Exploring New Models for Population Prediction in Detecting Demographic Phase Change for Sparse Census Data

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Abstract

Logistic model has long history regarding it's usefulness in population predictions. But the model has some limitations when applying for the sparse census data sets, typically available for developing countries. In such situation the relative growth rates (RGR) exhibit some unusual trend (increasing, primary increasing and then decreasing) which is different from the common decreasing trend of logistic law. To tackle those complicated demographic situations we have successfully explored a simplified version of Tsoularis and Wallace model (TWM) which can explain all of these feasible monotonic structures of RGR. In addition to this we have also proposed another model (PM) by assuming RGR as a direct function of time covariate but not the size. The model has some key advantages than the simplified TWM (STWM). It can detect the demographic phase change point at which the developing country switches over towards developed one. We performed RGR modelling (as a function of time) but not the size as neither TWM nor STWM is analytically solvable and the underlying population model is better identifiable in the former case but not in the later. The less number of parameters involve in both the STWM and PM ensure a better chance of convergence under non-linear least square estimation than the original TWM with more parameters.

Keywords and phrases: Population prediction, relative growth rate, Tsoularis-Wallace model, demographic phase change.

AMS Subject Classification: 62P25

1 Introduction

The problem of population prediction is concerned with the prediction of the size and composition of the future population on the basis of the size and composition of current population. Usually elaborate predictions are made of the population size at each age group and separately for male and female. Gradually different methods have been developed by the population scientists for this purpose. Among these, cohort component method is a very familiar one where predictions are first made separately for the three components viz. survivorship, migration and birth and then these three components are combined to predict the net size and composition of the future population.

In projecting population for small areas such as a city, state etc. the available techniques are very few in numbers. The cohort component method often fails due to the difficulties of prediction of the flow of migrants in and out of these areas. The estimates are often found to be unrealistic .

In the case of estimating Indian population or population of it's sub-regions, it is revealed that logistic model grossly underestimates true population. Additionally the logistic model has some limitations when applying for the scanty sparse census data set, typically available for developing countries. In such situations the relative growth rates (RGR) exhibit some unusual trend (increasing, primary increasing and then decreasing) which is different from the common decreasing trend of logistic law.

In population prediction problem, rate modeling may often be more powerful than the usual size modeling where rate is define as the relative change of size with respect to time. Sometime it is not easy or rather misleading for the experimenter to identify the proper underlying model by studying the shape of the growth profile curves among the plenty available growth curves. But in comparison if we plot the empirical estimate of RGR against time or size we can at least guess and identify the proper family of growth curves which is appropriate for the given data based on the monotonic structure of RGR. So elimination of improper model is comparatively easy from RGR profile than the size profile curves. This is particularly important when growth profile curves similar to common and existing growth curves though the RGR is not monotonically decreasing with time (Gompertz) / size (logistic, Richards, von-Bertallanfy etc.) or constant (exponential) - which is the basic property of the standard population growth curves. Corresponding to different census figures we observed different growth rates though growth profiles look alike (see Figure 1). The simulated Figure 2 (bell shaped) also illustrates the same property by showing the size profile, S-shaped Sigmoidal, whereas the RGR profile is not decreasing but primarily increasing then decreasing with size. So the underlying population model is better identifiable in the former case but not in the later. Finally, the most important point in favouring RGR modelling is that sometimes it is really impossible to solve the growth law analytically, by just integrating the rate differential equation. Basically these two shortcomings motivates us to introduce RGR modelling but not the size. We try to explore this by introducing Tsoullaris-Wallace model (2002) (TWM) which has the property to exhibit various uncommon structures of RGR. Tosoularis-Wallace model (2002) altogether consists of five unknown parameters to be estimated from the population census figures. Although for developed countries population figures are adequate to estimate all the parameters but it is not true for developing countries where population census figures are available only for 8-10 periods. The presence of less number of parameters in any standard model ensure a better chance of convergence in the non-linear least square estimates. So for the later case reduction of number of parameters is needed for using TWM without sacrificing it's advantages in representing all feasible monotonic structures of RGR. In the Section 3.2 we have discussed this model (STWM) elaborately.

In the spirit of TWM we also proposed a similar model, based on RGR as a function of time. This proposed model (PM) is simple and flexible enough to represent all type of feasible monotonic structures of RGR. The proposed model is more tractable than the TWM although both shares the property of having different monotonic structures of RGR (see Figure 2 and 3). In addition to this PM has some key advantages over the TWM and STWM which may be listed as: a) it can be solved analytically, b) it can detect the demographic phase change point at which the developing country switches over towards developed one. This phase change occurs at the time point at which RGR is maximized. The PM is discussed in Section 3.3.

The "Dynamic Logistic Model" (DLM) (Bhat (1999)) (described in Section 3.4) suffers from a serious drawback that it can not cover the situation when the population size ultimately explodes but not stabilizes after a specified time period. In comparison the STWM and the PM stabilizes ultimately which is one of the desired and expected property of the population growth curve model. The upper ceiling point of DLM is not constant but varies with time. So selection of the proper form of ceiling point is really difficult and arbitrary. The fitting of parameters of the model and ultimately the prediction of population size is largely dependent on the proper choice of this model although there does not exist any standard procedures so that we can choose a correct one. We prefer such a model which provides the fitting, at least as good as the DLM, but does not suffer from the above drawbacks. With a view to meet these purposes we have tried to explore STWM and PM.

The stochastic formulation of the STWM and PM is done by adding delta correlated and auto correlated errors with the deterministic counter part of the models. The estimates of different parameters are obtained by minimizing errors sum of squares and using non-linear least square theory. The empirical estimates of RGR (Fisher (1921)) is used as response variable which is basically a discretised version of the rate equation. The solution of the formulated stochastic differential equation (SDE) is also obtained using Statonovich calculus.

2 The Data

Census data

We have used the census data of West Bengal (from 1901 to 1981;9 time points), India (from 1901 to 1991;10 time points) and China (from 1949 to 1999; 50 time points). The nature of data are illustrated in figure 1. From Figure 1 corresponding to China (*size is measured per billion*) it may be observed that there is an increasing trend at the initial stage but a decreasing trend after reaching a maximum . At the initial stage a sharp decline is reflected at a particular time point(probably due to Government policy regarding family welfare). So an approximate bell shaped trend is expected if the sudden decline case is treated as an exceptional situation of the entire data frame. On the other hand for India (*size is measured per billion*) and one of it's province (state) West Bengal (*size is measured per million*) an overall increasing trend is observed although a slight decreasing tendency is reflected towards the end of the studied census time points.

Simulated data

Along with the census data we have also used the simulated data. To construct simulated data from Tsoularis and Wallace (2002)model we adopted the following algorithm.

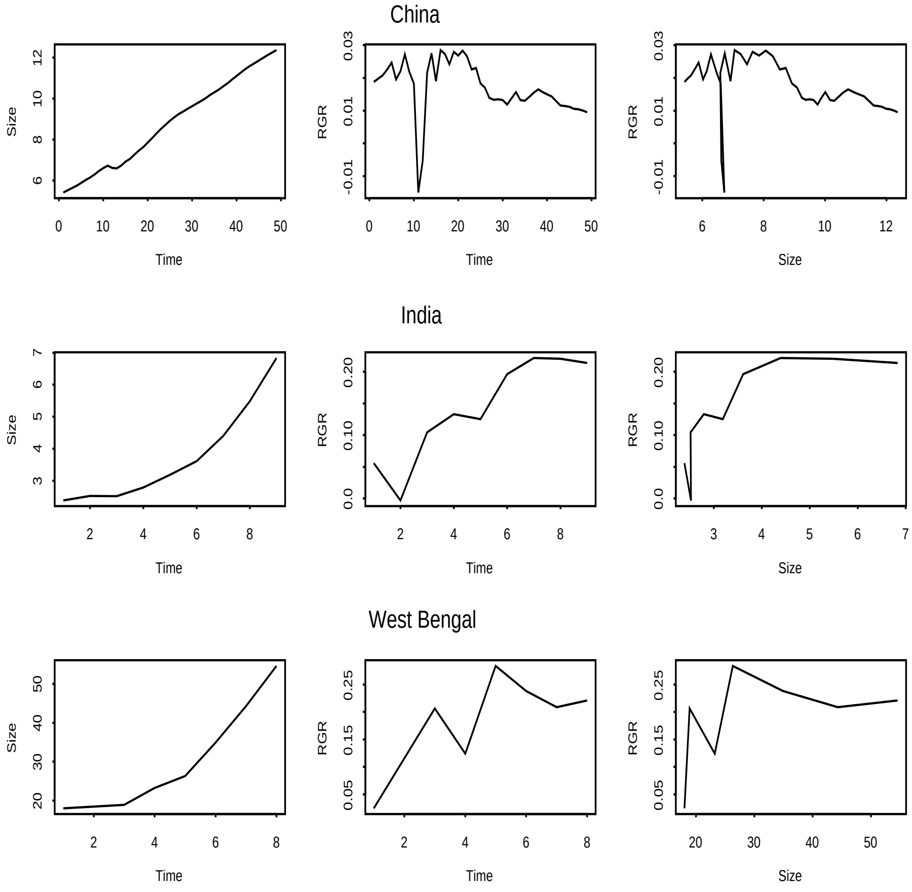

Figure 1: Size and RGR profile curves for census data of China, India and West Bengal

We can choose any real number sequence for x values. But from the demographic population the RGR is generally bounded and in rare situation it may exceed 1. So from practical point of view we have started with a sequence of feasible length (0.14 to 1.0). One may construct any other sequence based on the particular problem under consideration. Assuming this sequence as x values we evaluated the functional values of

$$\frac{\Gamma(m+n)}{(\Gamma(m)\Gamma(n))}x^{m-1}\left(1-\left(\frac{x}{k}\right)\right)^{n-1} \tag{1}$$

for different values of x. Considering these functional values as RGR, population size $P(t)$ are generated through either of the following recursive relationships of RGR and " Average Relative Growth Rate"(ARGR) (Fisher(1921)).

$$P(t+1) = P(t)\exp\left(R(t)\right) \tag{2}$$

$$P(t+1) = P(t)(1+R(t)). \tag{3}$$

Equations 2 and 3 are derived from discrete and continuous approximation of the term $\frac{1}{P(t)}\frac{dP(t)}{dt}$. For discrete approximation $\frac{1}{P(t)}\frac{dP(t)}{dt} = R(t) = \frac{P(t+1)-P(t)}{P(t)}$ (RGR) leads to equation 3 and for continuous approximation $R(t) = \int_t^{t+1}\left(\frac{1}{P(s)}\frac{dP(s)}{ds}\right)ds$(ARGR) leads to equation 2.

