{n}{\theta}\Big)^{k-j} \mu_{[1:n]}^{(j)}. \qquad \ldots(29)$$

Furthermore, from (7), multiplying both sides by y^k and integrating, we get

$$\mu_{[r:n]}^{(k)} = \sum_{i=n-r+1}^{n} (-1)^{i-n+r-1} \binom{i-1}{n-r} \binom{n}{i} \mu_{[1:i]}^{(k)}. \qquad \ldots(30)$$

6. Joint Distibution of Two Concomitants

Let $Y_{[r:n]}$ and $Y_{[s:n]}$ be concomitants of the r-th and s-th order statistics respectively. Then the joint pdf of $Y_{[r:n]}$ and $Y_{[s:n]}$ is given by (see David (1981), p. 110)

$$g_{[r,s:n]}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f(y_1|x_1) f(y_2|x_2) f_{r,s:n}(x_1, x_2) dx_1 dx_2,$$

where $f_{r,s:n}(x_1, x_2)$ is the joint pdf of $(X_{r:n}, X_{s:n})$, $1 \le r < s \le n$.
For the Gumbel's bivariate exponential distribution with pdf (2), we have

$$\begin{aligned} g_{[r,s:n]}(y_1, y_2) &= \int_0^\infty \int_0^{x_2} f(y_1|x_1) f(y_2|x_2) f_{r,s:n}(x_1, x_2) dx_1 dx_2 \\ &= \int_0^\infty \int_0^{x_2} exp\{-y_1(1+\theta x_1)\}\{(1+\theta x_1)(1+\theta y_1) - \theta\} \\ &\times \ exp\{-y_2(1+\theta x_2)\}\{(1+\theta x_2)(1+\theta y_2) - \theta\} \\ &\times \ C_{r,s:n}\Big(1 - exp\{-x_1\}\Big)^{r-1}\Big(exp\{-x_1\} - exp\{-x_2\}\Big)^{s-r-1} \\ &\times \ exp\{-(n-s)x_2\} exp\{-(x_1+x_2)\} dx_1 dx_2, \end{aligned} \qquad \ldots(31)$$

where $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$. Simplifying (31), we obtain

$$
\begin{aligned}
& g_{[r,s:n]}(y_1, y_2) = C_{r,s:n} exp\{-(y_1+y_2)\} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+s-r-j-1} \\
\times \; & \binom{r-1}{i} \binom{s-r-1}{j} \Big[\tfrac{(1+\theta y_1)(1+\theta y_2)}{K(y_1)} \\
\times \; & \Big\{ \theta^2 \Big(\tfrac{1}{K(y_1)L^2(y_2)} - \tfrac{2}{(K(y_1)+L(y_2))^3} - \tfrac{1}{K(y_1)(K(y_1)+L(y_2))^2} \Big) \\
+ \; & \theta \Big(\tfrac{1}{L^2(y_2)} - \tfrac{2}{(K(y_1)+L(y_2))^2} + \tfrac{1}{K(y_1)L(y_2)} - \tfrac{1}{K(y_1)(K(y_1)+L(y_2))} \Big) \\
+ \; & \tfrac{K(y_1)}{L(y_2)(K(y_1)+L(y_2))} \Big\} + \tfrac{\theta(1+\theta y_1)}{K(y_1)} \\
\times \; & \Big\{ \theta \Big(\tfrac{1}{(K(y_1)+L(y_2))^2} - \tfrac{1}{K(y_1)(K(y_1)+L(y_2))} - \tfrac{1}{K(y_1)L(y_2)} \Big) \\
- \; & \tfrac{K(y_1)}{L(y_2)(K(y_1)+L(y_2))} \Big\} + \tfrac{\theta(1+\theta y_2)}{K(y_1)} \\
\times \; & \Big\{ \theta \Big(\tfrac{1}{(K(y_1)+L(y_2))^2} - \tfrac{1}{L^2(y_2)} \Big) - \tfrac{K(y_1)}{L(y_2)(K(y_1)+L(y_2))} \Big\} \\
+ \; & \tfrac{\theta^2}{L(y_2)(K(y_1)+L(y_2))} \Big],
\end{aligned}
\qquad \ldots (32)
$$

where $K(y_1) = (y_1\theta + i + j + 1)$ and $L(y_2) = (y_2\theta + n - r - j)$.

Using (32) and (9) product moments of $Y_{[r:n]}$ and $Y_{[s:n]}$ can be evaluated.

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