Fast computation of smallest enclosing circle with center on a query line segment ☆

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A B S T R A C T

Here we propose an efficient algorithm for computing the smallest enclosing circle whose center is constrained to lie on a query line segment. Our algorithm preprocesses a given set of n points $P = \{p_1, p_2, \ldots, p_n\}$ such that for any query line or line segment L, it efficiently locates a point c on L that minimizes the maximum distance among the points in P from c. Roy et al. [S. Roy, A. Karmakar, S. Das, S.C. Nandy, Constrained minimum enclosing circle with center on a query line segment, in: Proc. of the 31st Mathematical Foundation of Computer Science, 2006, pp. 765–776] have proposed an algorithm that solves the query problem in $O(\log^2 n)$ time using $O(n \log n)$ preprocessing time and $O(n)$ space. Our algorithm improves the query time to $O(\log n)$; but the preprocessing time and space complexities are both $O(n^2)$.

1. Introduction

The 1-center problem was originally posed in 1857 by Sylvester [13]. Here the objective is to report the circle of the minimum radius which can enclose a given set P of n points. Elzinga and Hearn [4] first proposed an $O(n^2)$ time algorithm for this problem. Later, Shamos and Hoey [12], Preparata [9], and Lee [7] independently proposed $O(n \log n)$ time algorithms to solve this problem. Finally Megiddo [8] proposed an optimal $O(n)$ time algorithm using prune-and-search technique.

Megiddo [8] studied the constrained case of this problem where the center of the smallest enclosing circle of P lies on a given straight line. He gave an $O(n)$ time solution. Hurtado et al. [6] and Bose et al. [2] considered the problem where the center of the smallest enclosing circle of P is constrained to lie inside a given simple polygon of size m. The proposed algorithms run in $O(k + (n + m) \cdot \log(n + m))$ time, where k is the number of intersections of the boundary of the polygon with the furthest point Voronoi diagram of P. In the worst case, k may be $O(n^2)$. This result is later improved to $O((n + m) \log m + m \log n)$ [3]. In particular, if the polygon is a convex one, then the problem can be solved in $O((n + m) \log(n + m))$ time [2]. In a further generalization of this problem, r ($\geqslant 1$) simple polygons with a total of m vertices are given; one of them can contain the center of the smallest enclosing circle of the point set P [3]. The time complexity of this version is $O((n+m) \log n + (n\sqrt{r}+m) \log m + m\sqrt{r} + r^{3/2} \log r)$. The query version of the smallest enclosing circle was studied first by Roy et al. [11]. Their proposed algorithm reports the center and the radius of the smallest enclosing circle whose center is constrained to lie on a query line segment in $O(\log^2 n)$ time. The preprocessing time and space complexities of their algorithm are $O(n \log n)$ and $O(n)$, respectively.

In this work, we will consider the problem posed in [11]. We will reduce the query time complexity to $O(\log n)$ using the technique of geometric duality. But the preprocessing time and space complexities are both $O(n^2)$.

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2. Constrained 1-center problem

Given a point set $P = \{p_1, p_2, \ldots, p_n\}$ and a query line L, our objective is to enclose P with a minimum radius circle C whose center c is constrained to lie on L. Note that, the smallest circle enclosing the vertices of the convex hull of the point set P will also enclose all the points in P. So without loss of generality, we consider that the members of P are in convex position. We also assume, $p_1, p_2, \ldots, p_n$ are ordered in clockwise direction along the convex hull of P.

2.1. Basic results

The furthest point Voronoi diagram $\mathcal{V}(P)$ of a point set P partitions the plane into n unbounded convex regions, namely $\mathcal{R}(p_1), \mathcal{R}(p_2), \ldots, \mathcal{R}(p_n)$, such that for any point $p \in \mathcal{R}(p_j)$, $\delta(p, p_j) > \delta(p, p_k)$ for all $k = 1, 2, \ldots, n$, and $k \neq j$. Here $\delta(.,.)$ denotes the Euclidean distance between a pair of points. The furthest point Voronoi diagram $\mathcal{V}(P)$ can be constructed in $O(n \log n)$ time using $O(n)$ space and for any point p in the plane, it's furthest neighbor can be identified in $O(\log n)$ time by locating the region $\mathcal{R}(p_i)$ containing p [10].