Then using these $P(t)$ values different sets of RGR values with different monotonic structures are generated by choosing different sets of parameter values of the Tsoularis and Wallace (2002)model given by

$$R(t) = rP(t)^a(1-(P(t)/k))^c$$

The choices of different sets of parameters considered are ($r = 0.12, a = 0.6, k = 14, c = 0.01$) , ($r = 0.12, a = 0.01, k = 328, c = 1$) and ($r = 0.2, a = 0.8, k = 600, c = 20$) and initial values are chosen to be (3.4, 2.4 and 3.4) for increasing, decreasing and bell shaped trend of RGR respectively. The trends are exhibited in Figure 2. The size profiles look alike in bare eye for increasing and decreasing trend whereas the ARGR profiles showing completely reverse trend. It may also be noted from the last figure that although the size profile is similar to common growth law, ARGR profile is bell shaped.

3 The Model

To understand the taxonomy of the proposed model let us start with the exponential growth law. The exponential growth of multiplying

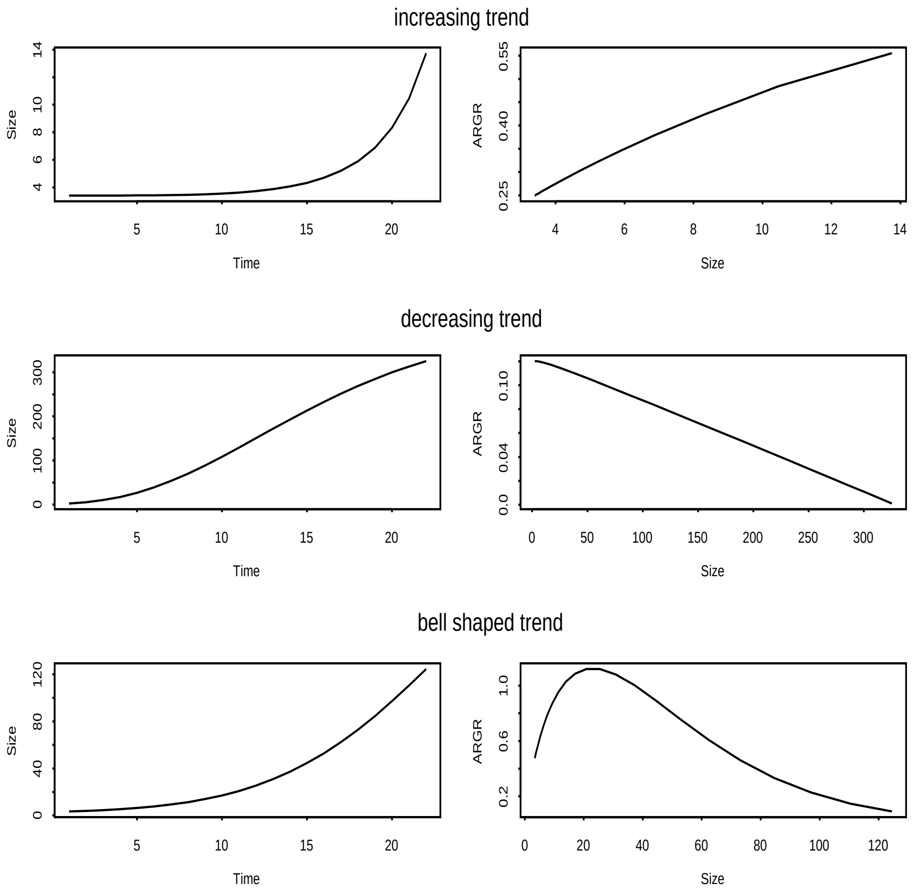

Figure 2: Figure showing growth and RGR profiles with size for simulated data under reduced TWM

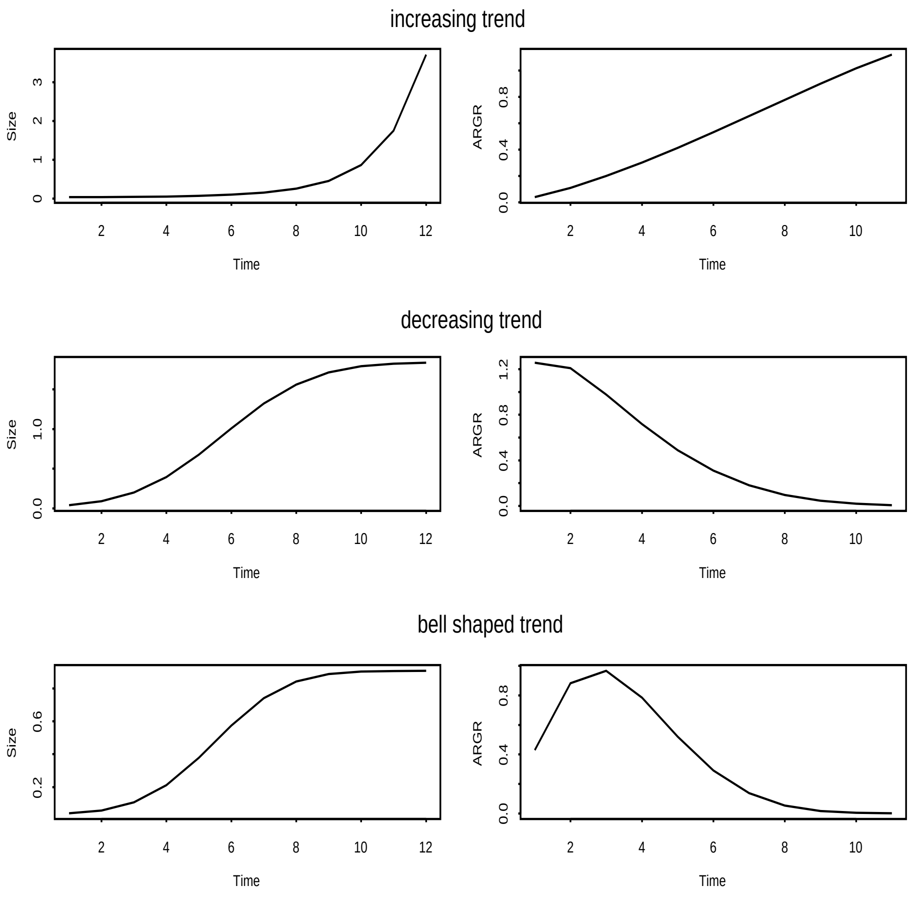

Figure 3: Figure showing growth and RGR profiles with time for simulated data under proposed model

organism is represented by a simple and widely used model that increases without bounds or limits as Figure 4(a) - illustrates. So, in mathematical terminology growth rate is proportional to population size (i.e. RGR is constant through out the whole growth process) and that leads to the exponential growth law represented by the following differential equation

$$\frac{dP(t)}{dt} = r\ P(t). \tag{4}$$

The familiar solution to (4) is

$$P(t) = m\ e^{r\ t}, \tag{5}$$

where r is the growth rate and m is the initial population $P(0)$. Although many population grow exponentially for a limited time period,no bounded system sustain exponential growth indefinitely, unless the parameters or boundaries of the system are changed. Because only a few, if any, systems are permanently unbounded and sustain exponential growth, equation (5) must be modified to have a limit or a carrying capacity that makes it more realistic sigmoidal shaped as illustrated in Figure 4(b). The most widely used modification of the exponential growth rate is the logistic growth rate. It was introduced by Verhulst(1838) but popularized by Lotka(1925), as Kingsland (1985) wrote in her comprehensive history of such models in Population ecology.

The logistic equation begins with $P(t)$ and r of the exponential curve but adds a "negative feedback" term $\left(1 - \frac{P(t)}{k}\right)$ that reduces the growth rate of a population as the limit k is approached :

$$\frac{dP(t)}{dt} = rP(t)\left[1 - \frac{P(t)}{k}\right]. \tag{6}$$

Note that the "negative feedback" term is closed to 1 when $P(t) \ll k$ and approaches to 0 as $P(t) \to k$. Thus, the growth rate begins exponentially then decreases to 0 as the population $P(t)$ approaches to the limit k, and producing an S-shaped (sigmoidal) growth trajectory. The term $r\ P(t)$ can be treated as a "positive feedback" term. So, in logistic law both the "positive and negative feedback" terms are linear in nature. In some of the practical scenario due to various environmental and demographical fluctuations these feedback terms might not be linear but a polynomial of suitable degree. That modifies the "positive and negative feedback" terms as $r\ P(t)^a$ and $\left(1 - \left(\frac{P(t)}{k}\right)^b\right)^c$ and leads to the TWM which is described elaborately in the next sub section. Since Tsoularis and Wallace (2002) derived the model only from mathematical tractability view point they have not well-described interpretations of different parameters.

