# INVERSION OF 25×25 MATRIX ON 602A CALCULATING PUNCH

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## Introduction

Systems of linear equations in several variables appear in analytical work relating to almost all branches of sciences. For their solution the inversion of a given matrix of coefficients of the unknowns is often found useful. Punched card machines have been used variously for the solution of such equations especially when there are large number of unknowns. A problem requiring the inversion of a matrix of order 25×25 recently arose in the course of certain econometric studies conducted in the Indian Statistical Institute under the guidance of Ragnar Frisch in connection with Planning for National Development. This paper deals with the punched card method adopted for the inversion of the matrix. F. M. Verzuh (1949) indicated a method of solution of simultaneous equations, with the aid of the 602 Calculating Punch, dealing with matrices of the order 10×10. We have adopted here a different means for the solution by inverting a matrix of order 25×25 through methods involving simpler machine operations and substantial saving in machine hours, although the plugging required has become little complicated.

## 1. Matrix inversion

The process of matrix inversion adopted is indicated below. Details are given by Rao (1952).

Let the matrix to be inverted be,

$$A = \begin{matrix} a_{11} \cdots\cdots\cdots\cdots a_{1n} \\ a_{21} \cdots\cdots\cdots\cdots a_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1} \cdots\cdots\cdots\cdots a_{nn} \end{matrix}$$

(a) Divide row 1 by $a_{11}$, multiply the new row 1 by $a_{i1}$ and subtract from this product row $i$ for $i = 2, 3, 4, \ldots, n$, successively. This set of operations result in 1 in the first position, and zeros elsewhere in the first column of the matrix. Again applying the same method on row 2, we get 1 in the second position of the main diagonal and zeroes everywhere else in the second column. By repeating this process on the subsequent rows, the given matrix is reduced to a triangular matrix of the following form having 1 in each of its diagonal elements.

$$\begin{matrix} 1 & * & * & \cdots\cdots\cdots & * \\ 0 & 1 & * & \cdots\cdots\cdots & * \\ 0 & 0 & 1 & \cdots\cdots\cdots & * \\ \cdots & \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots\cdots & 1. \end{matrix}$$

(b) Now take this triangular matrix and multiply row $n$ by the $n$-th element of row $i$ and subtract it from the $i$-th row where $i = 1, 2, \ldots, n-1$, this operation results in all zeros in the $n$-th column of row $1, 2, \ldots, n-1$. Again, applying this method on row $n-1$ we get all zeroes in the $(n-1)$th column of row $1, 2, \ldots, n-2$. Repeat this process on row $n-2, \ldots 2, 1$, until this traingular matrix is reduced to a unit matrix $I$.

Simultaneous application of the same sequence of operations on a unit matrix, when taken on the right-hand side along with the matrix $A$, leads to the transformation of the unit matrix into $A^{-1}$, the inverse of $A$.

The operational steps may be summarised here. First we take the matrix $A$ and a unit matrix $I$:

$$[A \mid I]$$

Performing the operation (a) above, we have,

$$[B \mid C]$$

where $B$ is a triangular matrix having 1 in each of its diagonal elements.

Then performing operation (b), we have,

$$[I \mid A^{-1}]$$

## 2. Machine operations

The detailed steps are stated below.

(1) Individual cards are punched for each of the elements of the matrices $A$ and $I$ with respective row and column numbers for identification according to the following punch card design.

DESIGN OF CARD

| sl. no. | item | card columns | remarks |
|---|---|---|---|
| 1. | step no. | 1–2 | |
| 2. | row no. ($i$) | 3–4 | |
| 3. | col. no. ($j$) | 5–6 | |
| 4. | value of the element ($a_{ij}$) | 7–15 | (algebraic sign in col. 7 : 0 for positive, 5 for negative; 6 places of decimals) |
| 5. | either quotient $\left(B = \frac{a_{ij}}{a_{ii}}\right)$ | | |
| | or product difference ($\pm S$) | 16–24 | (sign in col. 16, 6 places of decimals) |
| 6. | designation | 80 | ($X$ for $P$ cards, 1 for $Q$ cards, blank for $R$ cards) |

(2) The cards containing the elements of the matrices $A$ and $I$ are sorted according to row and column numbers (i.e., $i$ and $j$).