Observe that the circle C must pass through at least one point of P. If C passes through a single point p_i, c is in $\mathcal{R}(p_i)$ and c is the perpendicular projection of p_i on the query line L. We can conclude that $p_i \in P$ is the furthest point from line L. Let $p_a, p_{a+1}, \ldots, p_b$ define the upper hull, and $p_b, p_{b+1}, \ldots, p_a$ defines the lower hull of P with L as the x-axis. Here the indices of p_i are considered as modulo of n. Observe that the sequence of distances of the points $p_a, p_{a+1}, \ldots, p_b$ from the line L is unimodal. Similarly it is true for the points $p_b, p_{b+1}, \ldots, p_a$. Hence the furthest point p_i from L can be determined in $O(\log n)$ time.

When C passes through two points $p_i, p_j \in P$, the center c lies on the Voronoi edge separating $\mathcal{R}(p_i)$ and $\mathcal{R}(p_j)$. Since c also lies on L, it is the intersection point of the perpendicular bisector of p_i and p_j with the query line L. In the degenerate case, C may pass through three or more points of P where c will be a vertex of $\mathcal{V}(P)$ lying on L. Below we concentrate on the case where the minimum enclosing circle passes through two points.

For any point q in $\mathcal{R}(p_i)$, $\rho(q)$ is defined as the Euclidean distance between q and p_i. Thus $\rho(q) = \max_{i=1}^{n}(\delta(q, p_i))$. Let L intersect the edges $e_1, e_2, \ldots, e_k$ of $\mathcal{V}(P)$ in order at the points $a_1, a_2, \ldots, a_k$, respectively. For any a_i as center, $\rho(a_i)$ is the radius of the smallest enclosing circle of P, and the sequence $\{\rho(a_1), \rho(a_2), \ldots, \rho(a_k)\}$ is unimodal [11].

Let g be the center of the unconstrained smallest enclosing circle of P; g lies on an edge or a vertex of $\mathcal{V}(P)$. We may consider $\mathcal{V}(P)$ as a tree $\mathcal{T}$ with g as root node, all the Voronoi vertices are the internal nodes of $\mathcal{T}$, and all the Voronoi edges are edges of $\mathcal{T}$ (see Fig. 1(a)). We will use $\pi(v)$ to denote the path from g to a point v on an edge in $\mathcal{T}$. Observe that, as we traverse from g along a path in $\mathcal{T}$, the ρ-value of the nodes along that path increases monotonically.

3. Algorithm

For any constant $\alpha \geqslant 0$, let us define a region $Q(\alpha)$ such that for any point $q \in Q(\alpha)$, we have $\rho(q) \leqslant \alpha$. Now we have the following lemma.

Lemma 3.1. *If $\alpha < \rho(g)$ then $Q(\alpha) = \emptyset$; but if $\alpha > \rho(g)$ then $Q(\alpha)$ is a nonempty convex region containing the point g.*

Proof. The first part of the lemma is obvious. For the second part, let $D(p, \alpha)$ denote the circle centered at a vertex $p \in P$ with radius α. Since $\alpha > \rho(g)$, $Q(\alpha) = \bigcap_{i=1}^{n} D(p_i, \alpha)$ is a convex region. Again, since $g \in D(p_i, \alpha)$ for all i, we have $g \in Q(\alpha)$. □