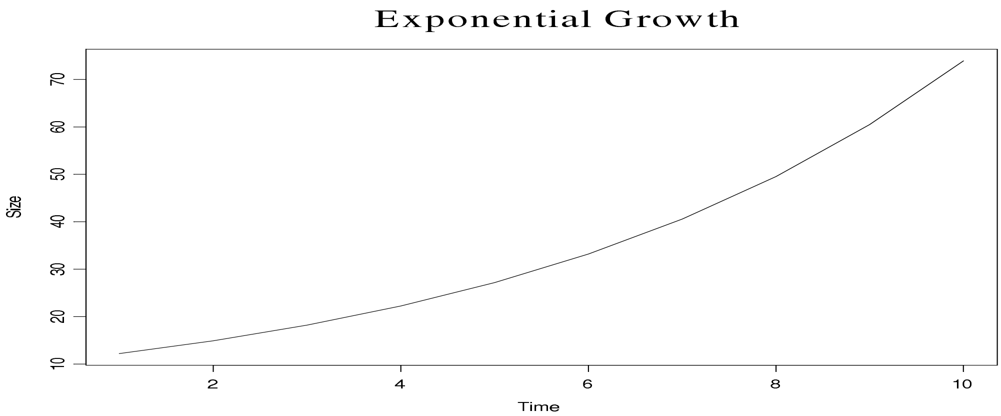

Figure 4(a): The size curve through Exponential growth

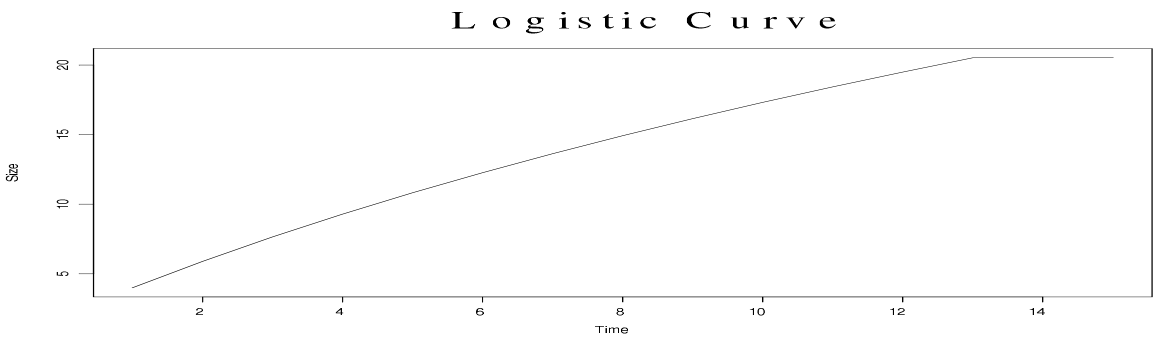

Figure 4(b): The size curve through Logistic growth

3.1 Tsoularis-Wallace Model

Tsoularis-Wallace modified Neldar (1961) and Richards' (1969) equations to generalize logistic law.

Their proposed model generalized the logistic equation which can incorporate most of the previously reported laws as special cases.

The Tsoularis-Wallace model with full set of parameters is characterized by the equation

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = rP(t)^a\left[1-\left(\frac{P(t)}{k}\right)^b\right]^c \tag{7}$$

where r, a, b, c and k are positive real numbers.

3.2 Simplified TWM

From the TWM it is difficult to estimate the five parameters as the chance of convergence of the nonlinear least square estimates is very low due the singularity problem in the gradient matrix.

It will be best if we can reduce the number of parameters of the TWM without loosing the representations of different shapes of RGR functions. In this paper we have introduced more simple form of TWM with reduced set of parameters which can still exhibit the different trends of RGR. STWM is more powerful in avoiding singularity problem which ensures the better chance of convergence than the original TWM.

We can propose the reduced form ($b = 1$) of the TWM defined as STWM as described below

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = rP(t)^a\left[1-\frac{P(t)}{k}\right]^c \tag{8}$$

The other form of STWM we use in this paper are respectively given by the following two equations where we have assumed $a = 1$ and $c = 1$ respectively

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = rP(t)\left[1-\frac{P(t)}{k}\right]^c \tag{9}$$

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = rP(t)^a\left[1-\frac{P(t)}{k}\right]. \tag{10}$$

3.3 Proposed Model

As $P(t)$ is not analytically solvable for STWM as defined in (8), it is very difficult to find the time point (t_{max}) at which RGR attains its maximum value. In some particular demographic situations RGR decreases with time after reaching its maximum value which clearly indicates that with limited source of environmental resources population is going to change it's phase from less developed to better one. So, it is quite reasonable to identify this change point as to explain that particular demographic phenomenon. This motivates us to develop such a model by which we can analytically find the time point (t_{max}) at which the RGR is maximized. To achieve this we proposed PM where RGR is explained through time covariate but not size. The PM is represented by

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = r\ t^a \left[1 - \frac{t}{d}\right]^c, \qquad 0 \le t \le d \tag{11}$$

$$= 0, \qquad\qquad t > d$$

It is to be noted that for both STWM and PM a and c are the key parameters which can efficiently tune the monotonicity of RGR. Different uncommon structures (increasing, primarily increasing then decreasing etc.) of the RGR are visible for small sparse data set available from census and that needs to tune the usual monotonicity of RGR. The unusual shape of RGR indicates serious environmental and demographic fluctuations. The key or in other words the tunning parameters a and c for both the STWM and PM play a vital role to explain these unusual shapes. So, these parameters can be interpreted as the environmental and demographic surrogates.

For reasonably large value of d population approaches to the upper ceiling point (say, $P(d)$) when $t \to d$. So basically d is the upper ceiling point of time but not the population. It is to be noted that RGR (see equation (11)) is zero at both $t = 0$ & d. So $P(t)$ stabilizes at $t = d$ and this obviously implies that $P(t)$ can be represented as a S-Shaped sigmoidal carve. In other words we try to transform the time frame within which $P(t)$ stabilizes, from $(0, \infty)$ to $(0, d)$ which is more realistic from practical point of view. Also, the PM can also shares all feasible monotonic structures of RGR similar to STWM(see Figure 2-3). Moreover it has one additional key advantage in expressing $P(t)$ analytically as a function of t.

In the following section we are going to discuss some main features of the proposed model.

The analytic solution of the proposed model (11) is given by

$$P(t) = P(0)exp\left(\frac{r\ t^{(1+a)}\text{Hypergeometric2F1}[1+a, -c, 2+a, \frac{t}{d}]}{(1+a)}\right) \tag{12}$$

where, $\text{Hypergeometric2F1}[m, n, p, q] = \sum_{r=0}^{\infty} \frac{(m)_r (n)_r}{(p)_r}\frac{q^r}{r!}$ and $P(0)$ is the population size at first census.

The three main features of the PM are

1. The RGR of the PM attains maximum at $t = \frac{a\ d}{a+c}$
2. The point of inflections of the curve (12) can be obtained as a solution of the equation (13) given by

$$a\ (t-d)\ +\ t\ \left(c - r\ (d-t)\ t^a\ \left(1-\frac{t}{d}\right)^c\right) = 0 \qquad (13)$$

3. The points of inflections of the RGR curve given by the equation (11)are
$\frac{a^2\ d+a\ d\ (c-1)-\sqrt{a}\ d\ \sqrt{c}\ \sqrt{a+c-1}}{a^2+(c-1)\ c+a\ (2\ c-1)}$ and $\frac{a^2\ d+a\ d\ (c-1)+\sqrt{a}\ d\ \sqrt{c}\ \sqrt{a+c-1}}{a^2+(c-1)\ c+a\ (2\ c-1)}$
4. At t_{max} the magnitude of RGR is

$$r\ \left(\frac{a\ d}{a+c}\right)^a \left(\frac{d\ c}{a+c}\right)^c \qquad (14)$$

   If any one of the parameters of (14) tends to 0, the RGR approaches 0.

For simulated data trends are exhibited in Figure 3 . It is to be noted that in the first figure the size profile is similar to existing growth law but ARGR profile is showing increasing trend which is not very common. For the last two figures both the size profiles are exhibiting the common S-shaped curve but ARGR profiles are not same. One shows decreasing but another shows bell shaped trend which is an exceptional phenomenon.

3.4 Dynamic Logistic Model

The traditional logistic curve provides a poor fit to past population trends mainely because of the underlying assumptions that the ceiling of the population size is time invariant. Owing to various reasons, the capacity of the same land to accommodate people may rise with time and thus explain the reason behind increasing or stationary growth rates seen in the population. The DLM (Bhat, 1999) based on the time dependent ceiling point, is characterized by the equation

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = b[k(t) - P(t)] \qquad (15)$$

For estimating population size we can assume different functional forms of $k(t)$. In this paper we have assumed the simplest form of $k(t)$ that is linear. The further complicated form requires a relatively straightforward extension of this theory, although actual computation of the population size is much messier.