(3) Separate the first row of the matrix (i.e. cards with $i = 1$).

(4) Remove the first card (i.e., $a_{11}$) and punch $X$ on it in col., 1.

(5) Divide the elements $a_{ij}$ on the other cards of the first row by $a_{11}$ on the leading column card (card with $X$ in col. 1). Punch the quotients on the dividend cards.

(6) Reproduce the dividend cards punching the quotients in the original field for elements. These cards will form the pivotal row. Punch 1 in col. 80 on these set of cards (call them $Q$ cards).

(7) Take the cards of the next row of the martices. Punch $X$ on col. 80 of the first card (call it $P$ card and the remaining cards of the row $R$ cards).

(8) Place the $Q$ cards followed by the $P$ and $R$ cards and sort them on cols., 5-6 (i.e., for $j$).

(9) Pass the cards through 602$A$ to calculate$(\pm P)$. $(\pm Q_j)-(\pm R_j) = \pm S_j$, $(\pm S_j$ being punched on the $R_j$ cards). Details of this step are given in Section 3.

(10) Separate the $Q$ cards from the $P$ and $R$ cards by sorting on col., 80.

(11) Reproduce the $R_j$ cards to bring back $S_j$ in the original field for elements.

(12) Perform sum-check on Tabulator.

(13) Take the cards of the next row of the composite matrix and repeat operations from 8 onwards till the last row is exhausted. Then the pivotal row ($Q$ cards) is eliminated.

(14) Take all reproduced $R$ cards (i.e., new $S$ cards) which forms a new matrix with the number of rows reduced by one. Repeat steps (3) to (14) till all the pivotal rows are eliminated.

(15) Take all the pivotal row cards ($Q$ cards)—eliminated by earlier operations. Sort them for $i$ and $j$. They will now form new matrices $B$ followed by $C$. The rectangular matrix constituted by the last column of the matrix $B$ and the entire matrix $C$ is separated out.

(16) Take the last row of the matrix so formed and treat it as the pivotal row. Punch 1 in col. 80 on this set of cards. Call them $Q$ cards as in (6). Repeat procedures (7) to (13) with one difference at the calculating stage (9) where the formula will be $-[(\pm P)(\pm Q_j)-(\pm R_j)] = \pm S_j$.

(17) Take all reproduced $R$ cards (i.e. new $S$ cards) which forms a new matrix with the number of rows reduced by one, as before. At this stage, insert the next to last column vector from the $B$ matrix in the matrix at hand, Then repeat step (16) till all the columns of the $B$ matrix are exhausted.

The matrix formed by these $Q$ cards will represent the required matrix $A^{-1}$.

## 3. Computation on 602A calculating punch

The calculation work would consist of two types of operations in the 602A calculating punch, the first involving the division $\frac{a_{ij}}{a_{ii}}$ for building up of the pivotal row and the second giving rise to the expression : $(\pm P).(\pm Q)-(\pm R) = \pm S$. The breaking up of the machine operations into two distinct parts would prove very advantageous in the setting up of the machine. The first operation is required 25 times for the building up of 25 pivotal rows. The second type of operations would continue for the rest of the calculations in two stages as indicated above.

The panel wiring required is shown in the diagram, where it may be observed that factor $P$ is indicated by $X$ in col. 80.

| | | | | |
|---|---|---|---|---|
| " | $Q$ | " | 1 | " |
| " | $R$ | " | blank | " |

0 is punched in col. 7 for positive sign and 5 punched for negative sign (so that $5+5 = 0$, $0+5 = 5$) as suggested by Matthai (1950).

*Read cycle :* 2 pilot selectors (nos. 1 and 2) are picked up from the control brush set on card column 80, by using $X$ and 1 in card column 80.

| | | | | |
|---|---|---|---|---|
| $P$ | is read in storage | $1R$, | its sign in | storage $1L$ |
| $Q$ | " | 2 | " | counter 4 |
| $R$ | " | 4 | " | storage $3R$ |

Cards for $P$ and $Q$ are read and skipped off. Cards for $R$ are held at the punch bed till the calculations are completed and the result $S$ punched on it.

*Programme* 1 : Multiplicand $Q$ is read out of storage $2R$ and read in counters $1+2+3$ to develope the product $PQ$.