Note that, for a given $\alpha > \rho(g)$, we can also write $Q(\alpha) = \bigcup_{i=1}^{n}(\mathcal{R}(p_i) \cap D(p_i, \alpha))$, and for a pair of points p_i and p_j, $i \neq j$, the interiors of the convex regions $\mathcal{R}(p_i) \cap D(p_i, \alpha)$ and $\mathcal{R}(p_j) \cap D(p_j, \alpha)$ are disjoint. The region $Q(\alpha)$ is bounded by circular arcs, and has at most n vertices. Each vertex of $Q(\alpha)$ lies on an edge of $\mathcal{V}(P)$ (see Fig. 1(b)). Let us now consider the regions $Q(\alpha_i)$, $i = 1, 2, \ldots, m$, where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the ρ-values of the vertices of $\mathcal{V}(P)$ in nondecreasing order. Thus $Q(\alpha_1) \subset Q(\alpha_2) \subset Q(\alpha_3) \subset \cdots \subset Q(\alpha_m)$. Let z be the smallest index, such that L intersects $Q(\alpha_z)$. If α denotes the length of the radius of the constrained minimum enclosing circle,

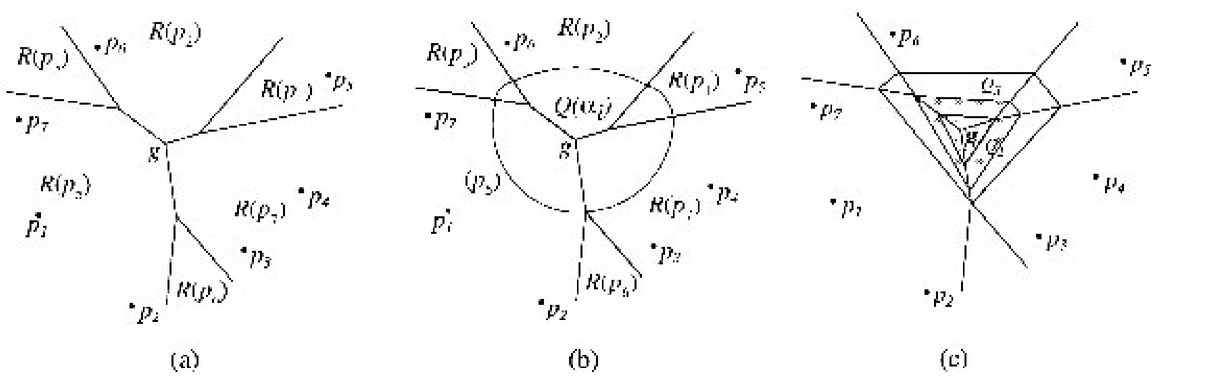

Fig. 1. The tree $\mathcal{T}$ and the regions Q_i's defined from $\mathcal{V}(P)$.

then $\alpha_{z-1} > \alpha \geqslant \alpha_z$. Now we need to consider the following two cases.

(i) L does not intersect any Voronoi edge inside $Q(\alpha_z)$.
(ii) L intersects some of the Voronoi edges inside $Q(\alpha_z)$.

In case (i), L intersects a single circular arc $D(p_k, \alpha_z)$ (say). This in turn, implies c is the projection of the vertex p_k on L, and the minimum enclosing circle passes through p_k only. In Section 2.1, we have discussed the method of obtaining the solution for this case.

In case (ii), the possible locations for c are the intersection points of L and the Voronoi edges inside $Q(\alpha_z)$. Let $\alpha_1^*, \alpha_2^*, \ldots, \alpha_{m^*}^*$ be the ρ-values attached with those intersection points. The objective is to compute $\alpha = \min(\alpha_1^*, \alpha_2^*, \ldots, \alpha_{m^*}^*)$. A straight forward computation of α needs $O(n)$ time. But we can expedite our search using the fact that the sequence $\{\alpha_1^*, \alpha_2^*, \ldots, \alpha_{m^*}^*\}$ is unimodal [11]. While searching for z we may simplify the situation by assuming $Q(\alpha_i)$'s are convex polygons. For each $Q(\alpha_i)$ its set of vertices remains same but each circular arc is replaced by a straight line segment connecting the corresponding pair of vertices.

We will use geometric duality to solve this problem. The *duality transform* maps a line $l: y = ax + b$ in the primal plane to a point $l' = (-a, b)$ in the dual plane and also maps a point $p = (a', b')$ of the primal plane to a line $p': y = a'x + b'$ in the dual plane [1].