Then we can predict the population sizes at different censuses by using the nonlinear least square estimates in the following recursive relation.

$$P(\widehat{t+1}) = \exp[\ln \widehat{P(t)} + \hat{b}\hat{k(0)} + \hat{b}\hat{k}t - \hat{b}\widehat{P(t)}] \tag{16}$$

The initial condition figure is obtained from the first census.

4 Parameter Estimation

In population dynamics the parameters are usually estimated from deterministic solution of the population differential equation. It is quite reasonable to insert the random component in the deterministic model due to the various demographic fluctuations . The correlated structure of ARGR / RGR between different time points are completely unknown in population dynamics. Even if sometimes the existence of this dependency is a big question. For some of the earlier problems Rao (1952, 1987) assumed independent structure but without any proper explanation. Hence for the parameter estimation it is better to assume a more general structure of errors so that the response variable ARGR / RGR are either correlated or uncorrelated when the errors are auto correlated or uncorrelated accordingly.

Consider the model described by the stochastic differential equation as follows

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = \frac{d\ \ln P(t)}{dt} = R(t) = h(P(t);\theta) + \sigma(t;\eta)\epsilon(t) \tag{17}$$

and

$$\frac{1}{P(t)}\frac{dP(t)}{dt} = \frac{d\ \ln P(t)}{dt} = R(t) = h(t;\theta) + \sigma(t;\eta)\epsilon(t), \tag{18}$$

where, $\epsilon(t)$ is a stationary "Gaussian Random Process" with $Var(\epsilon(t)) = 1$ and $\sigma(t;\eta)$ is included to allow the variance of the errors to change over time (heteroscedastic). Note that the model (17) is of the form (7) and (8) but the model (11) is the special case of the general model (18). Now let us consider two following cases.

(a) **Delta Correlated Process :** The error process $\epsilon(t)$ is called delta correlated if $Cov(\epsilon(t_i), \epsilon(t_j)) = 0$ for all $t_i \neq t_j$. That is $\epsilon(t)$ follows iid $N(0, \sigma^2)$ if $\sigma(t;\eta) = \sigma^2$ (homoscedastic). Then $\frac{1}{P(t)}\frac{dP(t)}{dt}$ follows independently $N(h(P(t);\theta), \sigma^2)$.

(b) **Auto Correlated Process :** The process is called auto correlated if $\epsilon(t)$ has stationary auto correlation function i.e., $Corr(\epsilon(t_i), \epsilon(t_j) = \rho(|t_i - t_j|)$. Then the problem is not a simple nonlinear regression problem even if we don't allow the heteroscedasticity in the process.

Now, the RGR for any size variable at a time point t is defined as the relative rate of change of size; this is mathematically defined as $\frac{1}{P(t)}\frac{dP(t)}{dt}$ where $P(t)$ is the size at time point t. If we consider a time interval (t_1, t_2), $t_1 < t_2$, rather than a specific time point, then the RGR can be approximated by

$$\frac{ln(P(t_2)) - ln(P(t_1))}{t_2 - t_1}$$

(Fisher, 1921), where $\Delta t = t_2 - t_1$ is the length of the time interval. This is basically the ARGR over the interval Δt. Levenbach and Reuter (1976), Sandland and Gilchrist (1979), Seber and Wild (1984), among others, used the same empirical estimate of RGR in their studies of growth curve analysis.

In our case population data are discrete and available for equispaced time points so without loss of generality we can replace t_2 by $(t+1)$ and t_1 by t and that leads $\Delta t = 1$.

If $P(t)$ is the population size at time point t then we can use ARGR as the estimate of RGR and is given by

$$\ln(P(t+1)) - \ln(P(t)) \tag{19}$$

In this paper we replace the left hand side of (17) and (18) by this empirical estimate of RGR.

The estimated value of θ can be obtained through "nonlinear least square" due to Marquardt (1963) principle by minimizing the error sum of squares.

Under auto-correlated error structure the parameters are estimated through "iterated non-linear least square" technique (Seber and Wild (1989)). Correlation structure is estimated by Prais-Winsten(1954) method.

It may be noted that this nonlinear least square estimates are also MLE estimates under the Guassian errors.

It is really difficult to estimate the parameters of STWM or PM from scanty and sparse census data. Singularity of the gradient matrix creates the problem in estimating the parameters of the model through nonlinear least square for this scanty census data set. So it is reasonable to study the existence and the consistency properties of the nonlinear least square estimates although it follows from the general results of the theory of estimating equations. For PM the existence and consistency property of least square estimates hold. These asymptotic properties are studied elaborately in the appendix (Section 10.1 - 10.2).

5 Solution of stochastic differential equation

Comparing the equation (18) and it's deterministic counter part (11) we have $h(t,\theta) = r\ t^a\ (1-t/d)^c$. Then the solution of (18) is a Gaussian process

$$ln(P(t)) = ln(P(0)) + \int_{t_0}^{t} r\ u^a\ \left(1 - \frac{u}{d}\right)^c du + \int_{t_0}^{t} \sigma(u,\eta)\epsilon(u)du \quad (20)$$

for which

$$E[ln(P(t))|P(0)] = ln(P(0)) + \int_{t_0}^{t} r\ u^a\ \left(1 - \frac{u}{d}\right)^c du. \quad (21)$$

The solution (20) is obtained using "Stratonovich's stochastic calculus", also discussed in Sandland and McGilchrist (1979); Seber and Wild (1988) (see the Section 10.3 in the appendix). We have evaluated the conditional varience-covarience structures of the responses $P(t)$ given the initial value $P(0)$ under uncorrelated homoscedastic structure of errors are given by

$$Var[ln(P_{t_i})|ln(P_{t_0})] = Var[ln(P_{t_i}) - ln(P_{t_0})|ln(P_{t_0})] = (t_i - t_0)\ \sigma^2 = \sigma_i^2, \quad (22)$$

and

$$Cov[ln(P_{t_i}), ln(P_{t_j})|ln(P_{t_0})] = Var[ln(P_{t_i})|ln(P_{t_0})] = \sigma_i^2. \quad (23)$$

For autocorrelated structure of errors, when $\epsilon(t)$ has autocorrelation function represented by $corr[\epsilon(u), \epsilon(v)] = \rho(|u - v|)$, we have (Seber and wild (1988)),

$$Var[\ln(P(t_i)) - \ln(P(t_j))] = \int_{t_i}^{t_j} \int_{t_i}^{t_j} \sigma(u,\eta)\ \sigma(v,\eta)\ \rho(|u-v|)dudv. \quad (24)$$

Also, for any two non overlapping time intervals $[t_2, t_1]$ and $[t_4, t_3]$,

$$Cov[\ln(P(t_2)) - \ln(P(t_1)), \ln(P(t_4)) - \ln(P(t_3))]$$
$$= \int_{t_1}^{t_2} \int_{t_3}^{t_4} \sigma(u, \eta)\ \sigma(v, \eta)\ \rho(|u - v|) du\ dv. \tag{25}$$

For special case, when ρ has some specific structure, the derived expressions for (22,23) are available in the appendix. As the census figures are available for equispaced time points, without loss of generality the time frame can be transformed in such a way so that the increment is unity. It converts the weighted least square estimation procedure to a more simple ordinary least square to estimate the model parameters (see the Section 10.3 in the appendix)

6 Advantage of PM over STWM in detecting demographic phase change

With a limited source of environmental resource a population going to change it's phase from developing to developed one when RGR switches over it's patteren from increasing to decreasing trend. It will happen when RGR is bell shaped. The two distinct phases are separated at the key time point where the RGR is maximized. So RGR is one of the vital tools in reflecting this phase change although there are also other responsible demographic parameters which can distinguish those phases in different ways.

This key time point varies from one population to another which clearly indicates that the occurrence of demographic phase change delayed in one of the two populations. The identification of demographic cause for this delay might be one of the very promising social problems of interest. So it is quite reasonable to frame and test the hypothesis whether the time points where RGR attains its' maximum value are same for the two populations against one is greater than other. The test can be constructed only when the census data are available at the same time periods for two populations.

The two point of inflexions of the RGR curve also have some demographic interpretations. When RGR is increasing with time initially the slope is flat but suddenly it grows up and the slope become steeper. But when RGR is decreasing the nature of the curve behaves reversely that is primarily the slope is steeper but then it is gradually decreasing in a slow rate. So the time points indicating a sharp change in terms of the slope of the RGR might have immense practical values for both the cases. More specifically these time points provide some caution to the developing countries in terms of the demographic change. In the first case when the slope of the RGR grows up, it indicates that the developing country gradually loosing it's status and is approaching towards an worse condition which would lead the population ultimately towards explosion. So some caution should be taken. Similarly for the second case when the decreasing rate of RGR slows down it would provide a negative feedback in the shifting process forcing a developing country towards a developed one. Under this scenario it is also important to identify some social and demographic factors so that some preventive measures can be taken. These changes might not be occurred at the same time and so one can think to construct a hypothesis whether the time points where the point of inflexions occurs for RGR curve are same for the two populations or not.

As STWM (or TWM) is not analytically solvable, it is not possible to evaluate the time points where the RGR is maximized or the point of inflexions of the RGR curve in terms of model parameters. So this is one of the major shortcomings of the STWM model. But in comparison we can easily evaluate a closed and simple analytical form of these time points from PM. So the related tests can also be constructed for the PM and which is the key advantage of the model.