*Programme* 2 : $PQ$ is transferred from counters $1+2+3$ to counters $5+6$ corrected to 6 places of decimals. The sign for $PQ$ is developed in counter 4 by adding digit in storage $1L$ to the digit in counter 4.

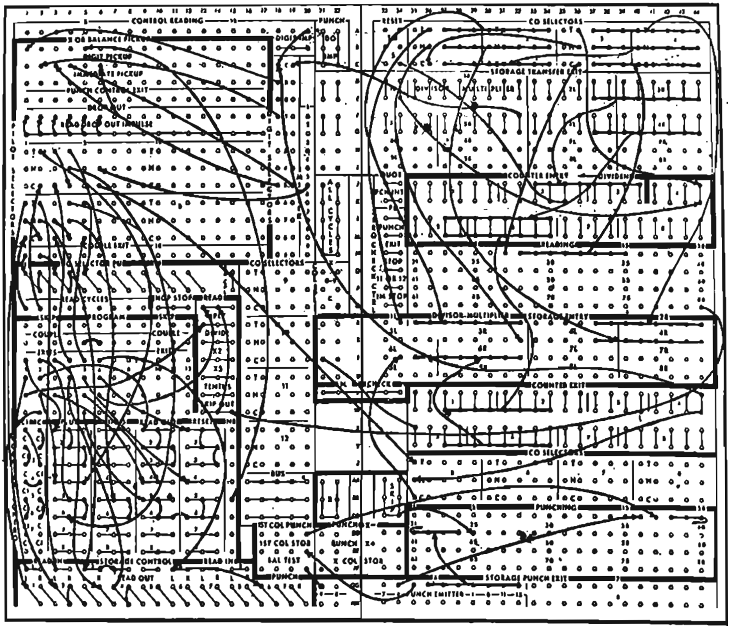
*Programme* 3 : Three pilot selectors (3, 4 and 6) energized by readings of digit 5 in counter 4 and storage $3R$. Two of the Pilot selectors are picked up by the digit in counter 4 and the third one by that of storage $3R$.

*Programme* 4 : $R$ from storage $4L+4R$ is read into counters $5+6$ in true form when the pair of selectors 4 and 5 are unequal (i.e. only one of them is picked up) and in complementary form when selectors are equal (both picked up or dropped).

*Programme* 5 : Co-selector number 5 is picked up through pilot selector 5 energized by complementary figures in counters $5+6$. Digit 5 from emitter is taken to the transfer side of co-selector 5 through Pilot selector 6 which moves by digit 5 in counter 4. Digit from counter 4 is read out and put directly to $N$ hub of the co-selector. $S$ is read out of counters $5+6$ and put in storage 6 for punching, while its sign is furnished to Punch Storage 6 through the co-selector 5. The card for $R$ is skipped off after $S$ is punched on it.

It will be seen that in the equation $(\pm P).(\pm Q)-(\pm R) = (\pm S)$. The sign of $S$ opposite to that of the product $PQ$ (as registered in counter 4 at Programme 2) when $PQ$ and $R$ are of same signs and $R$ is greater than $PQ$ (indicated by complementary figures in counters 5+6). In all the other cases, the sign of $S$ is same as that of $PQ$. This is effected by the wiring shown in the diagram.

WIRING DIAGRAM




## 4. Merits of the method

About 75 calculating machine hours are required to be spent for working out the 25×25 matrix. The merit of the procedure detailed above lies in the fact that machine operations are reduced to a simple circular flow of the following form:

Input → Reproducer → Tabulator → Sorter → 602A → Output
↑ ←——————————————— ↓

In contrast to Verzuh's method of punching two elements in one card, individual cards are punched for each of the elements, thus keeping the factors $P$ and $Q$ dissociated. Machine operations are made simpler resulting in a substantial saving of machine hours, while complication if any is introduced only in the wiring.

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References

Rao, C. R. (1952): *Advanced Statistical Methods in Biometric Research*, 30-31, John Wiley & Sons, New York.

Verzuh, Frank M. (1949): The solution of simultaneous equations with the aid of 602 calculating punch. *Mathematical Tables and Aids to Computations*, July 1949, 453-55.

Matthai, A. (1950): On methods of handling algebraic signs on the Hollerith multiplier. *Sankhyā*, 10, 124-128.

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