The dual of the upper (resp., lower) hull of a convex polygon is a polygonal chain representing the lower (resp., upper) envelope of the set of lines obtained by the dual transformation of the vertices of the polygon [1]. The lower hull and upper hull of a convex polygon $Q_i = Q(\alpha_i)$ maps to a pair of x-monotone chains (χ_i^l, χ_i^u), respectively, in the dual plane. Fig. 2 demonstrates the dual of Q_i's for $i = 1, 2, \ldots, m$. The interior region of the convex polygon Q_i in the primal plane is mapped to the region Q_i', bounded by the pair of chains (χ_i^l, χ_i^u) in the dual plane. The following lemma leads to the fact that $Q_1' \subset Q_2' \subset Q_3' \subset \cdots \subset Q_m'$ (see Fig. 2(b)).

Lemma 3.2. *Let the sets* $\Gamma_1 = \{\chi_1^l, \chi_2^l, \ldots, \chi_m^l\}$ *and* $\Gamma_2 = \{\chi_1^u, \chi_2^u, \ldots, \chi_m^u\}$. *Then,*

(a) *any pair of chains from the set* $\Gamma_1 \cup \Gamma_2$ *are non-intersecting,*
(b) *each chain in* Γ_1 *lies above all the chains in* Γ_2 *and each chain in* Γ_2 *lies below all the chains in* Γ_1,
(c) *for any* $i > j$, χ_i^l *lies above* χ_j^l *and similarly* χ_i^u *lies below* χ_j^u.

Proof. Let us consider a pair of chains χ_i^l and χ_j^l ($i < j$), which have a common intersection point. Then we can always find two points p and q in the dual plane such that p lies above χ_i^l but below χ_j^l and q lies above χ_j^l and below χ_i^l. Then the corresponding line p' in the primal plane intersect Q_j but does not intersect Q_i. Similarly q' intersect Q_i but does not intersect Q_j. This contradicts the fact that $Q_j \subset Q_i$. Thus part (a) of the lemma follows.

Parts (b) and (c) follow from the standard order preserving rule among points and lines in the primal plane and their corresponding duals in the dual plane [1]. □

Lemma 3.3. *Let z be the smallest index such that L intersects Q_z. The point L' in the dual plane corresponding to the line L, will lie in* $Q_z' \setminus Q_{z-1}'$.

Proof. If z is the smallest index, then L intersects Q_z, $Q_{z+1}, \ldots, Q_m$ but does not intersect Q_{z-1}. Hence L' lies inside Q_z' but does not lie in Q_{z-1}'. This corresponds to a pair of open regions A_z and B_z where A_z is bounded by χ_z^u and χ_{z-1}^u and B_z is bounded by χ_z^l and χ_{z-1}^l. □

3.1. Preprocessing

The ρ-values of nodes in $\mathcal{T}$ are sorted and these are $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. We compute Q_i for $i = 1, 2, \ldots, m$ as follows. We start from the root g of $\mathcal{T}$ and perform a breadth first search to identify all the edges of $\mathcal{T}$ that contains a point having ρ-value equal to α_1. These set of points define $Q(\alpha_1)$. After computing Q_i we compute Q_{i+1} by searching all the branches of $\mathcal{T}$ in a breadth first manner starting from their earlier positions.