Test for phase change and point of inflexions

Let us denote the theoretical solutions of the time points at which the RGR is maximized respectively by t_1 and t_2 for the two populations. Similarly, the theoretical points of inflexions of the RGR curves for both the two populations are respectively denoted by(t_{1inf}, t^*_{1inf}) and (t_{2inf}, t^*_{2inf}). From the feature (1) of the proposed model as described in Section 3.3, the magnitude of t_1 and t_2 are respectively $\frac{a_1 d_1}{a_1+c_1}$ and $\frac{a_2 d_2}{a_2+c_2}$. Where, the model parameters for the first and second populations are respectively (a_1, d_1, c_1) and (a_2, d_2, c_2). Similarly the evaluated expression (see feature 3) for (t_{1inf}, t^*_{1inf}) and (t_{2inf}, t^*_{2inf}) are
$(\frac{a_1^2\ d_1+a_1\ d_1\ (c_1-1)-\sqrt{a_1}\ d_1\ \sqrt{c_1}\ \sqrt{a_1+c_1-1}}{a_1^2+(c_1-1)\ c_1+a_1\ (2\ c_1-1)}, \frac{a_1^2\ d_1+a_1\ d_1\ (c_1-1)+\sqrt{a_1}\ d_1\ \sqrt{c_1}\ \sqrt{a_1+c_1-1}}{a_1^2+(c_1-1)\ c_1+a_1\ (2\ c_1-1)})$
and
$(\frac{a_2^2\ d_2+a_2\ d_2\ (c_2-1)-\sqrt{a_2}\ d_2\ \sqrt{c_2}\ \sqrt{a_2+c_2-1}}{a_2^2+(c_2-1)\ c_2+a_2\ (2\ c_2-1)}, \frac{a_2^2\ d_2+a_2\ d_2\ (c_2-1)+\sqrt{a_2}\ d_2\ \sqrt{c_2}\ \sqrt{a_2+c_2-1}}{a_2^2+(c_2-1)\ c_2+a_2\ (2\ c_2-1)})$
respectively.

In mathematical notation, the hypotheses based on the time point where RGR is maximized and points of inflexion of the RGR curve are stated as follows.

$$H_{10} : \frac{a_1 d_1}{a_1+c_1} = \frac{a_2 d_2}{a_2+c_2} \; ag \; H_{11} : \text{not} \quad H_{10},$$

$$H_{20} : \frac{a_1^2 \; d_1+a_1 \; d_1 \; (c_1-1)-\sqrt{a_1} \; d_1 \; \sqrt{c_1} \; \sqrt{a_1+c_1-1}}{a_1^2+(c_1-1) \; c_1+a_1 \; (2 \; c_1-1)}$$
$$= \frac{a_2^2 \; d_2+a_2 \; d_2 \; (c_2-1)-\sqrt{a_2} \; d_2 \; \sqrt{c_2} \; \sqrt{a_2+c_2-1}}{a_2^2+(c_2-1) \; c_2+a_2 \; (2 \; c_2-1)} \quad ag \quad H_{21} : \text{not} \quad H_{20},$$

$$H_{30} : \frac{a_1^2 \; d_1+a_1 \; d_1 \; (c_1-1)+\sqrt{a_1} \; d_1 \; \sqrt{c_1} \; \sqrt{a_1+c_1-1}}{a_1^2+(c_1-1) \; c_1+a_1 \; (2 \; c_1-1)}$$
$$= \frac{a_2^2 \; d_2+a_2 \; d_2 \; (c_2-1)+\sqrt{a_2} \; d_2 \; \sqrt{c_2} \; \sqrt{a_2+c_2-1}}{a_2^2+(c_2-1) \; c_2+a_2 \; (2 \; c_2-1)} \quad ag \quad H_{31} : \text{not} \quad H_{30}.$$

If we assume errors distribution to be $N(0, \sigma^2)$ then the nonlinear least square estimates (NLSE) and MLE are equivalent (Seber & Wild, 1989).

Simulation for the verification of the asymptotic normality of NLSE

It is really questionable whether the asymptotic normality of the nonlinear least square estimates really holds good for the scanty, sparse census data set. Extensive simulation is needed to verify it in a more precise way. In this context we can also think for the profile likelihood which is not symmetric but strongly skewed. Although the estimation through profile-likelihood may be an alternative approach but it is typically applicable in the situation where the number of nuisance parameters is significant. But in our situation the only nuisance parameter is r and this approach may not improve the estimation procedure. So we stick to asymptotic normality. For illustration we use the sparse and scanty census data of India and West Bengal which are available for the same time periods. Now let us denote the MLE (NLSE) of $(a_1, d_1, c_1, a_2, d_2, c_2)$ by $(\hat{a_1}, \hat{d_1}, \hat{c_1}, \hat{a_2}, \hat{d_2}, \hat{c_2})$. Since t_1 and t_2 are one to one function of the parameters $(a_1, d_1, c_1, a_2, d_2, c_2)$ the ML estimates of t_1 and t_2 are $\hat{t_1}$, $\hat{t_2}$ respectively with $\hat{t_1} = \frac{\hat{a_1}\hat{d_1}}{\hat{a_1}+\hat{c_1}}$ and $\hat{t_2} = \frac{\hat{a_2}\hat{d_2}}{\hat{a_2}+\hat{c_2}}$. For simplicity we have taken the parameter a to be known with magnitude 1.

At first using the census data the model parameters are estimated through non-linear least square (Marquandt, 1963). The response variable ARGR is normally distributed which holds from the normality assumption of the error terms. We can approximate the mean of ARGR by putting the estimated values of the parameters of non-linear function. Since the raw census data set of West Bengal consists of only eight observations, to identify the asymptotic behavior of the estimated parameters, we have drawn 1000 random samples each of size eight. As the parameters are very sensitive and some of them are very small in magnitude, the variance of errors or ARGR is to be chosen suitably small. For each of the 1000 sets of random samples two parameters d and c are estimated 1000 times. The histograms are shown in Figure 5 for the parameters d and c respectively. The simulated histograms are quite close to normal density. The p-values for goodness of fit test are 0.80941,0.93898 for the parameters d and c respectively. So, the assumption of normality for NLSE is quite reasonable.

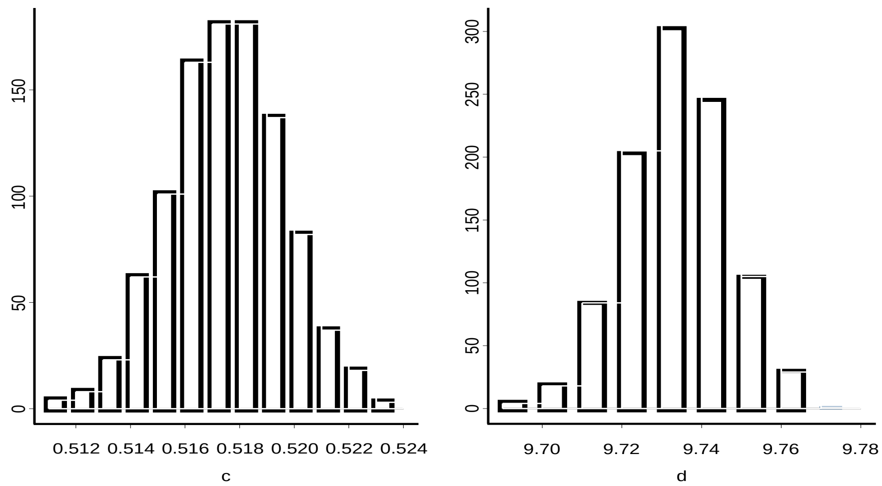

Figure5: Histograms of the distribution of parameters "c" and "d"

Test Statistics

From the simulation results asymptotically we have,
$\hat{\theta}_i \sim N(\theta_i, var(\hat{\theta}_i)), i = 1(1)3$, where $(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) = (\hat{a}, \hat{c}, \hat{d}) \Rightarrow \hat{t_k} \sim N(t_k, var(\hat{t_k})), k = 1, 2$, and $(t_k,\hat{t_k})$are the one to one function of the $(\theta_i, \hat{\theta_i})$.

The test statistics and critical regions for H_{10}, H_{20} and H_{30} are respectively

$$T_1 = \frac{(\hat{t_1} - \hat{t_2}) - (t_1 - t_2)}{\sqrt{\hat{Var}(\hat{t_1}) + \hat{Var}(\hat{t_2})}} \tag{26}$$

$$T_2 = \frac{(\widehat{t_{1inf}} - \widehat{t_{2inf}}) - (t_{1inf} - t_{2inf})}{\sqrt{\hat{Var}(\widehat{t_{1inf}}) + \hat{Var}(\widehat{t_{2inf}})}} \tag{27}$$

$$T_3 = \frac{(\widehat{t^*_{1inf}} - \widehat{t^*_{2inf}}) - (t^*_{1inf} - t^*_{2inf})}{\sqrt{\hat{Var}(\widehat{t^*_{1inf}}) + \hat{Var}(\widehat{t^*_{2inf}})}} \tag{28}$$

Under H_{01}, H_{02} and H_{03} all T_1, T_2 and $T_3 \sim N(0,1)$. So H_{01}, H_{02} and H_{03} are rejected at $\alpha\%$ level of significance if

$$|T_1| > \tau_{\alpha/2}, |T_2| > \tau_{\alpha/2} \text{ and } |T_3| > \tau_{\alpha/2}$$

where the variances can be estimated from the general results of asymptotic theory (Rao (1987)).

7 Data Analysis

Here the pattern of change of population trend as well as ARGR do not match with any standard growth curve models for our scanty and sparse census data sets. The famous logistic and other sigmoidal growth curve models often used in demographic problems are not applicable for this situation as the ARGR is not decreasing with time or size but rather increasing at some time points. Basically that motivates us to explore STWM and PM for our data.

On the hand we expect that the DLM will give a better prediction for any data set if we can choose the proper function for the upper ceiling point which is no doubt a hard job as it is purely arbitrary. The estimates of upper ceiling points for different census figures are completely impossible for DLM where as STWM and PM do not suffer from those drawbacks. Assuming linearity of the upper ceiling point in the DLM we have fitted DLM, STWM and PM for China (Figure 6), India (Figure 7) and West Bengal (Figure 8)data. Although STWM and PM smooth out the trend in comparison with the DLM but in terms of " Residual Sum of Squares" (RSS) of over all fitting, the differences are negligible in magnitude. For example if we consider the China data the RSS values of ARGR fittings are (0.002572744, 0.002631723) and (0.00180148, 0.00181418) for STWM and PM under uncorrelated and auto correlated error structures respectively. On the other hand the RSS value for DLM is 0.002528938. It has been observed that the RSS values of STWM and PM are pretty closed with that of DLM even if with auto correlated error structure it is less than DLM. The estimated upper ceiling point under STWM with uncorrelated and autocorrelated errors are approximately 30 and 21 (measured in billion) respectively where as for proposed model the upper ceiling point of population are observed at time points 54 (year 2003) and 55 (year 2004) under uncorrelated and autocorrelated errors respectively. The estimated population figures for these time points are respectively 32 and 28 (approximately)(measured in billion) as obtained from equation 14.

The asymptotic test for detecting the demographic phase change where the RGR is maximized, is carried out for the scanty census data set of India and West Bengal. The estimated values of the phase change for West Bengal and India occurred approximately at the 6th (year 1951) and 9th (year 1981)time point of the census data set. The asymptotic normality for the NLSE is verified through extensive simulations. The observed value of the test statistic (26) is 6.45 and the test is rejected at 1 % level of significance. It indicates that the phase changes occurred for West Bengal and India are not at the same time point for the census data. So this implies the initiation of development of one province does not necessarily mean the development process of the whole country has been started.

8 Concluding Remarks

In the present work we have tried to explain the phenomenon of population change through different types of growth curves. The previous models suffered from the difficulties that they could not explain and predict properly some special situations where the population changes occurred in some exceptional ways. In addition it is also an attempt to represent the relative growth rate as a function of time only which enabled us to estimate the parameters of the models in closed forms which is not possible in STWM model.

We hope that the present study will help the population scientists to rediscover different critical nature of population growth in several developed and developing countries which will emerge due to different family welfare programmes and changing nature of the conjugal life throughout the world.

Another important contribution of the present work is the identification of demographic phase change point for a particular region. This feature will help us to take proper actions beforehand so that ultimately the present world with severe inequalities will be a homogeneous one.

When the shape of the RGR curve is bimodal or multimodal, the prediction problem can be extended in a straight forward way, representing RGR as a weighted mixture of the proposed model. The weights are determined by treating it as the additional parameters for the nonlinear least square.

Acknowledgment

The work of first author was supported by a research fellowship from Council of Scientific and Industrial Research (Sanction No. 9/28(611)/2003-EMR-I),India.

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10 Appendix

10.1 Existence of the nonlinear least square estimator

TWM and proposed model ((4) and (6)) under SDE set up (equations (12) and (13)) with uncorrelated and homoscedastic error process can be written as

$$R(t) = rP(t)^a \left[1 - \frac{P(t)}{k}\right]^c + \epsilon(t) \tag{29}$$

or

$$R(t) = b\ t^a \left[1 - \frac{t}{d}\right]^c + \epsilon(t) \tag{30}$$

Here both the equations (24) and (25) can be written as

$$y = f(\theta, x) + \epsilon \tag{31}$$

where, $f(\theta, x) = rx^a \left[1 - \frac{x}{d}\right]^c$.

Now $\hat{\theta}$ is chosen to be the nonlinear -least-square estimator of the parameter θ if it minimizes the following function.

$$S_n(\theta) = \sum_i (y(i) - f(\theta, x(i))^2 = \sum (y - f(\theta, x))^2 \tag{32}$$

where $\theta = (r, a, d, c)'$.

In case of equation (27) we have the following restrictions

(1) ϵ's are iid with mean zero and variance σ^2 ($\sigma^2 > 0$).

(2) Now for each nonzero x, $f(\theta, x)$ is definitely a continuous function of θ defined in Θ. WLG we can take x as nonzero because if it is zero then the axis can be transformed in such a way so that it is nonzero.

(3) Θ is closed, bounded (i.e. compact) subset of $\Re^4$.

With the above restrictions which are obviously true for the model (24) and (25) and by following the same technique of Jennrich(1969) it can be proved that the nonlinear-least-square estimators exist for both the two regression models.

The boundedness of Θ as described above is not a serious restriction, as most parameters are bounded by the physical constraints of the system being modelled.

10.2 Consistency of the nonlinear least square estimate for the PM

To sketch the the proof we have considered heavily the works of Jennrich (1969), Malinvaud (1970a, b) and Wu (1981), and the helpful review papers of Gallant (1975) and Amemiya (1983). Henceforth we will consider only the equation (25) written in the form (26).

Let θ^{**} is the true value of the 4 dimensional vector θ. Then to proof the consistency of the nonlinear least square estimator θ the first step is to prove that θ^{**} uniquely minimizes $plimS_n(\theta)$. In that case, if n is sufficiently large so that $n^{-1}S_n(\theta)$ is close to $plimS_n(\theta)$, then $\hat{\theta}$ which minimizes the former, will be close to θ^{**}, which minimizes the latter. This gives the weak consistency. Now let us consider the equation (27), then

$$n^{-1}S_n(\theta) = n^{-1}\sum(y - f(\theta, x)^2$$

$$= n^{-1}\sum(y - f(\theta^{**}, x) + f(\theta^{**}, x) - f(\theta, x))^2$$

$$= n^{-1}\sum(\epsilon + f(\theta^{**}, x) - f(\theta, x))^2$$

$$= n^{-1}\sum\epsilon^2 + 2n^{-1}\sum\epsilon(f(\theta^{**}, x) - f(\theta, x)) + n^{-1}\sum(f(\theta^{**}, x) - f(\theta, x))^2$$

$$= C_1 + C_2 + C_3$$

By law of large numbers, $plimC_1 = \sigma^2$. Secondly, for fixed θ^{**} and θ, $plimC_2$ follows from the convergence of

$$n^{-1}\sum(f(\theta^{**}, x) - f(\theta, x))^2$$

by Chebyshev's inequality :

$$P(n^{-1}\sum(f(\theta^{**}, x) - f(\theta, x))\epsilon > \eta^2) < \frac{\sigma^2}{\eta^2 n^2}\sum(f(\theta^{**}, x) - f(\theta, x))^2 \tag{33}$$

Since the uniform convergence of C_2 follows from the uniform convergence of the right-hand side of (28), it suffices to assume

$$\frac{1}{n}\sum f(\theta_1,x)f(\theta_2,x) \text{ converges uniformly in } \theta_1,\theta_2 \in \Theta.$$

Hence $plimS_n(\theta)$ will have a unique minimum at θ^{**} if $limC_3$ has a unique minimum at θ^{**}. Weak consistency would then be proved.

With this idea in mind, we have defined

$$\begin{aligned} B_n(\theta_1,\theta_2) &= \sum f(\theta_1,x)f(\theta_2,x), \\ D_n(\theta_1,\theta_2) &= \sum(f(\theta_1,x)-f(\theta_2,x))^2, \end{aligned} \tag{34}$$

and made the following assumption with the three restrictions already defined above :

(4).(a) $n^{-1}B_n(\theta_1,\theta_2)$ converges uniformly for all θ_1,θ_2 in Θ to function $B(\theta_1,\theta_2)$.

This implies, by expanding (29), that $n^{-1}D_n(\theta_1,\theta_2)$ converges uniformly to $D(\theta_1,\theta_2)=B(\theta_1,\theta_1)+B(\theta_2,\theta_2)-2B(\theta_1,\theta_2)$.

(4).(b) It is now further assume that $D(\theta,\theta^{**})=0$ if and only if $\theta=\theta^{**}$ (i.e. $D(\theta,\theta^{**})$ is "positive definite".

Now for the model (26)

$$B_n(\theta_1,\theta_2)=\sum r_1r_2x^{a_1}\left(1-\frac{x}{d_1}\right)^{c_1}x^{a_2}\left(1-\frac{x}{d_2}\right)^{c_2} \tag{35}$$

The function $f(\theta,x)$ in the model (26) is uniquely maximized at $x=\frac{a\ d}{a+c}$ for all positive values of the parameters.

So, $B_n(\theta_1,\theta_2)<\sum\left(\frac{a_1\ d_1}{a_1+c_1}\right)\left(\frac{a_2\ d_2}{a_2+c_2}\right)$.

$\Rightarrow n^{-1}B_n(\theta_1,\theta_2) < \left(\frac{a_1\ d_1}{a_1+c_1}\right)\left(\frac{a_2\ d_2}{a_2+c_2}\right) = B(\theta_1,\theta_2)$. $\Rightarrow B_n(\theta_1,\theta_2)$ converges uniformly to $B(\theta_1,\theta_2)$. Hence the consistent estimator of θ exists.