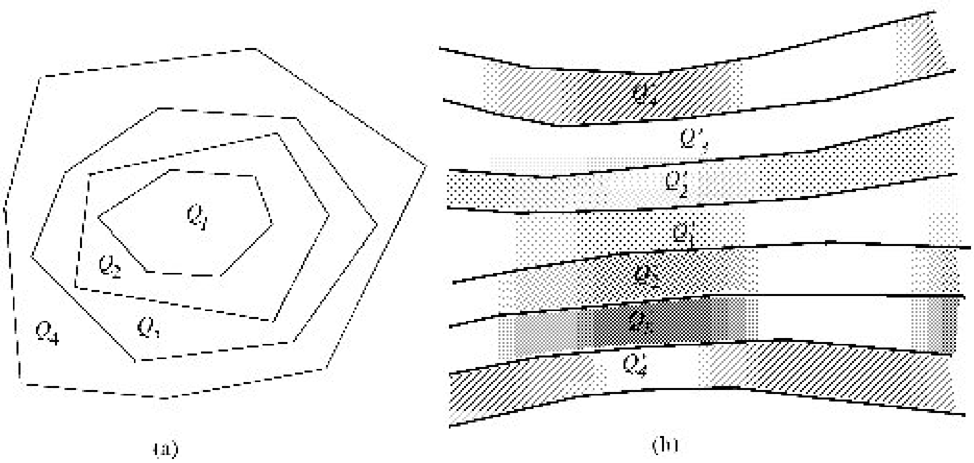

Fig. 2. (a) A set of concentric convex polygons and (b) their duals.

Next we compute the duals of $Q_1, Q_2, \ldots, Q_m$. These are a set of $2m$ x-monotone chains $\Gamma = \Gamma_1 \cup \Gamma_2 = \{\chi_m^l, \ldots, \chi_1^l, \chi_1^u, \ldots, \chi_m^u\}$. We rename the members in Γ as $\{\gamma_1, \gamma_2, \ldots, \gamma_{2m}\}$. These splits the plane into $2m+1$ disjoint x-monotone regions $R = \{r_1, r_2, \ldots, r_{2m+1}\}$ where r_i is bounded by γ_i and γ_{i+1}, $i = 2, \ldots, 2m-1$. r_1 is bounded above by γ_1 and r_{2m+1} is bounded below by r_{2m}. See Fig. 2(b) for demonstration. In order to perform a planar point location query among the monotone subdivisions R in the plane we construct a data structure as described in [5] which supports the query in $O(\log n)$ time. Thus we have the following theorem.

Theorem 1. *The time and space complexities of the preprocessing phase of our algorithm are $O(n^2)$.*

3.2. Query

Our algorithm for finding c proceeds in two phases as follows:

Phase 1: Compute the lowest index z, i.e., locate Q'_z and Q'_{z-1} such that the dual L' of the line L lies in Q'_z but it does not lie inside Q'_{z-1}.

Phase 2: Among the intersection points of L and the edges of $\mathcal{V}(P)$ inside Q_z, find the one whose distance is minimum from it's furthest point. This is the center c. As the sequence of ρ-values of the intersection points on the line L is unimodal we can apply binary search for finding c.

Lemma 3.4. *The query time complexity for searching the center c of the minimum enclosing circle of the pint set P is $O(\log n)$.*

Proof. In Phase 1, we identify Q_z from the preprocessed data structure in $O(\log n)$ time. Let G be the subset $v_1^*, v_2^*, \ldots, v_{m^*}^*$ of vertices of Q_z in clockwise direction that lie in the other side of g with respect to line L. Note that, v_1^* and $v_{m^*}^*$ can be located in $O(\log n)$ time. Consider e_i^* to be an edge of $\mathcal{V}(P)$ incident to v_i^*. Here, line L intersects the edges $e_1^*, e_2^*, \ldots, e_{m^*}^*$ of $\mathcal{V}(P)$ inside Q_z. Now c is an intersection of L and an edge in $e_1^*, e_2^*, \ldots, e_{m^*}^*$ that has minimum ρ-value. We search c among $e_1^*, e_2^*, \ldots, e_{m^*}^*$ using binary search. □

In case L is a line segment, then we first locate the center c on the line containing L. If c is not on the line segment L, then one of the end points of L nearer to c is the solution. We now have the following theorem.

Theorem 2. *Given a point set $P = \{p_1, p_2, \ldots, p_n\}$ and a query line L, the minimum enclosing circle having center on L can be reported in $O(\log n)$ time. Both the preprocessing time and space complexities are $O(n^2)$.*

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