10.3 Detail solution and estimation procedure of the SDE

The detail derivation of the solution of SDE (already described in Section 5) under two headings uncorrelated and autocorrelated errors are given below

Uncorrelated Error Process

If $\epsilon(t)$ is " white noise" (technically, the " derivative" of a Wiener process), then ln(P(t)) has independent increments. In particular, setting $z_{t_i} = ln(P_{t_{i+i}}) - ln(P_{t_i})$ [where $ln(P_{t_i}) = ln(P(t_i)$] for i=1,2,...,(n-1), the z_{t_i}'s are mutually independent. From (20) with t_{i+1} and t_i instead of t and t_0, and $E[\epsilon(t)]$=0, we have

$$E[z_{t_i}] = \int_{t_i}^{t_{i+1}} r\ u^a\ \left(1 - \frac{u}{d}\right)^c du, \tag{36}$$

Since $Var[\epsilon(t)] = 1$, we also have

$$Var[z_{t_i}] = \int_{t_i}^{t_{i+1}} \sigma^2(u, \eta) du \tag{37}$$

$$= (t_{i+1} - t_i)\sigma^2 \qquad \text{if } \sigma^2(u, \eta) = \sigma^2.$$

Hence, from (20),

$$z_{t_i} = \int_{t_i}^{t_{i+1}} r\ u^a\ \left(1 - \frac{u}{d}\right)^c du + \epsilon(t_i) \qquad (i = 1, 2, \ldots, (n-1)), \tag{38}$$

where the $\epsilon(t_i)$ are independently distributed [since $\epsilon(t)$ has independently increments], and (32) can be fitted by weighted least squares using weights from (31).

It is also useful to have varience-covarience matrix of the original responses $ln(P(t_i))$. Conditional on $ln(P(t_{t_0}))$, the size at some chosen time origin, we have from Seber and Wild (1989)

$$Var[ln(P_{t_i})|ln(P_{t_0})] = Var[ln(P_{t_i}) - ln(P_{t_0})|ln(P_{t_0})] = \int_{t_0}^{t_i} \sigma^2(u, \eta) du = \sigma_i^2, \tag{39}$$

say. Since

$$ln(P_{t_r}) - ln(P_{t_0}) = z_0 + z_1 + z_2 + \ldots + z_{r-1},$$

it follows that for $t_i \leq t_j$,

$$Cov[ln(P_{t_i}), ln(P_{t_j})|ln(P_{t_0})] = Cov[ln(P_{t_i}) - ln(P_{t_0}), ln(P_{t_j}) - ln(P_{t_0})|ln(P_{t_0})]$$
$$= Var[ln(P_{t_i}) - ln(P_{t_0})|ln(P_{t_0})]$$
$$= Var[ln(P_{t_i})|ln(P_{t_0})]. \quad (40)$$

The error structure considered in (18) introduces variation in the size of the growth increment due to stochastic variations in the environment. Garcia[1983] considered an additional source of error, namely errors in the measurement of $ln(P_{t_i})$. We measure $ln(P_{t_i})^*$ given by

$$ln(P_{t_i})^* = ln(P_{t_i}) + \tau_i, \quad (41)$$

where, τ_1,τ_2,...,τ_n are i.i.d. $N(0, \sigma_\tau^2)$ independently of the $\ln(P(t_i))$. Thus $E[\ln(P(t_i))]^* = E[\ln(P(t_i))]$ and,
for $\ln(P(t_i))^* = (\ln(P(t_1))^*, \ln(P(t_2))^*, \dots, \ln(P(t_n))^*)'$

$$D[\ln(P)^*] = D[\ln(P)] + \sigma_\tau^2 I_n \quad (42)$$

The model can then be fitted using maximum likelihood or generalized least squares.

Autocorrelated Error Processes

When $\epsilon(t)$ has autocorrelation function $Corr[\epsilon(u), \epsilon(v)] = \rho(|u - v|)$, we have from Seber and wild (1988),

$$Var[\ln(P(t_i)) - \ln(P(t_j))] = \int_{t_i}^{t_j} \int_{t_i}^{t_j} \sigma(u, \eta)\ \sigma(v, \eta)\ \rho(|u - v|) du dv. \quad (43)$$

Also, for any two non overlapping time intervals $[t_1, t_2]$ and $[t_4, t_3]$,

$$Cov[\ln(P(t_2)) - \ln(P(t_1)), \ln(P(t_4)) - \ln(P(t_3))]$$
$$= \int_{t_1}^{t_2} \int_{t_3}^{t_4} \sigma(u, \eta)\ \sigma(v, \eta)\ \rho(|u - v|) du dv. \quad (44)$$

Consider our case in which $\sigma(u, \eta) = \sigma$ and $\rho(|u - v|) = \exp^{-\lambda\ |u-v|}$ with $\lambda > 0$. Evaluating (38) when $t_1 < t_2 < t_3 < t_4$, we have

$$Cov[\ln(P(t_2)) - \ln(P(t_1)), \ln(P(t_4)) - \ln(P(t_3))]$$
$$= \int_{t_1}^{t_2} \left(\int_{t_3}^{t_4} \sigma^2\ \exp^{-\lambda\ (u-v)} du \right) dv. = \frac{\sigma^2}{\lambda^2} \left(e^{\lambda\ t_2} - e^{\lambda\ t_1} \right) \left(e^{\lambda\ t_3} - e^{\lambda\ t_4} \right) \quad (45)$$

From (37) and symmetry considerations,

$$Var[\ln(P(t_2)) - \ln(P(t_1))] = 2\int_{t_1}^{t_2}\int_{t_1}^{t_2} \sigma^2 \ \rho^{v-u} du dv$$

$$= 2\frac{\sigma^2}{\lambda^2}\left[\lambda(t_2 - t_1) + e^{\lambda(t_1-t_2)} - 1\right] = \sigma_*^2, \tag{46}$$

say. Suppose $t_{i+1}-t_i = \Delta(i = 0, 1, 2, \ldots, n-1)$, and let $z_{t_i} = ln(P_{t_{i+i}}) - ln(P_{t_i})$ as before. Then provided Δ is not too large, we would expect from (22) and (23) that $Corr[z_{t_i}, z_{t_j}] \approx e^{-\lambda\Delta|i-j|}$ which is of the form $\rho^{|i-j|}$. Hence we would expect the error structure of the Z_{t_i} to be approximately AR(1).

If we devide (38) by Δt then $R(t_i) = \frac{z_{t_i}}{\Delta t}$ is the ARGR and empirical fitted model is represented by

$$R(t_i) = \frac{1}{\Delta t}\int_{t_i}^{t_{i+1}} ru^a \left(1 - \frac{u}{d}\right)^d du + \epsilon(t_i) \tag{47}$$

where $\epsilon(t_i)$ are possibly autocorrelated errors. The trend relationship for $R(t_i)$ may be transformed to remove the heteroscedasticity before fitting a model with additive errors. As

$$\int_{t_i}^{t_{i+1}} ru^a \left(1 - \frac{u}{d}\right)^d du = rt_i^a \left(1 - \frac{t_i}{d}\right)^d \Delta t \tag{48}$$

implies $R(t_i) \approx h(t_i; \theta)$. Different parameters are estimated through (48) which we already discussed in Section 4.

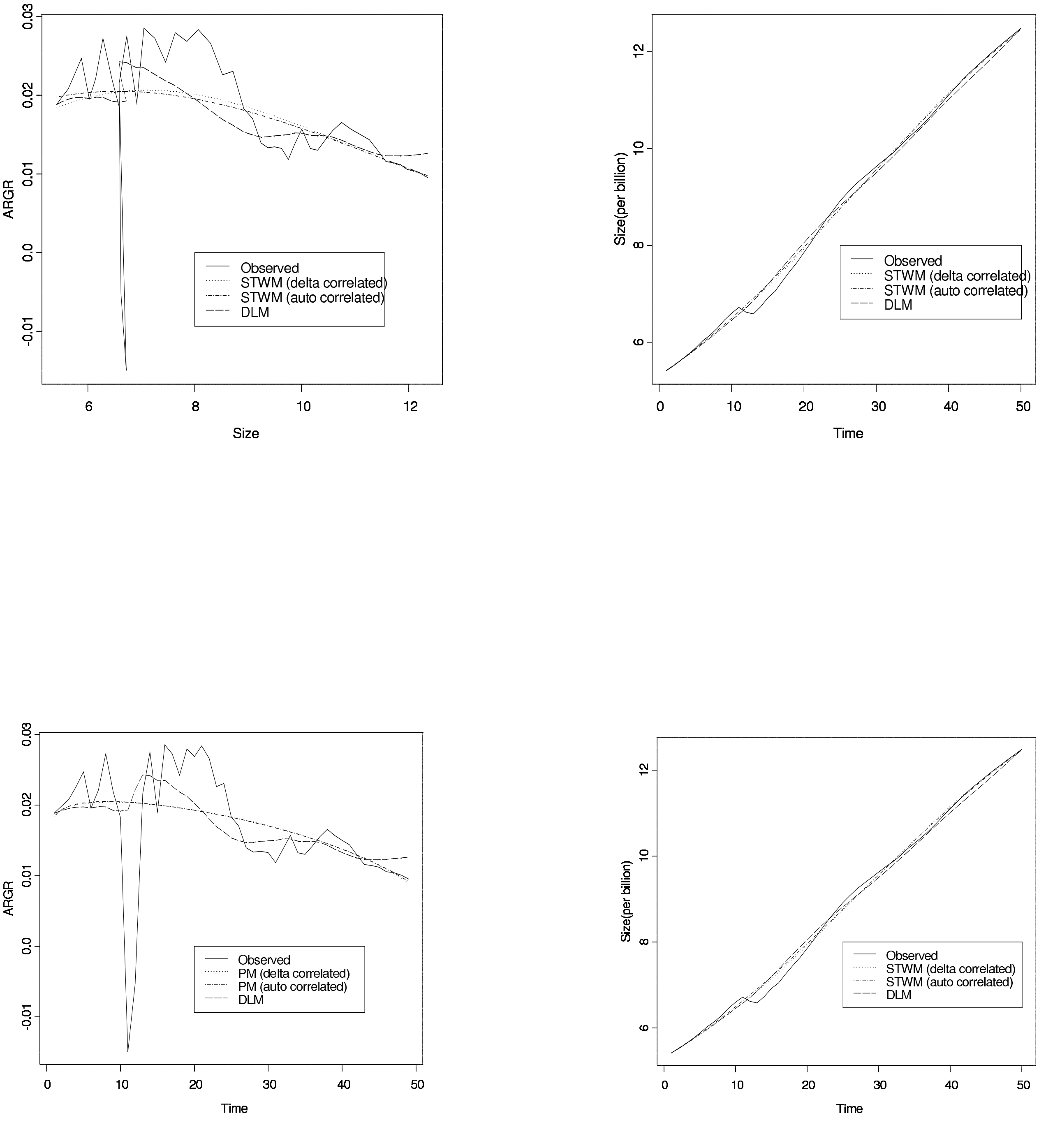

Figure 6: Observed and Fitted ARGR and Size through STWM, PM and DLM for China

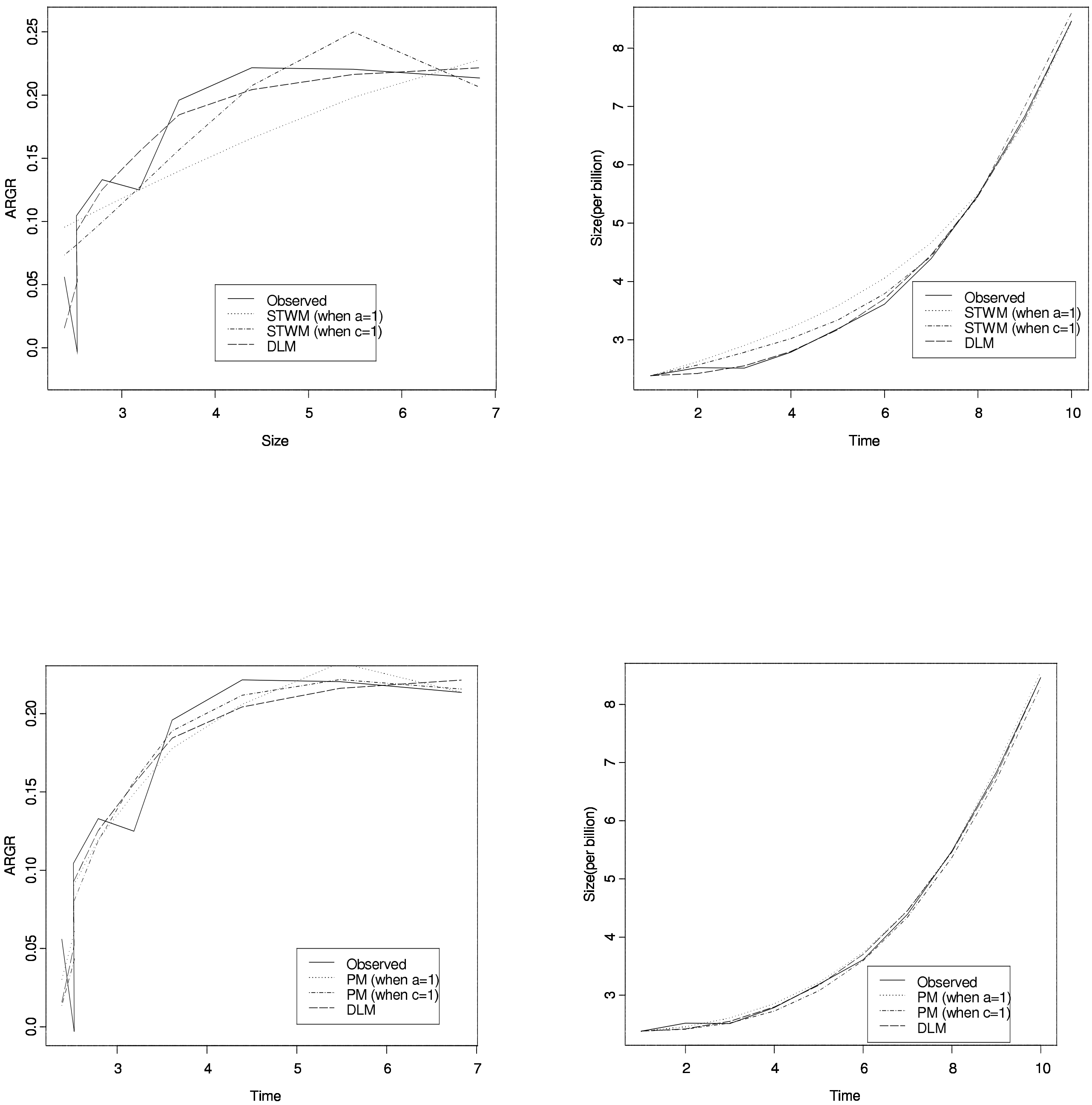

Figure 7: Observed and Fitted ARGR and Size through STWM, PM and DLM for India

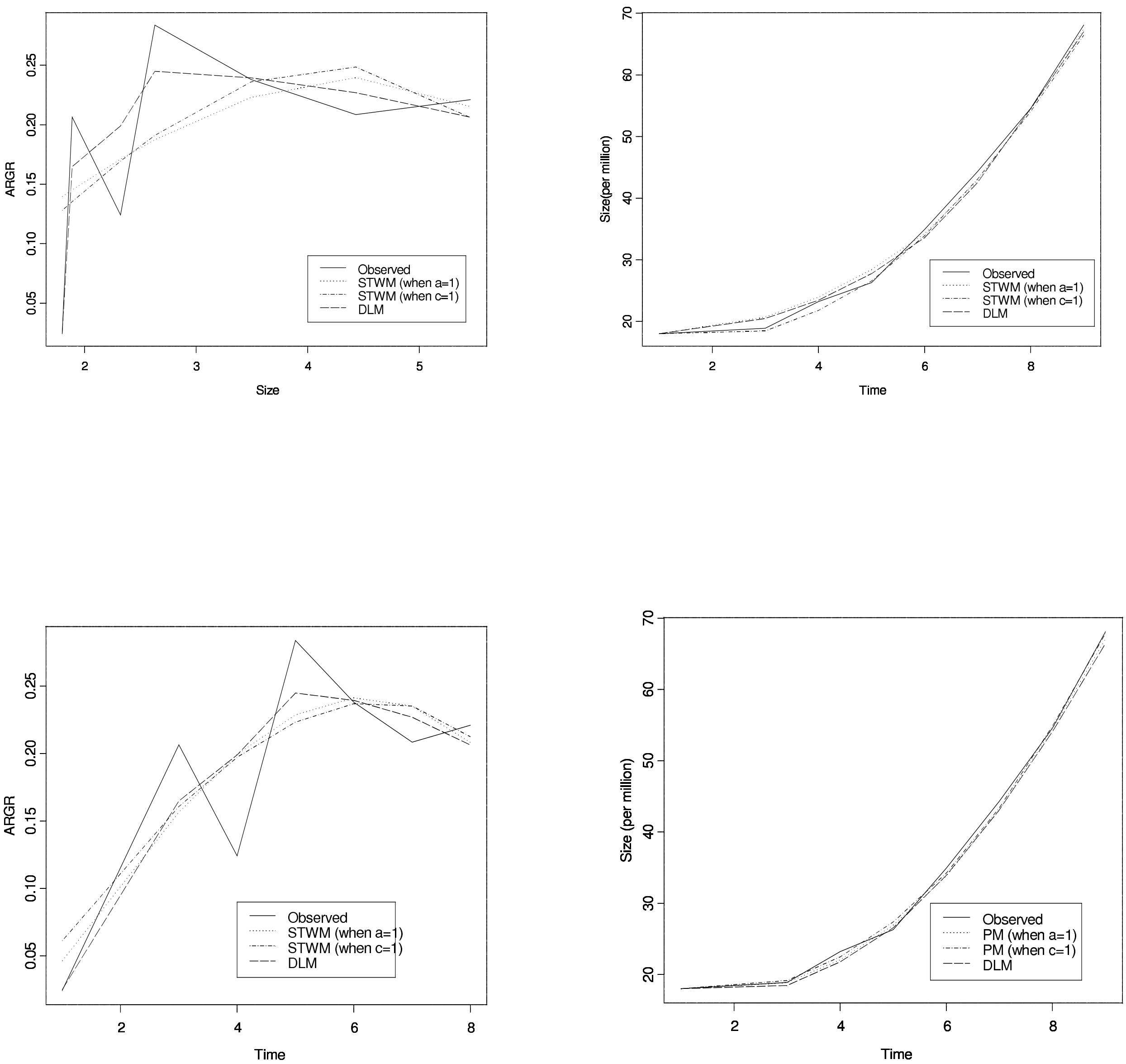

Figure 8: Observed and Fitted ARGR and Size through STWM, PM and DLM for West Bengal